VGIT PRESENTATION OF THE SECOND FLIP OF $\overline{M}_{2,1}$
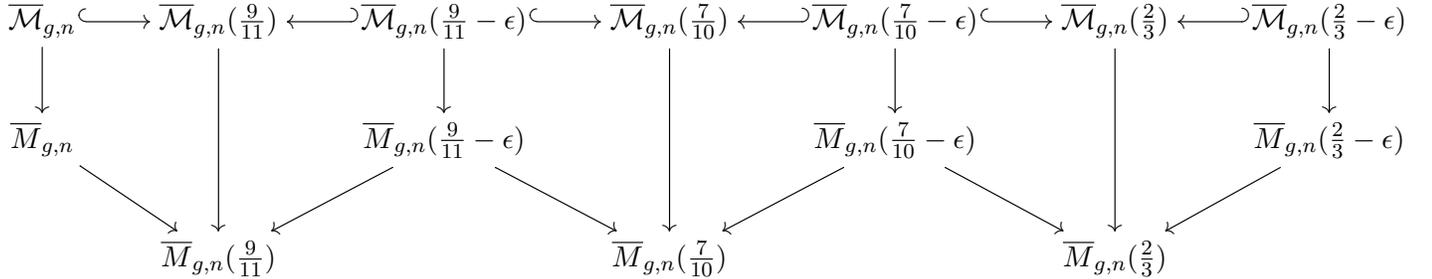
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ABSTRACT. We perform a variation of geometric invariant theory stability analysis for 2nd Hilbert points of bi-log-canonically embedded pointed curves of genus two. As a result, we give a GIT construction of the log canonical models $\overline{M}_{2,1}(\alpha)$ for $\alpha = 2/3 \pm \epsilon$, and obtain a VGIT presentation of the second flip in the Hassett-Keel program for the moduli space of pointed genus two curves.

1. INTRODUCTION

The Hassett-Keel program has a goal of inductively constructing stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ of singular curves whose moduli spaces are the following log canonical models of the stack $\overline{\mathcal{M}}_{g,n}$ of Deligne-Mumford stable curves:

$$\overline{M}_{g,n}(\alpha) = \mathrm{Proj} \bigoplus_{m=0}^{\infty} \mathrm{H}^0\big(\overline{\mathcal{M}}_{g,n}, m(K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta + (1-\alpha)\psi)\big).$$

Several stages of this program, over the field $\mathbb{C}$ of complex numbers, have been worked out to date in [HH09, HH13, AFSvdW17] for arbitrary g, and can be summarized by the following diagram (see [AFSvdW17, Main Theorem, p.3] for precise statements and notation):

$$\overline{\mathcal{M}}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n}(\tfrac{9}{11}) \hookleftarrow \overline{\mathcal{M}}_{g,n}(\tfrac{9}{11}-\epsilon) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) \hookleftarrow \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}-\epsilon) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\tfrac{2}{3}) \hookleftarrow \overline{\mathcal{M}}_{g,n}(\tfrac{2}{3}-\epsilon)$$

$$\overline{M}_{g,n} \qquad \overline{M}_{g,n}(\tfrac{9}{11}-\epsilon) \qquad \overline{M}_{g,n}(\tfrac{7}{10}-\epsilon) \qquad \overline{M}_{g,n}(\tfrac{2}{3}-\epsilon)$$

$$\overline{M}_{g,n}(\tfrac{9}{11}) \qquad\qquad \overline{M}_{g,n}(\tfrac{7}{10}) \qquad\qquad \overline{M}_{g,n}(\tfrac{2}{3})$$

Geometric Invariant Theory (GIT) constructions were given for $\overline{\mathcal{M}}_g(\alpha)$ for $\alpha > 2/3$ by Hassett and Hyeon [HH09, HH13] using Chow and asymptotic Hilbert stability analysis of pluricanonically embedded curves. The construction by [AFSvdW17] of the next step (the wall-crossing from $\alpha = 7/10 - \epsilon$ to $\alpha = 2/3 - \epsilon$) in the Hassett-Keel program required the machinery of stacks and good moduli spaces, and until now no GIT construction of this step was known for any (g, n). As one of our main results in this paper, we obtain

such a GIT construction for $(g, n) = (2, 1)$. Namely, restricting in the above diagram to $\alpha < 7/10$ and taking $g = 2$ and $n = 1$, we have the following picture:

$$
\begin{array}{c}
(1.1)
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathcal{M}}_{2,1}(\frac{7}{10} - \epsilon) \hookrightarrow \overline{\mathcal{M}}_{2,1}(\frac{2}{3}) \hookleftarrow \overline{\mathcal{M}}_{2,1}(\frac{2}{3} - \epsilon) \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\overline{M}_{2,1}(\frac{7}{10} - \epsilon) \qquad\qquad\qquad \overline{M}_{2,1}(\frac{2}{3} - \epsilon) \\
\searrow \qquad\qquad \downarrow \qquad\qquad \overset{\simeq}{\swarrow} \\
\overline{M}_{2,1}(\frac{2}{3})
\end{array}
$$

where $\overline{M}_{2,1}(\frac{2}{3} - \epsilon) \to \overline{M}_{2,1}(\frac{2}{3})$ is an isomorphism of projective good moduli spaces, and $\overline{M}_{2,1}(\frac{7}{10} - \epsilon) \to \overline{M}_{2,1}(\frac{2}{3})$ is a birational contraction of the Weierstrass divisor (see [AFSvdW17, Remark 1.1]). Our main result can now be stated (somewhat vaguely) as follows:

Main Theorem. *There is a projective master space $\overline{\overline{\mathbb{H}}}$ over $\operatorname{Spec}\mathbb{C}$ with a $\mathrm{PGL}(5)$-action and a family of ample linearizations $\mathcal{L}_\beta$ such that Diagram (1.1) is one of the two VGIT wall-crossings relating GIT quotients $\overline{\overline{\mathbb{H}}} /\!\!/_{\mathcal{L}_\beta} \mathrm{PGL}(5)$.*

We give a fuller and more technical statement of this result in Theorem 3.4 after all necessary notation is introduced. We note that the second VGIT wall-crossing from the above theorem gives rise to a new moduli stack of genus 2 pointed curves with at worst A_5-singularities, whose moduli space is a single point.

We have several reasons for focusing on $\overline{M}_{2,1}$. The first is largely historical: Hassett's paper [Has05] that laid foundations for the Hassett-Keel program was devoted to the log minimal model program for $\overline{M}_2$. The program is also complete for $\overline{M}_3$ [HL10], and so we naturally wondered what happens for $\overline{M}_{2,1}$, a space whose complexity sits between that of $\overline{M}_2$ and $\overline{M}_3$. The second reason is that, as we have discovered in the course of this project, the GIT approach to the Hassett-Keel program for $\overline{M}_{2,1}$ requires unorthodox constructions, which are unexpected from the point of view of classical GIT constructions of moduli spaces of curves.

To illustrate one such intricacy, we recall that in the case of pointed curves, we have a natural variation of the GIT problem due to linearizations coming from the parameter space of embedded curves and from the space of points. Since the moduli space $\overline{M}_{2,1}(7/10 - \epsilon)$ was constructed by Hyeon and Lee [HL14] as a GIT quotient where the parameter space of embedded curves was a Hilbert scheme of degree 6 genus 2 curves, one would expect that by varying the linearization in their GIT setup, one could obtain the next step in the Hassett-Keel program, namely $\overline{M}_{2,1}(2/3)$. This turned out not to be the case, as explained in §3.1. Instead, our construction requires a delicate analysis of finite Hilbert stability (cf. [AFS13]) of degree 6 genus 2 curves, and, as we will explain in the sequel, is impossible to replicate using the more classical Chow or asymptotic Hilbert stability.

Roadmap of the paper. In Section 2, we discuss various stability conditions for pointed genus 2 curves with at worst type A (i.e., $y^2 = x^n$) singularities, introduce bi-log-canonical curves, and their realizations as quadric sections of rational normal surface scrolls in $\mathbb{P}^4$. In Section 3, we set up our VGIT problem, state our main result (Theorem 3.4), and discuss its relation to prior works. The proof of Theorem 3.4 is given in Section 4.

We work over the field $\mathbb{C}$ of complex numbers throughout.

2. Geometry of genus 2 curves with A-singularities

In what follows, we collect together facts about the geometry of pointed genus 2 curves that are needed for our VGIT stability analysis; while these results are all well-known in the case of smooth curves, their extension to the case of mildly singular curves, e.g., to curves with higher A-singularities, including those introduced in [AFSvdW17], requires some care.

We will say that (C, p) is a *pointed genus* 2 *curve* if C is a projective Gorenstein curve of arithmetic genus 2, and $p \in C$ is its smooth point. We will say that p is a *Weierstrass point* of C if $\omega_C \simeq \mathcal{O}_C(2p)$.

We now introduce five notions of stability for pointed genus 2 curves. The first four of these appeared in [AFSvdW17], whose definitions of elliptic tails and bridges we keep (see [AFSvdW17, Definition 2.1]), while the second was introduced in [HL14].

Definition 2.1. Suppose that (C, p) is a pointed genus 2 curve with $\omega_C(p)$ ample. Then we will say that (C, p) is:

- A_2-*stable*: if C has only A_1, A_2-singularities, and C has no nodally attached elliptic tails;
- A_3-*stable*: if C has only A_1, A_2, A_3-singularities, and C has no nodally or tacnodally attached elliptic tails, and no nodally attached elliptic bridges;
- A_4-*stable*: if C has only A_1, A_2, A_3, A_4-singularities, and C has no nodally or tacnodally attached elliptic tails or nodally attached elliptic bridges;
- A_4^{non-W}-*stable*: if C is A_4-stable, and p is not a Weierstrass point.
- A_5-*stable*: if C has only A_1, A_2, A_3, A_4, A_5-singularities, C has no A_1 or A_3-attached elliptic tails or nodally attached elliptic bridges, and p is not a Weierstrass point of C.

Finally, we will say that (C, p) is a *Weierstrass curve* if (C, p) is A_4-stable and p is a Weierstrass point. In other words, a Weierstrass curve is an A_4-stable curve that is not A_4^{non-W}-stable.

These five notions of stability are modifications of the standard Deligne-Mumford stability [DM69], and arise naturally in the Hassett-Keel program for $\overline{M}_{2,1}$. The moduli space of A_3-stable pointed genus 2 curves was first constructed in [HL14] in the course of their analysis of the Hassett-Keel program for $\overline{M}_4$. The (generalizations of) moduli stacks of A_2, A_3, A_4, and A_4^{non-W}-stable curves, and their good moduli spaces, were constructed,

for an arbitrary genus and an arbitrary number of marked points, in [AFSvdW17] using stack-theoretic techniques. We note that in the terminology of [AFSvdW17], A_3-stable curves are $\left(\frac{2}{3} + \epsilon\right)$-stable, A_4-stable curves are $\frac{2}{3}$-stable, and, as we will see shortly in Corollary 2.6, A_4^{non-W}-stable curves are $\left(\frac{2}{3} - \epsilon\right)$-stable.

