GEOMETRIC INVARIANT THEORY OF SYZYGIES, WITH APPLICATIONS TO MODULI SPACES

MAKSYM FEDORCHUK

ABSTRACT. We define syzygy points of projective schemes, and introduce a program of studying their GIT stability. Then we describe two cases where we have managed to make some progress in this program, that of polarized K3 surfaces of odd genus, and of genus six canonical curves. Applications of our results include effectivity statements for divisor classes on the moduli space of odd genus K3 surfaces, and a new construction in the Hassett-Keel program for the moduli space of genus six curves.

CONTENTS

1. INTRODUCTION

Geometric Invariant Theory (GIT) was developed by Mumford [MFK94] in order to construct algebraic varieties parameterizing orbits of a reductive linear algebraic group acting on a scheme, and has since become an important tool in the algebraic geometer's arsenal, wielded to great effect in construction of various moduli spaces. A GIT method of constructing compact moduli spaces of polarized varieties was laid out in Gieseker's pioneering work [Gie77], and has been applied since then with minor variations.

Gieseker's approach is rooted in the notion of *Hilbert stability*, which we now recall. Suppose we are given a sufficiently nice class of polarized varieties with fixed numerical invariants,

$$\mathfrak{C} = \big\{ (X, \mathcal{L}) : X \text{ is projective with an ample line bundle } \mathcal{L} \big\},$$

where the objects whose moduli space we desire to construct live. The first step is a choice of an integer k so that $\mathcal{L}^k$ defines an embedding of X into a fixed projective space $\mathbb{P}^{r-1}$

1

for every $(X, \mathcal{L}) \in \mathfrak{C}$. The ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^{r-1}}$ then gives a point in the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^{r-1})$. More concretely, fix $q \gg 0$, and let $m = h^0(X, \mathcal{O}_X(q))$. Using the short exact sequence

$$0 \to \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{I}_X(q)) \to \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}(q)) \to \mathrm{H}^0(X, \mathcal{O}_X(q)) \to 0,$$

Gieseker defines *the qth Hilbert point of $X \hookrightarrow \mathbb{P}^{r-1}$* to be the point of the Grassmannian $\mathrm{Gr}(\mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}(q)), m)$ represented by the surjection $\mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}(q)) \to \mathrm{H}^0(X, \mathcal{O}_X(q))$ (see §1.1 for our conventions concerning Grassmannians). For $q \gg 0$, the qth Hilbert point determines (X, L) up to the action of $\mathrm{PGL}(r)$, and the assignment $\mathcal{I}_X \mapsto \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{I}_X(q))$ is a closed embedding of $\mathrm{Hilb}(\mathbb{P}^{r-1})$ into the Grassmannian

$$\mathrm{Gr}(\mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}(q)), m) = \mathrm{Grass}(h^0(\mathbb{P}^{r-1}, \mathcal{I}_X(q)), \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}(q))).$$

The pullback of the Plücker line bundle from this Grassmannian to $\mathrm{Hilb}(\mathbb{P}^{r-1})$ is an ample $\mathrm{SL}(r)$-linearization, so the general machinery of GIT produces a quotient parameterizing varieties with GIT semistable q^{th} Hilbert points.

If the qth Hilbert point of X is GIT semistable for $q \gg 0$, one says that X is *Hilbert semistable*. Many examples of successful GIT applications in moduli theory are obtained as GIT quotients of parameter spaces of Hilbert semistable objects in a suitably chosen $\mathfrak{C}$. These include GIT constructions of the moduli space $\overline{M}_g$ of Deligne-Mumford stable curves, the moduli space of surfaces of general type (both by Gieseker, in [Gie83] and [Gie77], respectively), and the moduli space of canonically polarized manifolds (Viehweg [Vie95]).

In this paper, we propose a modification of the Gieseker's method based on Koszul cohomology. Its basic idea is very simple. Since the qth Hilbert point of X is the vector space of degree q hypersurfaces containing X, it describes the degree q generators of the homogeneous ideal I_X of X. It is thus natural to look at higher syzygy modules of I_X, describing (higher) relations among the generators. Doing this gives rise to a notion of syzygy points, which depend on two parameters, p and q, and which generalize q^{th} Hilbert points. We formally introduce (p, q)-syzygy points in Section 2.