It was established in [AFSvdW17, Theorem 2.7], that the stacks $\overline{\mathcal{M}}_{2,1}(\alpha)$ of α-stable curves are algebraic, and there are open immersions of stacks

$$(2.2) \qquad \overline{\mathcal{M}}_{2,1}\left(\frac{2}{3} + \epsilon\right) \hookrightarrow \overline{\mathcal{M}}_{2,1}\left(\frac{2}{3}\right) \hookleftarrow \overline{\mathcal{M}}_{2,1}\left(\frac{2}{3} - \epsilon\right).$$

Our goal is to give a VGIT presentation for Diagram (2.2). Our approach to a GIT construction of the moduli stacks $\overline{\mathcal{M}}_{2,1}(\alpha)$ is to consider genus 2 curves equipped with a bi-log-canonical embedding, which we discuss below in §2.1.

Remark 2.3. A moment of reflection will convince the reader that the only A_5-stable but not A_4^{non-W}-stable curve is isomorphic to a union of two smooth rational curves along an A_5-singularity, with a marked point on one of the components, and that all such curves are isomorphic. We will prove later on that this isomorphism class gives a unique closed point in the stack of A_5-stable curves (see Lemma 4.15).

It is clear that both A_3-stability and A_4^{non-W}-stability imply A_4-stability, and that A_4^{non-W}-stability implies A_5-stability. Before we proceed, we will need the following technical result about A_4 and A_5-stable curves:

Proposition 2.4. *Every A_4-stable and A_5-stable pointed genus 2 curve is 3-connected in the sense of* [CFHR99, Definition 3.1].

Proof. Suppose $B \subset C$ is a strict subcurve such that $\deg \omega_C|_B - \deg \omega_B = 1$. Then B meets $\overline{C \setminus B}$ in a node and necessarily $p_a(B) = 1$. If B is unpointed, then B is an elliptic tail; otherwise, B is an elliptic bridge.

Suppose $B \subset C$ is a strict subcurve such that $\deg \omega_C|_B - \deg \omega_B = 2$. If $p_a(B) = 0$, then necessarily B is pointed. It follows that $\overline{C \setminus B}$ has either an elliptic bridge, or an A_3-attached elliptic tail. If $p_a(B) = 1$, then $p_a(\overline{C \setminus B}) = 0$ and so B is unpointed. It follows that B is either an elliptic bridge, or an A_3-attached elliptic tail. $\square$

Remark 2.5. The above result fails in higher genus, and in fact already in genus 3 as illustrated by a union of a smooth genus 2 curve and a pointed rational curve meeting along an A_3-singularity.

Corollary 2.6. *Suppose (C, p) is an A_4-stable or A_5-stable curve. Then C is honestly hyperelliptic, that is $|\omega_C|$ is a base-point-free linear system defining a finite flat degree 2 map $\kappa_C \colon C \to \mathbb{P}^1$. In particular, a point $p \in C$ is a Weierstrass point if and only if p is a ramification point of the canonical morphism κ_C, if and only if (C, p) is a Weierstrass curve.*

Proof. This follows from Proposition 2.4 and [CFHR99, Theorem 3.6]. $\square$

2.1. **Bi-log-canonical embeddings of A_4 and A_5-stable curves.** Given an A_4-stable or A_5-stable pointed genus 2 curve (C, p), the line bundle $\omega_C^2(2p)$ will be called a *bi-log-canonical line bundle*. In analogy with the standard case of smooth curves, we have the following simple result:

Lemma 2.7. *The line bundle $\omega_C^2(2p)$ is very ample for every A_4-stable or A_5-stable pointed curve (C, p).*

Proof. By [CFHR99, Theorem 1.1], it suffices to verify that $\deg \omega_C^2(2p)|_B \geq 2p_a(B) + 1$ for every strict subcurve $B \subset C$. This is straightforward using the definition of A_4 or A_5-stability. $\square$

A choice of an isomorphism $V \simeq \mathrm{H}^0\big(C, \omega_C^2(2p)\big)$ then gives an embedding $\phi \colon C \hookrightarrow \mathbb{P}V^\vee$ whose image is a degree 6 curve in $\mathbb{P}V^\vee \simeq \mathbb{P}^4$, which we call a *bi-log-canonical curve*. We denote by I_C the homogeneous ideal of $\phi(C)$. It is standard to verify that $\phi(C)$ is projectively normal, and so we can define *the mth Hilbert point of C* to be the short exact sequence

$$0 \to (I_C)_m \to \mathrm{Sym}^m V \to \mathrm{H}^0\big(C, \omega_C^{2m}(2mp)\big) \to 0$$

considered as a point in the Grassmannian

$$\mathrm{Grass}\big((I_C)_m, \mathrm{Sym}^m V\big).$$

Our focus will be entirely on the 2nd Hilbert points of bi-log-canonically embedded curves, and so we note that for $m = 2$, we have $\dim \mathrm{Sym}^2 V = 15$, while $h^0(C, \omega_C^4(4p)) = 11$, and $\dim(I_C)_2 = 4$.

2.2. **Surfaces containing bi-log-canonical curves.** Our analysis of GIT semistability of 2nd Hilbert points of bi-log-canonical curves is rooted in the analysis of minimal surfaces on which these curves and their degenerations lie, and so we pause to describe them. Whether the marked point p is a Weierstrass point on C determines which minimal surface the bi-log-canonical curve lies on.

2.2.1. *Non-Weierstrass curves.* First, we see that A_5-stable (and hence A_4^{non-W}-stable) curves have planar models given by cuspidal plane quartics. This observation and the existence of a GIT moduli space for plane quartics will be essential in our analysis of GIT stability for these curves.

Lemma 2.8. *Suppose that (C, p) is an A_5-stable curve. Then the line bundle $\omega_C(2p)$ is base-point-free and defines a morphism $\psi \colon C \to \mathbb{P}^2$ such that $\psi(C)$ is a plane quartic. Moreover, ψ is an isomorphism onto its image away from p and $\psi(p)$ is a cusp of $\psi(C)$.*

Proof. Let $\overline{C}$ be the unique arithmetic genus 3 curve obtained from C by imposing a cusp at p. Since p is not a Weierstrass point of C, we have that $\overline{C}$ is not strictly hyperelliptic. It follows by [CFHR99, Theorem 3.6] that $\omega_{\overline{C}}$ is very ample and embeds $\overline{C}$ as a quartic curve in $\mathbb{P}^2$. The claim follows by noting that the partial normalization morphism $\nu \colon C \to \overline{C}$ satisfies $\nu^* \omega_{\overline{C}} = \omega_C(2p)$ and that $h^0\big(C, \omega_C(2p)\big) = 3$. $\square$

Definition 2.9. We call the image $\psi(C)$ *the cuspidal quartic model of* (C, p).

Let $\psi(C) \hookrightarrow \mathbb{P}^2$ be the cuspidal quartic model of (C, p). Blowing-up the cusp $q := \psi(p)$ in $\mathbb{P}^2$ realizes C as a smooth curve in the class $|4H - 2E|$ on the Hirzebruch surface $\mathrm{Bl}_q \mathbb{P}^2 \simeq \mathbb{F}_1$, tangent to the exceptional divisor E at the point $p \in C$. The embedding of $\mathbb{F}_1$ by the complete linear system $|2H - E|$ realizes the surface as a *cubic surface scroll* $S_1 \hookrightarrow \mathbb{P}^4$ (the reader can find an overview of the properties of cubic surface scrolls in [HRS05, §6.1], which we use in what follows without further comments). The directrix of S_1 is the line given by the image of E, and for this embedding of S_1 into $\mathbb{P}^4$, the curve C becomes a quadric section of S_1 tangent to the directrix at $p \in C$.

Very concretely, we can realize S_1 as the image of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ given by

$$(2.10) \qquad\qquad [x : y : z] \mapsto [xz : yz : x^2 : xy : y^2],$$

which leads to a determinantal presentation of the quadric generators of the ideal of S_1. Namely, $(I_{S_1})_2$ is given by the 2×2 minors of

$$\begin{pmatrix} z_0 & z_2 & z_3 \\ z_1 & z_3 & z_4 \end{pmatrix}.$$

We denote the resulting net of quadrics by N_1, so that

$$(2.11) \qquad N_1 := (z_0 z_3 - z_1 z_2, z_0 z_4 - z_1 z_3, z_2 z_4 - z_3^2) \in \mathrm{Grass}(3, \mathrm{Sym}^2 V).$$

The equation of the directrix of S_1 is $z_2 = z_3 = z_4 = 0$ in these coordinates.

2.2.2. *Weierstrass curves.* Suppose (C, p) is an A_4-stable curve and p is its Weierstrass point. Since C is honestly hyperelliptic by Corollary 2.6, it admits a finite flat degree 2 map to $\mathbb{P}^1$, and so we can represent the affine curve $C \setminus \{p\}$ by the equation

$$(2.12) \qquad\qquad y^2 = x^5 + c_3 x^3 + c_2 x^2 + c_1 x + c_0, \quad c_i \in \mathbb{C},$$

and take p to be the point at infinity. It is easy to check that dx/y is a Rosenlicht differential on C with a double zero at infinity, and so is a global regular section of $\omega_C(-2p)$. Note that x has a double pole at infinity and is regular everywhere else, and y has a pole of order 5 at infinity and is regular everywhere else. It follows that

$$(2.13) \quad \mathrm{H}^0\big(C, \omega_C^2(2p)\big) = \left\langle x^3 \left(\frac{dx}{y}\right)^2, \ y \left(\frac{dx}{y}\right)^2, \ x^2 \left(\frac{dx}{y}\right)^2, \ x \left(\frac{dx}{y}\right)^2, \ \left(\frac{dx}{y}\right)^2 \right\rangle.$$

Denoting the listed generators of $\mathrm{H}^0\big(C, \omega_C^2(2p)\big)$ by $z_0, \ldots, z_4$, we see that the resulting homogeneous ideal of the bi-log-canonical curve $\phi(C)$ in $\mathbb{P}^4$ is

$$(2.14) \quad (z_0 z_3 - z_2^2, z_0 z_4 - z_2 z_3, z_2 z_4 - z_3^2, z_1^2 - z_0 z_2 - c_3 z_2 z_3 - c_2 z_2 z_4 - c_1 z_3 z_4 - c_0 z_4^2),$$

and the marked point is $p = [1 : 0 : 0 : 0 : 0]$. The first three quadrics in (2.14) cut out a singular minimal surface in $\mathbb{P}^4$ given by a *cone over a rational normal cubic curve*. We

call this surface S_2. The quadric generators of the homogeneous ideal of S_2 are given by the net

$$(2.15) \qquad N_2 := (z_0 z_3 - z_2^2, z_0 z_4 - z_2 z_3, z_2 z_4 - z_3^2) \in \mathrm{Grass}(3, \mathrm{Sym}^2 V),$$

which is of course given by the 2×2 minors of

$$\begin{pmatrix} z_0 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}.$$

2.3. **The ramphoid cuspidal atom.** Next, we introduce a special Weierstrass curve, called *the ramphoid cuspidal atom*, and denoted by C_R. This curve is defined by the following equation

$$(2.16) \qquad I_{C_R} = (z_0 z_3 - z_2^2, z_0 z_4 - z_2 z_3, z_2 z_4 - z_3^2, z_1^2 - z_0 z_2).$$

As above, we mark the point $\infty := [1 : 0 : 0 : 0 : 0] \in C_R$. Note that C_R is simply a compactification of the affine curve $y^2 = x^5$ given by Equation (2.12) with $c_3 = c_2 = c_1 = c_0 = 0$, and marked by the point at infinity. It is evident that the connected component of the identity in the automorphism group of (C_R, ∞) is $\mathrm{Aut}(C_R, \infty)^\circ \simeq \mathbb{G}_m$, where $\mathbb{G}_m$ acts by

$$t \cdot (x, y) = (t^2 x, t^5 y).$$

In particular, the 1-PS of $\mathrm{GL}(5)$ (respectively, the induced 1-PS of $\mathrm{SL}(5)$) acting on the generators $(z_0, \ldots, z_4)$ of $\mathrm{H}^0\big(C_R, \omega_{C_R}^2(2\infty)\big)$ as given by (2.13) with weights $(0, -1, -2, -4, -6)$ (resp., $(13, 8, 3, -7, -17)$) is the connected component of the identity in the stabilizer of (C_R, ∞) in $\mathrm{GL}(5)$ (resp., $\mathrm{SL}(5)$).

We will make use of the first-order deformation theory of (C_R, ∞), and so we pause to describe it here. Let $\mathrm{Def}^1(C_R, \infty)$ and $\mathrm{Def}^1(A_4)$ be the first-order deformation spaces of the pointed curve (C_R, ∞) and the A_4-singularity, respectively. By standard deformation theory, we have $\dim_{\mathbb{C}} \mathrm{Def}^1(C_R, \infty) = 5$ and $\dim_{\mathbb{C}} \mathrm{Def}^1(A_4) = 4$. Furthermore, we have a short exact sequence

$$(2.17) \qquad 0 \to T_\infty \to \mathrm{Def}^1(C_R, \infty) \to \mathrm{Def}^1(A_4) \to 0,$$

where T_∞ is the tangent space to C_R at ∞ naturally identified with the first-order deformations of the point ∞ in C_R. (A priori, T_∞ is only a subspace of the kernel of the surjection $\mathrm{Def}^1(C_R, \infty) \to \mathrm{Def}^1(A_4)$, but in this case exactness follows by dimension reasons.) Let W be the tangent space to the locus of Weierstrass curves (that is, those given by Equation (2.12)) in the miniversal deformation space of (C_R, ∞). Since Equation (2.12) also describes the miniversal deformation of the A_4-singularity, we conclude that the subspace $W \hookrightarrow \mathrm{Def}^1(C_R, \infty)$ maps isomorphically onto $\mathrm{Def}^1(A_4)$ in (2.17). We also note that (2.17) is a short exact sequence of $\mathbb{G}_m$-representations, where $\mathbb{G}_m \simeq \mathrm{Aut}(C_R, \infty)^\circ$ acts on $\mathrm{Def}^1(A_4)$ with weights $(-4, -6, -8, -10)$, and on T_∞ with weight 1.

3. VGIT PROBLEM AND MAIN THEOREM

As discussed in §2.1, every A_4 or A_5-stable pointed genus 2 curve (C, p) is equipped with a bi-log-canonical embedding $C \hookrightarrow \mathbb{P}V^\vee$, where $V \simeq \mathrm{H}^0\big(C, \omega_C^2(2p)\big) \simeq \mathbb{C}^5$. By §2.2, to each such bi-log-canonical curve $C \hookrightarrow \mathbb{P}^4$ we can associate a unique (either smooth or singular) cubic surface scroll S containing it. We now introduce an element $h(C, p)$ of $\mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee$, which we call *the H-point of* (C, p), defined by

$$(3.1) \qquad h(C, p) := \big((I_C)_2, p\big) \in \mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee,$$

where $[(I_C)_2 \subset \mathrm{Sym}^2 V] \in \mathrm{Grass}(4, \mathrm{Sym}^2 V)$ is the 2nd Hilbert point of C. Note that we always have

$$(3.2) \qquad (I_S)_2 \subset (I_C)_2, \text{ where either } (I_S)_2 = N_1 \text{ or } (I_S)_2 = N_2, \text{ up to } \mathrm{SL}(V)\text{-action.}$$

Denote by $\mathbb{H}^{sm}$ the constructible locus inside $\mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee$ consisting of all H-points $h(C, p)$, as (C, p) varies over all smooth bi-log-canonically embedded pointed genus 2 curves, and by $\overline{\mathbb{H}}$ the Zariski closure of $\mathbb{H}^{sm}$. The standard action of $\mathrm{SL}(V) \simeq \mathrm{SL}(5)$ on $\mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee$ restricts to an action on $\overline{\mathbb{H}}$, with the orbits in $\mathbb{H}^{sm}$ being in bijection with abstract isomorphism classes of smooth pointed genus 2 curves.

We will construct a compactification of $M_{2,1}$ by taking a GIT quotient of $\overline{\mathbb{H}}$. Moreover, since $h(C, p)$ is well-defined for every bi-log-canonically embedded A_4-stable curve C, we will obtain the moduli space of A_4-stable curves as a GIT quotient of $\overline{\mathbb{H}}$ after choosing an appropriate linearization of the $\mathrm{SL}(5)$-action on $\overline{\mathbb{H}}$. Note that because the space of ample linearizations of the $\mathrm{SL}(5)$-action on $\overline{\mathbb{H}}$ contains a two-dimension subspace spanned by the pullbacks of $\mathcal{O}_{\mathrm{Grass}(4, \mathrm{Sym}^2 V)}(1)$ and $\mathcal{O}_{\mathbb{P}V^\vee}(1)$, we have a natural VGIT problem (see [DH98] and [Tha96] for an introduction to VGIT), which we proceed to describe.

For an ample $\mathbb{Q}$-line-bundle

$$(3.3) \qquad \mathcal{L}_\beta := \mathcal{O}_{\mathrm{Grass}(4, \mathrm{Sym}^2 V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^4}(\beta), \quad \text{where } \beta \in (0, \infty) \cap \mathbb{Q},$$

we let $\overline{\mathbb{H}}(\beta)$ be the semistable locus inside $\overline{\mathbb{H}}$ with respect to the linearization given by $\mathcal{L}_\beta$. The points of $\overline{\mathbb{H}}(\beta)$ will be called β-*stable pairs*. We then have a sequence of GIT quotients

$$\overline{H}_\beta := \overline{\mathbb{H}}(\beta) /\!\!/_{\mathcal{L}_\beta} \mathrm{SL}(5).$$

Our first goal is to show that for a range of β-values, the semistable locus $\overline{\mathbb{H}}(\beta)$ parameterizes only curves. Namely, we will show that for $\beta \in (1/7, 1/2]$, every β-stable pair $(I, p) \in \mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}^4$ is such that $(I, p) = h(C, p)$ for some bi-log-canonically embedded curve (C, p). Incidentally, we will see that for $\beta \in (1/7, 1/2)$, the semistable locus $\overline{\mathbb{H}}(\beta)$ consists exactly of H-points of A_4-stable curves, thus establishing a posteriori that the locus of H-points of A_4-stable curves is locally closed in $\mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}^4$.

Our second goal is to explicitly describe curves parameterized by $\overline{\mathbb{H}}(\beta)$ for all $\beta \in (1/7, 1/2]$. Our third and final goal is to show that the resulting GIT quotients are identified with the moduli spaces of A_3, A_4, and A_4^{non-W}-stable curves, respectively, thus giving a VGIT presentation for the 2nd flip of $\overline{M}_{2,1}$.

We denote by $\overline{\overline{\mathbb{H}}}(a,b)$ the locus of β-stable pairs if the stability does not change as β varies in the interval $(a,b) \subset (0,\infty) \cap \mathbb{Q}$. With this, we can state our main theorem as follows:

Theorem 3.4. *For $\beta \in (1/7, 1/2]$, the β-stable pairs have the following description:*

(A) *We have that*

$$\overline{\overline{\mathbb{H}}}(1/7, 4/13) = \big\{ h(C,p) \mid (C,p) \text{ is a bi-log-canonically embedded } A_3\text{-stable curve}\big\},$$

and, consequently,

$$\big[\overline{\overline{\mathbb{H}}}(1/7, 4/13)/\operatorname{PGL}(5)\big] \simeq \overline{\mathcal{M}}_{2,1}(2/3 + \epsilon).$$

(B) *We have that*

$$\overline{\overline{\mathbb{H}}}(4/13) = \big\{ h(C,p) \mid (C,p) \text{ is a bi-log-canonically embedded } A_4\text{-stable curve}\big\},$$

and, consequently,

$$\big[\overline{\overline{\mathbb{H}}}(4/13)/\operatorname{PGL}(5)\big] \simeq \overline{\mathcal{M}}_{2,1}(2/3).$$

(C) *We have that*

$$\overline{\overline{\mathbb{H}}}(4/13, 1/2) = \Big\{ h(C,p) \mid (C,p) \text{ is a bi-log-canonically embedded } A_4^{non-W}\text{-stable curve}\Big\},$$

and, consequently,

$$\big[\overline{\overline{\mathbb{H}}}(4/13, 1/2)/\operatorname{PGL}(5)\big] \simeq \overline{\mathcal{M}}_{2,1}(2/3 - \epsilon).$$

(D) *We have that*

$$\overline{\overline{\mathbb{H}}}(1/2) = \big\{ h(C,p) \mid (C,p) \text{ is a bi-log-canonically embedded } A_5\text{-stable curve}\big\},$$

and, consequently,

$$\big[\overline{\overline{\mathbb{H}}}(1/2)/\operatorname{PGL}(5)\big]$$

is the moduli stack of A_5-stable curves.

(E) *The moduli spaces of the GIT quotient stacks described in Parts (A), (B), (C), (D) are isomorphic to the log canonical models of $\overline{M}_{2,1}$ as follows:*

(3.5) $$\overline{\overline{\mathbb{H}}}(1/7, 4/13) /\!\!/ \operatorname{SL}(5) \simeq \overline{M}_{2,1}(2/3 + \epsilon),$$

(3.6) $$\overline{\overline{\mathbb{H}}}(4/13) /\!\!/ \operatorname{SL}(5) \simeq \overline{M}_{2,1}(2/3),$$

(3.7) $$\overline{\overline{\mathbb{H}}}(4/13, 1/2) /\!\!/ \operatorname{SL}(5) \simeq \overline{M}_{2,1}(2/3 - \epsilon),$$

(3.8) $$\overline{\overline{\mathbb{H}}}(1/2) /\!\!/ \operatorname{SL}(5) \simeq \overline{M}_{2,1}(19/29) \simeq \{point\},$$

and fit into a commutative diagram

$$\begin{array}{ccccccc}
\overline{\mathbb{H}}(4/13-\epsilon) & \hookrightarrow & \overline{\mathbb{H}}(4/13) & \hookleftarrow & \overline{\mathbb{H}}(4/13+\epsilon) & \hookrightarrow & \overline{\mathbb{H}}(4/13) \\
\downarrow & & & & \downarrow & & \\
\overline{M}_{2,1}(2/3+\epsilon) & & & & \overline{M}_{2,1}(2/3-\epsilon) & & \\
& \searrow & \downarrow & \swarrow^{\simeq} & & \searrow & \downarrow \\
& & \overline{M}_{2,1}(2/3) & & & & \overline{M}_{2,1}(19/29)=\{point\}
\end{array}$$

(3.9)

The birational morphism $\overline{M}_{2,1}(2/3+\epsilon) \to \overline{M}_{2,1}(2/3)$ contracts the Weierstrass divisor to a point, while $\overline{M}_{2,1}(2/3-\epsilon) \to \overline{M}_{2,1}(2/3)$ is an isomorphism.