Once a syzygy point is defined, GIT enters the picture in two different but related ways. The most classical scenario would be to first consider a component T of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^{r-1})$ of those polarized schemes in class $\mathfrak{C}$ that live in a fixed projective space $\mathbb{P}^{r-1}$, just as above. But instead of looking at Hilbert semistable objects in T, we now consider the locus of $t \in T$ such that X_t has a semistable (p, q)-syzygy point. If this locus is non-empty, we consider its GIT quotient by the action of $\mathrm{SL}(r)$ and ask what functor does the resulting GIT quotient represent. This question has not been systematically studied before for any value of p other than $p = 0$, which is the case of Hilbert stability. In Section 6, we give a non-trivial instance of a reasonably complete answer to this question for genus six curves.

The other question that we can ask, in the spirit of a well-known work of Cornalba and Harris [CH88], is what does GIT semistability of a (p, q)-syzygy point for a *single* $(X, L) \in \mathfrak{C}$ say about the birational geometry of the moduli space for objects in $\mathfrak{C}$? We first recast the Cornalba-Harris approach in the language of syzygy points in Section 3.

Two examples of $\mathfrak{C}$ to which we apply it in this paper are polarized K3 surfaces of odd genus and Deligne-Mumford stable curves; considered in Sections 4 and 5, respectively.

A disclaimer is in order: We do not prove anything new about Koszul cohomology of curves and K3 surfaces. For an overview of very recent progress in this area the reader is encouraged to consult [Far17] and [Kem17]. Rather, the aim of this paper is to illustrate that a study of *GIT stability of syzygy points*, even in such well-mined cases as curves and K3 surfaces, might turn out to be just as fruitful as that of Hilbert points.

1.1. **Notation and conventions.** We work over $\mathbb{C}$. A flat family $\pi \colon \mathcal{X} \to T$ of schemes over an irreducible one-dimensional base is called an *isotrivial specialization* if for some $0 \in T$, we have that $\mathcal{X} \times_T (T \setminus \{0\})$ is an isotrivial family over $T \setminus \{0\}$. Given an isotrivial specialization, we say that X_t (where $t \neq 0$) isotrivially specializes to X_0.

If V is a vector space, we define by $\mathbb{S}^\lambda(V)$ its Schur functor given by a partition λ. We denote by $\mathrm{Grass}(k, V)$ the Grassmannian of k-dimensional subspaces of V and by $\mathrm{Gr}(V, k)$ the Grassmannian of k-dimensional quotients of V. In Section 6, we use ϵ to denote a sufficiently small positive rational number.

We assume familiarity with GIT, VGIT, and Koszul cohomology of curves and K3 surfaces, as discussed, for example, in [MFK94], [DH98], and [AF11], respectively.

2. Syzygy points

A precise machinery for defining syzygy points was developed by Green under the name of Koszul cohomology [Gre84]. We briefly sketch it in the way most amenable to the application of GIT in what follows.

Let V be a $\mathbb{C}$-vector space of dimension r and $S = \mathrm{Sym}\, V$ be its symmetric algebra. Then the minimal graded free resolution of $\mathbb{C}$ as an S-module is given by the Koszul complex $\mathbb{K}(S)$:

$$0 \to \bigwedge^r V \otimes S(-r) \to \bigwedge^{r-1} V \otimes S(-r+1) \to \cdots \to \bigwedge^2 V \otimes S(-2) \to V \otimes S(-1) \to S.$$

In homological degree p, and graded degree $p + q$, the differential $d_{p,q} \colon \wedge^p V \otimes \mathrm{Sym}^q V \to \wedge^{p-1} V \otimes \mathrm{Sym}^{q+1} V$ is given by

$$d_{p,q}(x_0 \wedge x_1 \wedge \cdots \wedge x_{p-1} \otimes y) = \sum_{i=0}^{p-1} (-1)^i \, x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{p-1} \otimes x_i y.$$