Remark 3.10. We emphasize that the identification of the GIT quotient stacks and the stacks of α-stable curves in Parts (A–D) of Theorem 3.4 are formal consequences of the GIT stability analysis. Namely, suppose we have established for some two values of α and β that

(3.11) $\qquad \overline{\mathbb{H}}(\beta) = \{h(C,p) \mid (C,p) \text{ is a bi-log-canonically embedded } \alpha\text{-stable curve}\}.$

We claim this implies that

$$\left[\overline{\mathbb{H}}(\beta)/\operatorname{PGL}(5)\right] \simeq \overline{\mathcal{M}}_{2,1}(\alpha).$$

Indeed, by (3.11), we have a flat family of α-stable curves over $\overline{\mathbb{H}}(\beta)$. This family induces a PGL(5)-equivariant morphism $\overline{\mathbb{H}}(\beta) \to \overline{\mathcal{M}}_{2,1}(\alpha)$ that descends to give a morphism $f \colon [\overline{\mathbb{H}}(\beta)/\operatorname{PGL}(5)] \to \overline{\mathcal{M}}_{2,1}(\alpha)$. Conversely, let $\pi \colon \mathcal{C} \to \overline{\mathcal{M}}_{2,1}(\alpha)$ be the universal family of α-stable curves, with the universal section σ. Let $\mathcal{V} := \pi_*\left(\omega_\pi^2(2\sigma)\right)$, which is a vector bundle of rank 5. By Lemma 2.7, we have a morphism $\mathcal{C} \to \mathbb{P}(\mathcal{V})$ over $\overline{\mathcal{M}}_{2,1}(\alpha)$ given by a relatively very ample line bundle $\omega_\pi^2(2\sigma)$. Let $\mathcal{P} \to \overline{\mathcal{M}}_{2,1}(\alpha)$ be the PGL(5)-torsor associated to $\mathbb{P}(\mathcal{V})$. Then the pullback of $\mathbb{P}(\mathcal{V})$ to $\mathcal{P}$ is a trivial $\mathbb{P}^4$-bundle and so we obtain a morphism $\mathcal{P} \to \overline{\mathbb{H}}(\beta)$ given by taking the H-points of the fibers in $\mathbb{P}^4$. This defines a morphism $g \colon \overline{\mathcal{M}}_{2,1}(\alpha) \to [\overline{\mathbb{H}}(\beta)/\operatorname{PGL}(5)]$. That $g \circ f$ and $f \circ g$ are identities follows from the fact that $(C,p) \simeq (C',p')$ if and only if $\operatorname{PGL}(5) \cdot h(C,p) = \operatorname{PGL}(5) \cdot h(C',p')$ for any two α-stable curves.

We summarize the relationship between the notions of stability from Definition 2.1 (and [AFSvdW17]) and the GIT stability for H-points of bi-log-canonical curves in Table 1.

Remark 3.12. For a general (g,n), the notion of α-stability for curves arising in the Hassett-Keel program for $\overline{\mathcal{M}}_{g,n}$ has been defined only for $\alpha > 2/3-\epsilon$ (see e.g., [AFSvdW17]). There does exist a conjectural notion of $\frac{19}{29}$-stability (see [AFS16, Table 3]), and as this paper illustrates, our A_5-stability coincides with it for $(g,n) = (2,1)$.

Stability	New singularities	Newly disallowed configurations	Range of α-stability cf. Remark 3.12	GIT stability range
DM-stable	A_1		$\alpha \in [1, 9/11)$	
A_2-stable	A_2	A_1-attached elliptic tails	$\alpha \in (9/11, 7/10)$	
A_3-stable	A_3	A_3-attached elliptic tails, A_1-attached elliptic bridges	$\alpha \in (7/10, 2/3)$	$\beta \in (1/7, 4/13)$
A_4-stable	A_4		$\alpha = 2/3$	$\beta = 4/13$
A_4^{non-W}-stable		p is a Weierstrass point	$\alpha \in (2/3, 19/29)$	$\beta \in (4/13, 1/2)$
A_5-stable	A_5		$\alpha = 19/29$	$\beta = 1/2$

TABLE 1. Stability notions for pointed genus two curves.

3.1. **Relation to prior works.** As discussed in the introduction, moduli stacks $\overline{\mathcal{M}}_{2,1}(\alpha)$ for $\alpha > 2/3 - \epsilon$ have been studied before in other contexts. Here we give a brief overview of these works and relate them to our construction.

Hyeon and Lee gave a GIT construction of $\overline{\mathcal{M}}_{2,1}(7/10)$ and $\overline{\mathcal{M}}_{2,1}(7/10 - \epsilon)$ by taking GIT quotients of the Chow and Hilbert schemes of bi-log-canonical genus 2 curves in $\mathbb{P}^4$ [HL14, Theorems 2.2 and 4.2] in the course of their study of the Hassett-Keel program for $\overline{\mathcal{M}}_4$. Initially, we hoped to emulate their approach to the construction of $\overline{\mathcal{M}}_{2,1}(\alpha)$ for $\alpha \leq 2/3$. However, as we have discovered, the Chow point (in the sense of [HL14, Section 4]) of the ramphoid cuspidal atom (C_R, ∞) is unstable with respect to any linearization and so Chow or asymptotic Hilbert stability cannot be used to construct the moduli stack $\overline{\mathcal{M}}_{2,1}(2/3)$ of A_4-stable curves.

In [Pol18], Polishchuk introduces two moduli stacks, $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ and $\overline{\mathcal{U}}_{2,1}^{ns}(2)$, and proves that there is a morphism $\overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \to \overline{\mathcal{U}}_{2,1}^{ns}(2)$ contracting the Weierstrass divisor to a single point [Pol18, Theorem 2.4.5]. Both of these stacks are smooth, Deligne-Mumford, and contain $\mathcal{M}_{2,1}$ as an open substack. The first, $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$, is a stable modular compactification of $\mathcal{M}_{2,1}$ for an extremal assignment $\mathcal{Z}$ over $\overline{\mathcal{M}}_{2,1}$ given by all unmarked irreducible components (cf. [Smy13, Definition 1.5 and Example 1.12]). The second, $\overline{\mathcal{U}}_{2,1}^{ns}(2)$, parameterizes non-special marked curves of genus 2 in the sense of [Pol17]. Here we identify these moduli stacks with those appearing in the Hassett-Keel program for $\overline{\mathcal{M}}_{2,1}$, and hence with our GIT quotient stacks.

Proposition 3.13. *We have isomorphisms* $\overline{\mathcal{M}}_{2,1}(7/10 - \epsilon) \simeq \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$, *and* $\overline{\mathcal{U}}_{2,1}^{ns}(2) \simeq \overline{\mathcal{M}}_{2,1}(2/3 - \epsilon)$.

Proof. We first prove that $\overline{\mathcal{M}}_{2,1}(7/10-\epsilon) \simeq \overline{\mathcal{M}}_{2,1}(\mathcal{Z})$. The isomorphism types of A_3-stable curves that are not $\mathcal{Z}$-stable are as follows:

(a1) A union of two smooth rational curves along A_1 and A_3-singularities, with one of the components marked. This curve has no automorphisms (as the crimping parameter of the tacnode is killed by the automorphisms of the unmarked component).

(a2) A union of two smooth rational curves along three A_1-singularities, with one of the components marked. Such curves are Deligne-Mumford stable and also have no automorphisms.

The isomorphism types of $\mathcal{Z}$-stable curves that are not A_3-stable are as follows:

(b1) An irreducible curve with a single two-branched singularity where one branch is a cusp, and the other branch is smooth and transversal to the tangent space of the cusp (cf. the first curve in [Pol18, §2.3 Case IIb]). There is exactly one isomorphism class of such a curve, and this curve has no automorphisms.

(b2) An irreducible curve with a single rational 3-fold singularity (the coordinate cross singularity of [Pol18, §2.3 Case IIa]). Such curves also have no automorphisms.

We see that the complement of the union of (a1)-(a2) loci in $\overline{\mathcal{M}}_{2,1}(7/10-\epsilon)$ is isomorphic to the complement of the union of (b1)-(b2) loci in $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$. Since these loci lie in the schematic part of the respective Deligne-Mumford stacks, it suffices to show that the rational map $f\colon \overline{\mathcal{M}}_{2,1}(\mathcal{Z}) \dashrightarrow \overline{\mathcal{M}}_{2,1}(7/10-\epsilon)$ has no indeterminacy. Towards this, we note that Deligne-Mumford stable replacement of a curve C of type (b2) is a unique curve of type (a2), given by normalizing C and attaching to C a smooth rational component along the preimages of the rational 3-fold singularity. Hence the only possible indeterminacy point of f is the (a1)-curve and the only possible indeterminacy point of f^{-1} is the (b1)-curve. Since both stacks are normal and schematic at these points, we conclude that f is an isomorphism.

The proof of $\overline{\mathcal{U}}_{2,1}^{ns}(2) \simeq \overline{\mathcal{M}}_{2,1}(2/3-\epsilon)$ is exactly the same: Away from the union of (a1)-(a2) loci $\overline{\mathcal{M}}_{2,1}(2/3-\epsilon)$ is isomorphic to the complement of the union of (b1)-(b2) loci in $\overline{\mathcal{U}}_{2,1}^{ns}(2)$. $\qquad\square$

Remark 3.14. One can regard $\overline{\mathcal{M}}_{2,1}(7/10-\epsilon)$ (resp., $\overline{\mathcal{M}}_{2,1}(2/3-\epsilon)$) and $\overline{\mathcal{M}}_{2,1}(\mathcal{Z})$ (resp., $\overline{\mathcal{U}}_{2,1}^{ns}(2)$) as two different approaches of compactifying the moduli space of crimping (i.e., attaching data) of a tacnodal singularity (cf. the discussion in [Smy13, p.468]). The presence of an A_5-singularity also shows that $\overline{\mathcal{M}}_{2,1}(\alpha)$ are not stable modular compactifications of $\mathcal{M}_{2,1}$ for $\alpha \leq 2/3$ (see [Smy13, Corollary 1.15]).

We also note that, thanks to Corollary 2.6, the stack $\overline{\mathcal{M}}_{2,1}(2/3-\epsilon)$ is isomorphic to the stack $\mathcal{H}_5[4]$ of quasi-admissible genus 2 hyperelliptic covers with at worst A_4-singularities, as constructed in [Fed14, Proposition 4.2(1)], which leads to a different proof of [Pol18, Proposition 2.1.1] giving an isomorphism

$$\overline{\mathcal{M}}_{2,1}(2/3-\epsilon) \simeq \overline{\mathcal{U}}_{2,1}^{ns}(2) \simeq \mathcal{P}(2,3,4,5,6).$$

4. Proof of Main Theorem

Our GIT analysis of H-points of bi-log-canonical curves predicates on the study of GIT stability of (quadric sections of) degenerations of rational normal scrolls in $\mathbb{P}^4$, and so in §4.1 we describe possible nets of quadrics in $\mathbb{P}^4$ cutting out such degenerations. We recall the Hilbert-Mumford numerical criterion as it applies to the elements of $\mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee$ in §4.2, and use it in §4.3 to narrow down potential β-stable pairs to those of H-pairs of bi-log-canonical curves. In §4.4, we establish generic stability of smooth bi-log-canonical curves. All these preliminary results are bootstrapped to give a proof of Theorem 3.4 in the latter parts of this section.

4.1. Orbits of nets of quadrics.

It is easy to see that the $\mathrm{SL}(5)$-orbit of N_1 (given by (2.11)) contains N_2 (given by (2.15)) in its closure. Indeed,

$$\lim_{t \to 0} \begin{pmatrix} z_0 & & z_2 & z_3 \\ tz_1 + (1-t)z_2 & z_3 & z_4 \end{pmatrix} = \begin{pmatrix} z_0 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}.$$

In fact, we have a complete characterization of the orbit closure of N_1 in the Grassmannian of nets of quadrics in $\mathbb{P}^4$.

Proposition 4.1. *The closure of the $\mathrm{SL}(5)$-orbit of $[N_1] \in \mathrm{Grass}(3, \mathrm{Sym}^2 V)$ is*

$$\overline{\mathrm{SL}(5) \cdot [N_1]} = \mathrm{SL}(5) \cdot [N_1] \cup \mathrm{SL}(5) \cdot [N_2] \cup \mathrm{SL}(5) \cdot [N_3] \cup \mathrm{SL}(5) \cdot [N_4] \cup \mathrm{SL}(5) \cdot [N_5]$$

where

$$(4.2) \qquad \begin{aligned} N_3 &= (z_0 z_3, z_0 z_4, z_2 z_4 - z_3^2), \\ N_4 &= (z_0^2, z_0 z_3, z_0 z_4), \\ N_5 &= (z_0 z_4 - z_1 z_3, z_1 z_2, z_2 z_4). \end{aligned}$$

Moreover,

$$\overline{\mathrm{SL}(5) \cdot [N_2]} = \mathrm{SL}(5) \cdot [N_2] \cup \mathrm{SL}(5) \cdot [N_3] \cup \mathrm{SL}(5) \cdot [N_4].$$

Proof. Since we work in $\mathbb{P}^4$, the net N_1 is the generic net of determinantal quadrics, i.e., quadrics given by the minors of a 2×3 matrix with linear entries. It follows by [EPS81, Theorem 2] that the closure of the orbit $\mathrm{SL}(5) \cdot [N_1]$ is a smooth subvariety of $\mathrm{Grass}(3, \mathrm{Sym}^2 V)$ whose boundary points also parameterize determinantal nets. With this in mind, let $D(L_0, \ldots, L_5) \in \mathrm{Grass}(3, \mathrm{Sym}^2 V)$ be the net of quadrics spanned by the 2×2 minors of the matrix

$$M(L_0, \ldots, L_5) = \begin{pmatrix} L_0 & L_2 & L_4 \\ L_1 & L_3 & L_5 \end{pmatrix},$$

where $\{L_i\}_{i=0}^5 \in V$.

If the entries of $M(L_0, \ldots, L_5)$ span V, then one verifies that

$$\mathrm{SL}(5) \cdot D(L_0, \ldots, L_5) = \mathrm{SL}(5) \cdot D(z_0, z_1, z_2, \sum_{i=0}^4 a_i z_i, z_3, z_4), \quad \text{where } a_i \in \mathbb{C}.$$

Suppose not all the a_i are 0. Then

$$\mathrm{SL}(5) \cdot D\left(z_0, z_1, z_2, \sum_{i=0}^{4} a_i z_i, z_3, z_4\right) = \mathrm{SL}(5) \cdot D(z_0, z_1, z_2, z_3, z_3, z_4) = [N_1].$$

If all the a_i are 0, then $\mathrm{SL}(5) \cdot D(z_0, z_1, z_2, 0, z_3, z_4) = [N_5]$.

Suppose the dimension of the span of the entries of $M(L_0, \ldots, L_5)$ is at most four. Then $D(L_0, \ldots, L_5)$ is (a cone over) a net of determinantal quadrics in $\mathbb{P}^3$. Generically, this net cuts out a twisted cubic in $\mathbb{P}^3$, and so corresponds to N_2. The remainder of the proposition follows from the analysis of the $\mathrm{SL}(4)$-orbit closure of $[N_2] \in \mathrm{Grass}\big(3, \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\big)$ performed in [EPS81].

$\square$

4.2. Numerical criterion.

Our convention for the Hilbert-Mumford indices are as follows. Given a one-parameter subgroup (1-PS) ρ of $\mathrm{SL}(5)$ acting diagonally on a basis $z_0, \ldots, z_4$ of $V = \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}(1))$ with weights $w_0, \ldots, w_4$, the Hilbert-Mumford index $\mu_\rho(I)$ of $I \in \mathrm{Grass}(k, \mathrm{Sym}^m V)$ will be the sum of ρ-weights of the k *leading* (i.e., greatest ρ-weight) degree m monomials of I with respect to ρ. This also applies to the mth Hilbert points $(I_X)_m$ of closed subschemes $X \hookrightarrow \mathbb{P}^4$.

In the case when $X = p = (a_0, \ldots, a_4) \in \mathbb{P}^4$ is a point and $m = 1$, this translates into

$$\mu_\rho(p) := - \min\{w_i \mid a_i \neq 0\} = \max\{-w_i \mid a_i \neq 0\}.$$

We say that the point $(I, p) \in \mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}V^\vee$ is ρ-unstable under linearization $\mathcal{L}_\beta$ if

$$\mu_\rho\big(I, p\big) = \mu_\rho(I) + \beta \mu_\rho(p) < 0.$$

The Hilbert-Mumford numerical criterion can then be translated as saying that $(I, p) \in \overline{\mathbb{H}}(\beta)$ if and only if $\mu_\rho(I) + \beta \mu_\rho(p) > 0$ for all non-trivial 1-PS ρ of $\mathrm{SL}(5)$.

Note that if $\rho \subset \mathrm{Stab}\big(I, p\big)$ is a subgroup of the stabilizer of (I, p), and ρ^{-1} is its inverse subgroup, then

$$\mu_\rho\big(I, p\big) = -\mu_{\rho^{-1}}\big(I, p\big),$$

and so the necessary condition for (I, p) to be semistable is that $\mu_\rho\big(I, p\big) = \mu_{\rho^{-1}}\big(I, p\big) = 0$.

4.3. Preliminary stability results.

Consider a point $(I, p) \in \overline{\mathbb{H}} \subset \mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}^4$. Since (I, p) lies in the closure of the locus of H-pairs of smooth pointed genus 2 curves and a generic such curve lies on a smooth cubic scroll S_1, we conclude that there is a dimension 3 linear subspace $N \subset I$ such that $N \in \overline{\mathrm{SL}(5) \cdot [N_1]}$ as an element of $\mathrm{Grass}(3, \mathrm{Sym}^2 V)$. The orbit closure $\overline{\mathrm{SL}(5) \cdot [N_1]}$ has been described in Proposition 4.1. With this in mind, we have the following instability result:

Proposition 4.3. *Let $(I, p) \in \overline{\mathbb{H}} \subset \mathrm{Grass}(4, \mathrm{Sym}^2 V) \times \mathbb{P}^4$. We have that*

    (1) $(I, p) \notin \overline{\mathbb{H}}(\beta)$ *for $\beta < 1$, if $N_3 \subset I$,*
    (2) $(I, p) \notin \overline{\mathbb{H}}(\beta)$ *for $\beta < 2$, if $N_4 \subset I$,*
    (3) $(I, p) \notin \overline{\mathbb{H}}(\beta)$ *for $\beta < 2$, if $N_5 \subset I$.*

Proof. (1) Suppose that in coordinates $\{z_i\}_{i=0}^4$, we have that $N_3 = (z_0z_3, z_0z_4, z_2z_4 - z_3^2) \subset I$. Consider the 1-PS ρ acting on $\{z_i\}_{i=0}^4$ with weights $(-1, 1, 1, 0, -1)$. Then

$$\mu_\rho(I) \le \mu_\rho(N_3) + 2 = (-1 + (-2) + 0) + 2 = -1,$$
$$\mu_\rho(p) \le 1.$$

Hence the pair (I, p) is unstable with respect to $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(\beta)$ for $\beta < 1$.

(2) Suppose that in coordinates $\{z_i\}_{i=0}^4$, we have that $N_4 = (z_0^2, z_0z_3, z_0z_4) \subset I$. Consider the 1-PS ρ acting on these coordinates with weights $(-1, 1, 0, 0, 0)$. Then

$$\mu_\rho(I) \le \mu_\rho(N_4) + 2 = -2,$$
$$\mu_\rho(p) \le 1.$$

Hence the pair is unstable with respect to $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(\beta)$ for $\beta < 2$.

(3) Suppose that in coordinates $\{z_i\}_{i=0}^4$, we have that $N_5 = (z_1z_2, z_0z_4 - z_1z_3, z_2z_4) \subset I$. Note that the point p lies on the flat limit of the directrix of N_1 in N_5, which is given by the line $z_2 = z_3 = z_4 = 0$. Consider the 1-PS ρ acting on the coordinates $\{z_i\}_{i=0}^4$ with weights $(3, -1, -4, 3, -1)$. Let $Q \in I \setminus N_5$ so that $I = Q + N_5$. Then

$$\mu_\rho(I) \le \mu_\rho(N_5) + \mu_\rho(Q) \le (-5 + 2 - 5) + 6 = -2,$$
$$\mu_\rho(p) \le 1.$$

Hence the pair (I, p) is unstable with respect to $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(\beta)$ for $\beta < 2$. $\qquad\square$

It follows from the above proposition that:

Corollary 4.4. *For $\beta \in (0, 1)$, the semistable locus $\overline{\mathbb{H}}(\beta)$ contains only pairs $((I_2)_C, p)$, where C is either a quadric section of the smooth cubic scroll S_1 or a quadric section of the singular cubic scroll S_2.*

From now on, we identify points of $\overline{\mathbb{H}}(\beta)$ with the pointed curves they represent.

Lemma 4.5. *For $\beta \in (0, 1)$, all curves in $\overline{\mathbb{H}}(\beta)$ are locally planar.*

Proof. In view of Corollary 4.4, it suffices to show that $\overline{\mathbb{H}}(\beta)$ contains no quadric sections of S_2 passing through the vertex of the cone. Recall that the equation of S_2 is given by

$$N_2 = (z_0z_3 - z_2^2, z_0z_4 - z_2z_3, z_2z_4 - z_3^2).$$

Suppose that $I = N_2 + Q$, where $Q \in I \setminus N_2$, defines a quadric section passing through the vertex of S_2. Then $Q \in (z_0, z_2, z_3, z_4)$. Consider the 1-PS ρ acting with weights $(-1, 4, -1, -1, -1)$. Then $\mu_\rho(I) \le -3$, and $\mu_\rho(p) \le 1$. This shows that (I, p) is unstable for $\beta < 3$. $\qquad\square$

Lemma 4.6. *If $((I_C)_2, p) \in \overline{\mathbb{H}}(\beta)$ for some $\beta \in (1/7, 1)$, then p is a smooth point of C.*

Proof. For $\beta > 1/7$, we are going to destabilize the pair $((I_C)_2, p)$ where C is a quadric section of the smooth scroll meeting the directrix of the scroll in a point p such that p is a singular point of C. From the above discussion, this is a generic case, and so instability

of all other pairs where p is a singular point of C will follow. Assuming that the smooth scroll S_1 is cut out by the equations (2.11), that $p = [1:0:0:0:0]$ and that the tangent space to C at p is $z_3 = z_4 = 0$, we see that the quadrics cutting out C are

$$(4.7) \qquad \left(z_0 z_3 - z_1 z_2, z_0 z_4 - z_1 z_3, z_2 z_4 - z_3^2, z_1^2 + g(z_1, z_2, z_3, z_4)\right),$$