Now given an arbitrary graded S-module R, the Koszul complex of R is defined simply as $\mathbb{K}(R) := \mathbb{K}(S) \otimes R$. If $d_{p,q}^R$ are the Koszul differentials in $\mathbb{K}(R)$, then the Koszul cohomology groups of R are defined to be

$$K_{p,q}(R) := \mathrm{H}_p(\mathbb{K}(R))_{p+q} = \ker d_{p,q}^R / \operatorname{im} d_{p+1,q-1}^R \simeq \mathrm{Tor}_p^S(R,\mathbb{C})_{p+q}.$$

If $\varphi \colon S \to R$ is a ring homomorphism giving R the structure of a graded S-algebra, and $I = \ker(\varphi)$, then comparing the Koszul complexes of S, R, and I, we obtain in graded degree $p + q$, and homological degrees $p + 1, p, p - 1$, the following commutative diagram of vector spaces

$$(2.1) \quad \begin{array}{ccccc}
\bigwedge^{p+1} V \otimes I_{q-1} & \xrightarrow{d_{p+1,q-1}^I} & \bigwedge^p V \otimes I_q & \xrightarrow{d_{p,q}^I} & \bigwedge^{p-1} V \otimes I_{q+1} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^{p+1} V \otimes \mathrm{Sym}^{q-1} V & \xrightarrow{d_{p+1,q-1}} & \bigwedge^p V \otimes \mathrm{Sym}^q V & \xrightarrow{d_{p,q}} & \bigwedge^{p-1} V \otimes \mathrm{Sym}^{q+1} V \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^{p+1} V \otimes R_{q-1} & \xrightarrow{d_{p+1,q-1}^R} & \bigwedge^p V \otimes R_q & \xrightarrow{d_{p,q}^R} & \bigwedge^{p-1} V \otimes R_{q+1}
\end{array}$$

Suppose now that the following is satisfied:

Assumption 2.2. The map $\varphi_{q-1} \colon \mathrm{Sym}^{q-1} V \to R_{q-1}$ is surjective, and $K_{p,q}(R) = 0$.

Then from Diagram (2.1), we obtain a surjection $\ker(d_{p,q}) \twoheadrightarrow \ker(d_{p,q}^R)$. The decomposition of the Koszul complex of S into $\mathrm{SL}(V)$-representations is well-known. In particular, $\ker(d_{p,q})$ is an irreducible $\mathrm{SL}(V)$-representation $\mathbb{S}^{\lambda_{p,q}}(V)$ given by the partition $\lambda_{p,q} := (q, \underbrace{1, \ldots, 1}_{p})$ (see [FH91, p.83, Exercise 6.20(c)]). With this in mind, we make the following definition:

Definition 2.3. Under Assumption 2.2, we define the (p,q)-*syzygy point* of R to be the point of the Grassmannian

$$\mathrm{Gr}\left(\mathbb{S}^{\lambda_{p,q}}(V), \dim \ker(d_{p,q}^R)\right)$$

given by the exact sequence $\mathbb{S}^{\lambda_{p,q}}(V) \to \ker d_{p,q}^R \to 0$.

By construction, there is a natural $\mathrm{SL}(V)$-action on the set of (p,q)-syzygy points of all S-algebras R with a fixed value of $\dim \ker(d_{p,q}^R)$, induced by the $\mathrm{SL}(V)$-action on $\mathbb{S}^{\lambda_{p,q}}(V)$. We now transition to geometry.

Definition 2.4. Suppose X is a projective scheme and $\mathcal{L}$ is a line bundle on X. Set $V = \mathrm{H}^0(X, \mathcal{L})$, $S = \mathrm{Sym}\, V$, and let $R(X, \mathcal{L}) = \oplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^n)$ be the graded section ring of $\mathcal{L}$. If Assumption 2.2 is satisfied for the S-algebra $R(X, \mathcal{L})$, then the (p,q)-syzygy point of $R(X, \mathcal{L})$ will be called the (p,q)-syzygy point of $(X, \mathcal{L})$, and denoted $\mathrm{Syz}_{(p,q)}(X, \mathcal{L})$, or $\mathrm{Syz}_{(p,q)}(X)$ if the line bundle $\mathcal{L}$ is understood.