Consider now the 1-PS ρ acting with weights $(7, 2, 2, -3, -8)$ on the chosen coordinates. Then the 2nd Hilbert point of C has $\mu_\rho((I_C)_2) \leq 4 + (-1) + (-6) + 4 = 1$, and $\mu_\rho(p) = -7$. The claim follows. $\qquad\square$

Remark 4.8 (GIT quotients for $\beta < 1/7$)**.** Although we will not need this for the purposes of this paper, one can show that for $\beta < 1/7$ the generic curves given by Equation (4.7) and marked by $p = [1:0:0:0:0]$ are in fact GIT stable. The corresponding β-stable pairs in the GIT quotient parameterize irreducible arithmetic genus 2 curves with a non-separating inner node p and *a choice* of a Cartier divisor of degree 2 supported at p. Because linear equivalence classes of such divisors form a one-dimensional family, a simple dimension count shows that such β-stable pairs form a codimension one locus in the GIT quotient $\overline{\mathbb{H}}(\beta)/\!\!/\operatorname{SL}(5)$ for $\beta < 1/7$. It follows that the rational map $\overline{M}_{2,1} \dashrightarrow \overline{\mathbb{H}}(\beta)/\!\!/\operatorname{SL}(5)$ is not a contraction for $\beta < 1/7$. Instead, this map is a blow up of the codimension two locus of elliptic bridges in $\overline{M}_{2,1}$. We conjecture that there is a diagram

$$(4.9)$$

$$
\begin{array}{ccc}
\overline{\mathbb{H}}(1/7 - \epsilon)/\!\!/\operatorname{SL}(5) & & \overline{\mathbb{H}}(1/7 + \epsilon)/\!\!/\operatorname{SL}(5) \\
\downarrow & \searrow & \quad \overset{\simeq}{\searrow} \quad \downarrow{\scriptstyle \simeq} \\
\overline{M}_{2,1}(A_2) = \overline{M}_{2,1}(9/11 - \epsilon) & \overline{\mathbb{H}}(1/7)/\!\!/\operatorname{SL}(5) & \overline{M}_{2,1}(7/10 + \epsilon),
\end{array}
$$

where the triangle connecting the GIT quotients is a VGIT wall-crossing at $\beta = 1/7$ and the isomorphism $\overline{\mathbb{H}}(1/7 + \epsilon)/\!\!/\operatorname{SL}(5) \simeq \overline{M}_{2,1}(7/10 + \epsilon)$ is given by our Theorem 3.4(A).

4.4. **Stability of smooth non-Weierstrass curves.** In this section, we prove the following stability result:

Proposition 4.10. *Let* (C, p) *be a bi-log-canonically embedded* smooth *curve, where p is not a Weierstrass point. Then the H-point $h(C, p)$ is β-stable for all $\beta \leq 1/3$.*

Let $I_2 := (I_C)_2$ be the linear system of quadrics cutting out the bi-log-canonical embedding of (C, p). We begin with a preliminary observation:

Claim 4.11. *No two quadrics in I_2 have a common singular point.*

Proof. Suppose they do. Then C lies on a cone over a complete intersection of two quadrics in $\mathbb{P}^3$, which is a degree 4 curve (possibly not integral). Since C is a smooth scheme-theoretic intersection of the quadrics in I_2, it cannot pass through the vertex of the cone. It follows that the projection of C to $\mathbb{P}^3$ must be supported on a non-degenerate integral curve whose degree divides 6. Since this curve lies on two linearly independent quadrics, we conclude that the projection is necessarily a twisted cubic. This contradicts the assumption that (C, p) is not a Weierstrass curve. $\qquad\square$

Proof of Proposition 4.10. We use the following elementary fact repeatedly:

Claim 4.12. *Suppose $a \geq b \geq c \geq d \geq e$ are rational numbers with $a+b+c+d+e = 0$. Then*

$$\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d + \lambda_5 e \geq 0$$

for all non-negative rational numbers $\{\lambda_i\}_{i=1}^5$ satisfying

$$\sum_{i=1}^k \lambda_i > \frac{k}{5} \sum_{i=1}^5 \lambda_i \quad \text{for all } k = 1, \ldots, 4.$$

We apply the Hilbert-Mumford numerical criterion. Suppose the 1-PS ρ of $\mathrm{SL}(5)$ acts with weights $a \geq b \geq c \geq d \geq e$ on a basis $z_0, \ldots, z_4$ of $\mathrm{H}^0\big(C, \omega_C^2(2p)\big)$. By the assumption $\beta \leq 1/3$, we always have

$$\beta \mu_\rho(p) \geq -\frac{a}{3}.$$

Let now q be the point $z_1 = z_2 = z_3 = z_4 = 0$, let ℓ be the line $z_2 = z_3 = z_4 = 0$, and let π be the plane $z_3 = z_4 = 0$. Let m_1, m_2, m_3, m_4 be the initial monomials of the quadrics in I_2 with respect to the lexicographic order, and Q_1, Q_2, Q_3, Q_4 be respectively the generators of I_2 with those initial monomials.

Suppose first that $q \notin C$. Then $m_1 = z_0^2$. By Claim 4.11, $\dim(I_2 \cap (z_1, z_2, z_3, z_4)^2) \leq 1$, and so we must have $m_2 \geq z_0 z_3$, and $m_3 \geq z_0 z_4$. Furthermore, since C does not lie on a rank 2 quadric, we must have either $m_4 \geq z_2 z_3$, or, after an appropriate change of basis, $Q_4 = z_2 z_4 + z_3^2$. In the former case,

$$\mu_\rho(I_2) \geq a^2 + (a+d) + (a+e) + (c+d) = 4a + c + 2d + e,$$

and so

$$\mu_\rho(C, p) = \mu_\rho(I_2) + \beta \mu_\rho(p) \geq \frac{11}{3}a + c + 2d + e > 0,$$

by Claim 4.12. In the latter case, using the fact that no three linearly independent quadrics in I_2 can contain the plane π, we see that we must have a z_2^2 term in either Q_2 or Q_3. Therefore,

$$\mu_\rho(I_2) \geq 2a + \frac{1}{2}((a+d) + 2c) + (a+e) + \frac{1}{2}(c + e + 2d) = \frac{7}{2}a + \frac{3}{2}c + \frac{3}{2}d + \frac{3}{2}e,$$

and so

$$\mu_\rho(C, p) \geq \frac{19}{6}a + \frac{3}{2}c + \frac{3}{2}d + \frac{3}{2}e > 0,$$

by Claim 4.12. This concludes the proof of stability when $q \notin C$.

Suppose now that $q \in C$. Then because the curve C is smooth at q, we have at least 3 initial monomials divisible by z_0, namely $m_1 \geq z_0 z_2$, $m_2 \geq z_0 z_3$, and $m_3 \geq z_0 z_4$. As before, we must have either $m_4 \geq z_2 z_3$, or $Q_4 = z_2 z_4 + z_3^2$. Summarizing,

$$\mu_\rho(I_2) \geq (a+d) + (a+c) + (a+e) + \frac{1}{3}(2(c+e) + 2d) = 3a + \frac{5}{3}(c + d + e),$$

and so

$$\mu_\rho(C, p) \geq \frac{8}{3}a + \frac{5}{3}(c + d + e) = a - \frac{5}{3}b.$$

If $a > \frac{5}{3}b$, we are done. Suppose $a \leq \frac{5}{3}b$, and in particular $b > 0$. Since not all quadrics can contain the line ℓ, at least one of the quadrics contains a term $\geq z_1^2$. It follows that

$$\mu_\rho(I_2) \geq 2b + (a+c) + (a+e) + \frac{1}{3}((c+e) + 4d) = 2a + 2b + \frac{4}{3}(c+d+e),$$

and so

$$\mu_\rho(C,p) \geq \frac{5}{3}a + 2b + \frac{4}{3}(c+d+e) = \frac{1}{3}a + \frac{2}{3}b > 0.$$

$\square$

Proposition 4.13. *For $\beta \in (1/7, 1/2]$, every β-stable pair in $\overline{\mathbb{H}}$ is the H-point $h(C,p)$ where (C,p) is a pointed genus 2 curve with at worst A_5-singularities, and no nodally or tacnodally attached elliptic tails or nodally attached elliptic bridges. Moreover, for $\beta \in (1/7, 1/2)$, the curve (C,p) has at worst A_4-singularities and hence is A_4-stable.*

Proof. By Corollary 4.4 and Lemma 4.5, for $\beta \in (0,1)$ every β-stable pair has form $((I_C)_2, p)$ where C is a quadric section of the smooth cubic scroll S_1 or a quadric section of the singular cubic scroll S_2, not passing through the vertex. Moreover, for $\beta > 1/7$, Lemma 4.6 implies that C is tangent to the directrix of S_1 (resp., to a ruling of S_2) at a smooth point p. The adjunction formula now gives that

$$\omega_C^2(2p) = \mathcal{O}_{S_i}(1)|_C,$$

and so (C,p) is a bi-log-canonically embedded locally planar genus 2 pointed curve.

The ruling of the scroll S_i cuts out a complete linear system $|\omega_C|$ that defines a finite flat degree 2 morphism to $\mathbb{P}^1$. It follows that the singularities of C are of type A, and hence at worst A_5 by genus considerations. It also follows that C is either irreducible or has two rational components.

Assume that $\beta \in (1/7, 1/2)$. To prove that C is A_4-stable, it remains to show that C has no A_5-singularities. Suppose to the contrary. Then by genus considerations, C is a union of two rational components meeting in a single A_5 singularity. Reducible β-stable curves cannot lie on S_2 because such curves cannot be tangent to a ruling at a smooth point. Therefore, C must lie on S_1. In this case, the plane quartic model of (C,p) is a union of a cuspidal cubic and a flex line, given by the equation $z(zy^2 - x^3) = 0$. The equations cutting out C in $\mathbb{P}^4$ are then given by

$$(4.14) \qquad I_{A_5} := (z_0z_3 - z_1z_2, z_2z_4 - z_3^2, z_0z_4 - z_1z_3, z_1^2 - z_0z_2),$$

while the marked point p is given by $z_1 = z_2 = z_3 = z_4 = 0$.

We note that I_{A_5} is fixed by the 1-PS of $\mathrm{SL}(5)$ acting on $z_0, \ldots, z_4$ with weights $(-2, -1, 0, 1, 2)$. The generators of I_{A_5} as listed in (4.14) have weights $-1, 2, 0, -2$, respectively. Recalling that $p = [1:0:0:0:0]$, we see that under the linearization $\mathcal{L}_\beta$ the Hilbert-Mumford index of (I_{A_5}, p) with respect to this one-parameter subgroup is $-1 + 2\beta$. The pair (C,p) is then destabilized by the 1-PS $(-2, -1, 0, 1, 2)$ for $\beta < 1/2$, and so we are done. $\square$

We will call the curve described by Equation (4.14) the A_5-*curve*. As the following lemma shows, every A_4^{non-W}-stable curve isotrivially specializes to the A_5-curve.

Lemma 4.15. *The* SL(5)-*orbit closure in $\overline{\mathbb{H}}$ of every A_4^{non-W}-stable curve contains the orbit of the A_5-curve. Furthermore, for $\beta \in (1/2, 1)$, we have $\overline{\mathbb{H}}(\beta) = \varnothing$.*

Proof. It suffices to show that all quadric sections of the smooth cubic scroll S_1 that are simply tangent to the directrix $z_2 = z_3 = z_4 = 0$ isotrivially specialize to the A_5-curve and are unstable for $\beta > 1/2$. Assuming that $p = [1 : 0 : 0 : 0 : 0]$, the equation of such quadric section will necessarily be of the form

$$(z_0z_3 - z_1z_2, z_2z_4 - z_3^2, z_0z_4 - z_1z_3, z_1^2 - z_0z_2 + Q(z_1, \ldots, z_5)),$$

where $Q(z_1, \ldots, z_5) \in (z_2, z_3, z_4)$. Per our conventions from §4.2, under the linearization $\mathcal{L}_\beta$, the Hilbert-Mumford index of the H-point of (C, p) with respect to the one-parameter subgroup $(2, 1, 0, -1, -2)$ is $1 - 2\beta$. Hence this H-point is unstable once $\beta > 1/2$. The flat limit of (C, p) under the above one-parameter subgroup is the A_5-curve, marked by $[1 : 0 : 0 : 0 : 0]$. $\square$

Next we prove that $\beta = 4/13$ and $\beta = 1/2$ are the only values in the interval $(1/7, 1/2]$ at which the semistable loci could change. This will prove that $\overline{\mathbb{H}}(\beta)$ is constant for $\beta \in (1/7, 4/13)$ and for $\beta \in (4/13, 1/2)$.

Lemma 4.16. *Suppose (C, p) is a β-semistable curve for some $\beta \in (1/7, 1/2]$ with $\mathbb{G}_m$-action. Then either $\beta = 4/13$ and $(C, p) = (C_R, \infty)$ is the ramphoid cuspidal atom (cf. §2.3), or $\beta = 1/2$ and (C, p) is the A_5-curve given by Equation (4.14) and marked by $[1 : 0 : 0 : 0 : 0]$.*

Proof. By Proposition 4.13 and Corollary 2.6, for $\beta \in (1/7, 1/2]$, every β-semistable curve is honestly hyperelliptic. If $\kappa_C \colon C \to \mathbb{P}^1$ is the canonical morphism, then the branch locus of κ_C is $\mathbb{G}_m$-invariant. Since p is a smooth point of C and $\kappa_C(p)$ is also $\mathbb{G}_m$-invariant, we conclude that the branch locus consists either of $\kappa_C(p)$ and another point of multiplicity 5, or of a single point of multiplicity 6, which is distinct from p. In the former case, we obtain $(C, p) \simeq (C_R, \infty)$ as given by (2.16) and in the latter case, we obtain the A_5-curve.

Recall from §2.3 (whose notation we keep) that I_{C_R} is fixed by the 1-PS acting on $z_0, \ldots, z_4$ with weights $(13, 8, 3, -7, -17)$, and that the generators of I_{C_R} have weights $6, -4, -14, 16$, respectively. Since $\infty = [1 : 0 : 0 : 0 : 0]$, we conclude that under linearization $\mathcal{L}_\beta$ the Hilbert-Mumford index of (I_{C_R}, ∞) with respect to this 1-PS is

$$(6 + (-4) + (-14) + 16) - 13\beta = 4 - 13\beta.$$

This shows that (C_R, ∞) can be semistable only for $\beta = 4/13$.

Similarly, we saw in the proof of Proposition 4.14 that the A_5-curve (C, p) is fixed by a 1-PS of SL(5) with respect to which the Hilbert-Mumford index of (C, p) is $2\beta - 1$ under the linearization $\mathcal{L}_\beta$. This shows that (C, p) can be semistable only for $\beta = 1/2$.

$\square$

Remark 4.17. Note that we do not yet claim that (C_R, ∞) is semistable at $\beta = 4/13$. This will be proved later in Lemma 4.29.

4.5. **Proof of Theorem 3.4 Part (C).** Suppose $\beta \in (4/13, 1/2)$. We begin by showing that Weierstrass curves are destabilized for $\beta > 4/13$.

Lemma 4.18. *Every Weierstrass curve isotrivially degenerates to the ramphoid cuspidal atom (C_R, ∞), and is unstable with respect to $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(\beta)$ for $\beta > 4/13$.*

Proof. Recall from §2.2.2 that every Weierstrass curve (C, p) is given by the equations

$$I_C = (z_0 z_3 - z_2^2, z_0 z_4 - z_2 z_3, z_2 z_4 - z_3^2, z_1^2 - z_0 z_2 - c_3 z_2 z_3 - c_2 z_2 z_4 - c_1 z_3 z_4 - c_0 z_4^2),$$
$$p = [1 : 0 : 0 : 0 : 0],$$

after an appropriate choice of a basis $z_0, \ldots z_4$ of $\mathrm{H}^0\big(C, \omega_C^2(2p)\big)$. Let $\rho(t)$ be the one-parameter subgroup of $\mathrm{SL}(5)$ acting as

$$\rho(t) \cdot (z_0, \ldots z_4) = (t^{13} z_0, t^8 z_1, t^3 z_2, t^{-7} z_3, t^{-17} z_4).$$

Then the flat limit of $\rho(t) \cdot I_C$ as $t \to \infty$ is precisely I_{C_R}. Since $[1 : 0 : 0 : 0 : 0]$ is fixed by ρ, the first claim follows.

Next, the leading monomials of $(I_C)_2$ have ρ-weights $6, -4, -14, 16$, so that $\mu_\rho((I_C)_2) = 4$. On the other hand, $\mu_\rho(p) = -13$. The instability claim follows. □

By Proposition 4.13, every β-stable pair (C, p) is A_4-stable. Applying Lemma 4.18, we see that (C, p) is in fact A_4^{non-W}-stable.

By Proposition 4.10, all smooth non-Weierstrass curves (C, p) are β-stable for $\beta \leq 1/3$. Since the semistable locus is constant for $\beta \in (4/13, 1/2)$, we conclude that all *smooth* non-Weierstrass pointed genus 2 curves are in $\overline{\mathbb{H}}(4/13, 1/2)$.

To complete the proof of Theorem 3.4 (C), it remains to show that *singular A_4^{non-W}-stable* curves are also β-stable for $\beta \in (4/13, 1/2)$. For this, we make use of the GIT stack of semistable plane quartics. Recall from [MFK94] that a plane quartic D is semistable with respect to the standard $\mathrm{PGL}(3)$-action if and only if D has no multiplicity three singularities, and D is not a union of a plane cubic and its flex line. Moreover, D is strictly semistable if and only if D has a tacnode, or is a double conic. A strictly semistable quartic with a closed orbit is either a union of two conics along two tacnodes or a double conic.

Lemma 4.19. *Suppose (C, p) is an A_4^{non-W}-stable curve. Let $\psi \colon C \to \mathbb{P}^2$ be the morphism given by $\omega_C(2p)$, constructed in Lemma 2.8. Then:*
  (1) *$\psi(C)$ is a GIT semistable plane quartic with a cusp at $\psi(p)$.*
  (2) *$\psi(C)$ is a strictly semistable quartic if and only if one of the following holds:*
      (a) *$\psi(C)$ is an irreducible plane quartic with a cusp and a tacnode as singularities.*
      (b) *$\psi(C)$ is an irreducible plane quartic with a cusp and A_4-singularity.*

Proof. (1) follows from Lemma 2.8 and the classification of GIT semistable quartics; and (2) from the classification of strictly semistable plane quartics. □

Lemma 4.20. *There is a unique isomorphism class of a plane quartic in $\mathbb{P}^2$ with a type A_2 (a cusp) and a type A_4 (a ramphoid cusp) singularities.*

Proof. Let C be a plane quartic with singularities of type A_2 and A_4. Then by genus considerations, C is irreducible and its normalization is a rational curve. Assume that the cusp of C is $[0:0:1]$, that the ramphoid cusp is at $[0:1:0]$, that the line $z = 0$ is the tangent cone of the ramphoid cusp, and that the line $y = 0$ is the tangent cone of the cusp. Then the normalization morphism $\mathbb{P}^1 \to \mathbb{P}^2$ is necessarily given by

$$[s:t] \mapsto [s^2t^2 : st^3 + t^4 : s^4].$$

The equation of the image quartic in $\mathbb{P}^2$ is then

(4.21) $$(yz - x^2)^2 - x^3 z = 0.$$

$\square$

Definition 4.22 (A_4-curve). We call the pointed genus 2 curve obtained by normalizing the plane quartic described by Lemma 4.20 at the cusp, and marking the preimage of the cusp, *the A_4-curve.*

Define by Z the locally closed subvariety of semistable plane quartics consisting of semistable quartics with a cusp, and by $\overline{Z}$ its closure.

Proposition 4.23. *We have that $Z /\!\!/ \operatorname{PGL}(3) \simeq \overline{Z} /\!\!/ \operatorname{PGL}(3)$.*

Proof. What we need to show is that for every polystable quartic $[D]$ in $\overline{Z}$, there exists a unique orbit $\operatorname{PGL}(3) \cdot [E]$ in Z such that $[D] \in \overline{\operatorname{PGL}(3) \cdot [E]}$.

If $[D] \in \overline{Z}$ is a stable quartic, then D must be a reduced quartic with a cusp, and no tacnodes. In this case, $\operatorname{PGL}(3) \cdot [D]$ is the requisite orbit in Z.

Suppose $[D]$ is polystable with a positive-dimensional stabilizer. Then the existence and uniqueness of $[E]$ follows by inspecting the basins of attraction of D as worked out in [HL10]. Namely, if D is a double conic $(yz - x^2)^2 = 0$, then E will be the cuspidal plane model of the A_4-curve, i.e., the A_4-quartic $(yz - x^2)^2 - x^3 z = 0$ from Lemma 4.20. If D is a union of two conics along two tacnodes, i.e., the quartic given by an equation $(yz - x^2)(yz - \alpha x^2) = 0$, then E will be $(yz - x^2)(yz - \alpha x^2) - x^3 z = 0$.

The coordinate-free proof of the existence and uniqueness of $[E]$ is obtained by observing that the orbit (closure) of the tacnodal quartic D which is not the A_4-curve is determined by a *crimping parameter*: If $\widetilde{D}$ is the normalization of D at the tacnode, then $\widetilde{D}$ is an arithmetic genus 1 curve with a hyperelliptic involution exchanging the preimages q_1, q_2 of the tacnode of D. The crimping parameter of D is then given by a scalar defining a vector space automorphism $T_{q_1} \simeq T_{q_2} \simeq T_{q_1}$ where the first isomorphism is the canonical isomorphism given by the identification of T_{q_i} with the tangent cone of the tacnode of D, and the second isomorphism is given by the hyperelliptic involution on $\widetilde{D}$. $\square$

From Proposition 4.23, we immediately obtain the following two corollaries:

Corollary 4.24. *Suppose (C, p) and (C', p') are A_4^{non-W}-stable curves, and let $\psi \colon C \to \mathbb{P}^2$ (resp., $\psi' \colon C' \to \mathbb{P}^2$) be the plane cuspidal model of (C, p) (resp., (C', p')). Then*

$$\overline{\mathrm{SL}(3) \cdot \psi(C)} = \overline{\mathrm{SL}(3) \cdot \psi'(C')}$$

if and only if $\mathrm{SL}(3) \cdot \psi(C) = \mathrm{SL}(3) \cdot \psi'(C')$ if and only if $(C, p) \simeq (C', p')$.

Corollary 4.25. *Every A_4^{non-W}-stable curve has β-stable H-point for $\beta \in (4/13, 1/2)$.*

Proof. We have that

$$\overline{\mathbb{H}}(\beta) \subset \left\{ h(C, p) \mid (C, p) \text{ is } A_4^{non-W}\text{-stable} \right\} \subset Z.$$

Furthermore, by Corollary 4.24, we have birational morphisms of GIT quotients

$$\overline{\mathbb{H}}(\beta) /\!\!/ \mathrm{SL}(5) \to Z /\!\!/ \mathrm{SL}(3) \simeq \overline{Z} /\!\!/ \mathrm{SL}(3),$$

where the last isomorphism is given by Proposition 4.23. If some $h(C, p)$, where (C, p) is an A_4^{non-W}-stable curve, was not a β-stable pair, then $\overline{\mathbb{H}}(\beta) /\!\!/ \mathrm{SL}(5) \to Z /\!\!/ \mathrm{SL}(3)$ would not be a surjective morphism, which is absurd since both the domain and the target are projective. $\qquad\square$

4.6. **Proof of Theorem 3.4 Parts (B) and (A).** We have seen in Lemma 4.16 that the only wall in $(1/7, 1/2)$ at which the semistable locus changes is $\beta = 4/13$. We will next show that stability indeed changes at $\beta = 4/13$ and determine the variation of GIT picture around this wall. Let $\mathcal{W} \subset \overline{\mathbb{H}}$ be the locus of Weierstrass A_4-stable curves. Let

$$\mathcal{Z}^- := \overline{\mathbb{H}}(4/13) \setminus \overline{\mathbb{H}}(4/13 - \epsilon), \quad \text{and} \quad \mathcal{Z}^+ := \overline{\mathbb{H}}(4/13) \setminus \overline{\mathbb{H}}(4/13 + \epsilon).$$

Since the action of $\mathrm{Aut}(C_R, \infty)^\circ \simeq \mathbb{G}_m$ on $\mathrm{Def}^1(C_R, \infty)$ has no weight-spaces with weight 0, we conclude that

$$\mathcal{Z}^- \cap \mathcal{Z}^+ = \mathrm{SL}(5) \cdot (C_R, \infty).$$

We give the following exact description of the VGIT chambers:

Proposition 4.26 (VGIT chambers). *$\mathcal{Z}^+ = \mathcal{W}$, and $\mathcal{Z}^-$ consists of a single orbit of the A_4-curve (cf. Definition 4.22).*

By construction, the A_4-curve is A_4^{non-W}-stable, as it has a well-defined cuspidal plane quartic model given by Equation (4.21). It follows that the A_4-curve is β-stable for $\beta \in (4/13, 1/7)$ by the already established Part (C) of Theorem 3.4. We next observe that the A_4-curve isotrivially specializes to C_R:

Lemma 4.27. *The A_4-curve isotrivially specializes to (C_R, ∞) and is unstable for $\beta < 4/13$.*

Proof. Under the standard embedding of S_1 into $\mathbb{P}^4$ (cf. Equation (2.10), the A_4-curve, given by Equation (4.21)), is the intersection of the scroll S_1 with the quadric $(z_1 - z_2)^2 - z_0 z_2 = 0$. The four quadrics cutting out the A_4-curve are thus:

$$(4.28) \qquad z_0 z_3 - z_1 z_2, \ z_0 z_4 - z_1 z_3, \ z_2 z_4 - z_3^2, (z_1 - z_2)^2 - z_0 z_2,$$

and the marked point p is $[1 : 0 : 0 : 0 : 0]$.

Now consider the 1-PS ρ acting with weights $(-13, -8, -3, 7, 17)$ on $z_0, z_1 - z_2, z_2, z_3, z_4$. Then the flat limit of the ρ-translate of the A_4-curve as $t \to 0$ is precisely C_R. Moreover, this one-parameter subgroup destabilizes the A_4-curve for $\beta < 4/13$. □

Next, we analyze GIT stability of the ramphoid cuspidal atom (C_R, ∞) (cf. §2.3).

Lemma 4.29. *The pair (C_R, ∞) is strictly semistable with a closed orbit with respect to the linearization $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(4/13)$, and unstable with respect to $\mathcal{O}_{\mathrm{Grass}}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(\beta)$ for $\beta \neq 4/13$.*

Proof. We give two proofs of this key lemma. For the first, note that the A_4-curve is semistable for $\beta > 4/13$ (Part (C) of Theorem 3.4), but is unstable for $\beta < 4/13$ (Lemma 4.27). This implies that for $\beta = 4/13$, there must exist a strictly semistable point with a closed orbit. By Lemma 4.16, the only possibility for such a curve is the ramphoid cuspidal atom (C_R, ∞).

For a different proof, completely independent of our previous GIT analysis, note that $\mathrm{H}^0\big(C_R, \omega^2_{C_R}(2\infty)\big)$ is a multiplicity-free representation of $\mathrm{Aut}^0(C_R, \infty)^\circ = \mathbb{G}_m$ and this $\mathbb{G}_m$ acts diagonally on the basis of $\mathrm{H}^0\big(C_R, \omega^2_{C_R}(2\infty)\big)$ given by (2.13). Thus it suffices to verify stability with respect to the torus acting diagonally on the same basis (see [MS11]). This is a straightforward calculation, which we omit. □

We are now ready to finish our analysis of the VGIT chambers around (C_R, ∞).

Proof of Proposition 4.26. We have established that the ramphoid cuspidal atom (C_R, ∞) is strictly semistable for $\beta = 4/13$. Let O denote the orbit of (C_R, ∞) in $\overline{\overline{\mathbb{H}}}$. Note that $\overline{\overline{\mathbb{H}}}$ is smooth along this orbit, since the singularities of C_R are planar. Since $\mathrm{Aut}(C_R, \infty)^\circ = \mathbb{G}_m$, and this $\mathbb{G}_m$ acts with non-zero weights on the first-order deformation space of (C_R, ∞), it follows by general VGIT that $\mathcal{Z}^- \cup \mathcal{Z}^+$ consist of curves that isotrivially specialize to (C_R, ∞) (cf. [Tha96, Lemma 1.2 and Lemma 4.3]), and that $\mathbb{G}_m$ acts with negative (resp., positive) weights on the normal bundle $N_{O/\mathcal{Z}^-}$ (resp., $N_{O/\mathcal{Z}^+}$) at (C_R, ∞) (see [Tha96, Proposition 4.6]).

We have seen in Lemma 4.18 that $\mathcal{W} \subset \mathcal{Z}^+$, and the already established Theorem 3.4 Part (C) implies that $\mathcal{Z}^+ \subset \mathcal{W}$. We conclude that $\mathcal{Z}^+ = \mathcal{W}$.

By Luna's étale slice theorem [Lun73], $N_{O/\overline{\mathbb{H}}, x}$ is identified with the first-order deformation space of $x = (C_R, \infty)$, and under this identification, $N_{O/\mathcal{Z}^+, x}$ is identified with the four-dimensional space W, the tangent space to the Weierstrass locus. It follows that $N_{O/\mathcal{Z}^-, x}$ must have dimension at most one. On the other hand, by Lemma 4.27, the A_4-curve lies in $\mathcal{Z}^-$, and so we conclude that $\mathcal{Z}^-$ consists precisely of the orbit of the A_4-curve. □

By VGIT, we have $\overline{\overline{\mathbb{H}}}(4/13 + \epsilon) \subset \overline{\overline{\mathbb{H}}}(4/13)$ and by Lemma 4.29, we have that every Weierstrass A_4-stable curve has $4/13$-semistable H-point. Part (B) of Theorem 3.4 now follows from Proposition 4.13 and Part (C).

Since the only polystable curve at $\beta = 4/13$ is the ramphoid cuspidal atom (C_R, ∞), we have that

$$\overline{\mathbb{H}}(4/13 - \epsilon) = \overline{\mathbb{H}}(4/13) \setminus \mathcal{Z}^-.$$

Proposition 4.26 now gives Part (A) of Theorem 3.4.

4.7. **Proof of Theorem 3.4 Part (D).** Part (D) follows immediately from Part (C) and Lemma 4.15.

4.8. **Proof of Theorem 3.4 Part (E).** Let $\mathcal{U}_{2,1}(A)$ be the stack of pointed genus 2 curves (C, p) with at worst A singularities and ample $\omega_C(p)$. Then we have the following relations in the Picard group of $\mathcal{U}_{2,1}(A)$:

$$10\lambda = \delta_{\text{irr}} + 2\delta_{1,1},$$
$$\kappa = 12\lambda - \delta_{\text{irr}} - \delta_{1,1}.$$

This follows from the same relations on $\overline{\mathcal{M}}_{2,1}$ (see [AC98, Proposition 1.9] and [Mum83]) and the fact that the complement of $\overline{\mathcal{M}}_{2,1}$ in $\mathcal{U}_{2,1}(A)$ is of codimension at least 2.

Restricting to the open substack of curves with no elliptic tails, we obtain $\delta = 10\lambda$ and $\kappa = 2\lambda$. Since for $\beta \in (1/7, 1/2]$, all β-stable pairs in $\overline{\mathbb{H}}(\beta)$ are H-points of A_4 or A_5-stable curves, and the non-nodal locus in $\overline{\mathbb{H}}(\beta)$ has codimension at least two, a standard computation, as in [Mum77], shows that the polarization $\mathcal{L}_\beta$ (cf. Equation (3.3)) descends to the following ample line bundle on the GIT quotient $\overline{\mathbb{H}}(\beta) /\!\!/ \operatorname{SL}(5)$:

(4.30) $$L_\beta := \frac{1}{5}(-\lambda + 8\psi) - \frac{\beta}{5}(3\lambda + \psi).$$

Clearly, $L_{4/13}$ is a positive rational multiple of

$$(13\lambda - 2\delta + \psi) + \frac{2}{3}\delta + \frac{1}{3}\psi.$$

Recalling that also $K_{\overline{\mathcal{M}}_{2,1}} = 13\lambda - 2\delta + \psi$ (see [Log03, Theorem 2.6]), a standard discrepancy computation now shows that the isomorphism in Equation (3.6) holds. The fact that

$$\overline{\mathbb{H}}(4/13 - \epsilon) /\!\!/ \operatorname{SL}(5) \to \overline{\mathbb{H}}(4/13) /\!\!/ \operatorname{SL}(5)$$

is the contraction of the Weierstrass divisor follows from Proposition 4.26.

The remaining identifications of $\overline{\mathbb{H}}(\beta) /\!\!/ \operatorname{SL}(5)$ with the log canonical models of $\overline{\mathcal{M}}_{2,1}$ follow by a similar computation. For example, since the Weierstrass divisor on $\overline{\mathcal{M}}_{2,1}$ has class $3\psi - \lambda - \delta$ (see [HM98, Proposition 6.70]) and it is contracted in $\overline{\mathbb{H}}(4/13, 1/2) /\!\!/ \operatorname{SL}(5)$, we have a further relation $\lambda = 3\psi$ in the Picard group of $\overline{\mathbb{H}}(4/13, 1/2) /\!\!/ \operatorname{SL}(5)$. It then follows that for $\beta = 1/2$, we have

$$L_{1/2} = \frac{3}{2}\psi - \frac{1}{2}\lambda = 0.$$

At the same time, $L_{1/2}$ is a positive rational multiple of

$$(13\lambda - 2\delta + \psi) + \frac{19}{29}\delta + \frac{10}{29}\psi,$$

and so we obtain the isomorphism

$$\overline{\mathbb{H}}(1/2) /\!\!/ \operatorname{SL}(5) \simeq \overline{M}_{2,1}(19/29) \simeq \{\text{point}\}.$$

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