We will be almost exclusively concerned with the situation where $\mathcal{L}$ is a very ample line bundle embedding X as a projectively normal subscheme of a projective space $\mathbb{P}V^\vee$ with

a homogeneous ideal $I_X \subset S$. In this case, the syzygy point $\mathrm{Syz}_{(p,q)}(X)$ is well-defined if and only if $K_{p,q}(X) = 0$.

While there is no a priori reason not to consider all allowable values of (p, q), we will see that one obtains a lot of mileage out of just two special cases. The first one is $p = 0$. In this case, we see from Definition 2.4 that the $(0, q)$-syzygy point of X is the short exact sequence

$$0 \to (I_X)_q \to \mathrm{Sym}^q V \to \mathrm{H}^0(X, \mathcal{L}^q) \to 0,$$

which is exactly the qth Hilbert point of $X \hookrightarrow \mathbb{P}V^\vee$. The second case is $q = 2$ and arbitrary p, which corresponds to p^{th}-order linear syzygies among the quadrics cutting out X. Note that in this case

$$\ker(d_{p,2}^{I_X}) = \ker\left(\wedge^p V \otimes (I_X)_2 \to \wedge^{p-1}V \otimes (I_X)_3\right) = K_{p,2}(I_X) \simeq K_{p+1,1}(X).$$

It follows that $\mathrm{Syz}_{(p,2)}(X)$ is a point of $\mathrm{Grass}\left(\dim K_{p+1,1}(X), \ \mathbb{S}^{\lambda_2,1,\dots,1}(V)\right)$, where

$$\mathbb{S}^{\lambda_2,1,\dots,1}(V) = \frac{\wedge^{p+1}V \otimes V}{\wedge^{p+2}V}.$$

Having defined syzygy points, we will be guided by a series of the following open ended questions:

Question 2.5. When is the (p, q)-syzygy point of $X \subset \mathbb{P}^{r-1}$ semistable with respect to the $\mathrm{SL}(r)$-action? Are there geometric manifestations of this stability? Is there a reasonable moduli space of schemes with semistable (p, q)-syzygy points?

While GIT and Koszul cohomology date back more than 30 years ago, the problem of GIT stability of syzygy points of projectively normal varieties, already alluded to by Green [Gre84, Problem (5.21)], has not attracted much attention. A possible reason for this is the enormous difficulty in verifying stability of syzygy points even for such basic objects as canonical curves. Aside from powerful results of Kempf [Kem78] and Luna [Lun73] that allow to bypass the numerical criterion entirely, and often to immediately obtain semistability of *all* syzygy points at once, there is virtually nothing else that is known about stability of syzygy points when $p \geq 1$, or even when $p = 0$ and q is small. We summarize these results as follows:

Theorem 2.6 (Kempf-Luna semistability criterion). *Let $X \hookrightarrow \mathbb{P}V^\vee$ be a linearly normal subscheme with a well-defined (p, q)-syzygy point. Let $\mathrm{Stab}(X) \subset \mathrm{SL}(V)$ be the stabilizer of X.*

(a) *Suppose $G \subset \mathrm{Stab}(X)$ is a linearly reductive subgroup, and that $V = \mathrm{H}^0\left(X, \mathcal{O}(1)\right)$ is an irreducible representation of G. Then $\mathrm{Syz}_{(p,q)}(X)$ is semistable with respect to the $\mathrm{SL}(V)$-action.*

(b) *$\mathrm{Syz}_{(p,q)}(X)$ is semistable with respect to the $\mathrm{SL}(V)$-action if and only if it is semistable with respect to the action of the centralizer $C_{\mathrm{SL}(V)}(\mathrm{Stab}(X))$.*

Proof. Part (a) follows from [Kem78, Cor. 5.1] and Part (b) from [Lun73, Cor. 2 and Rem. 1]. □

3. Families of polarized varieties

In this section, we recall the Cornalba-Harris method [CH88] of producing effective divisors on families of polarized varieties using GIT stability of a generic fiber. Suppose $\pi \colon \mathcal{X} \to T$ is a flat family of projective schemes, over an integral base T, and $\mathcal{L}$ is a line bundle on $\mathcal{X}$. We assume further that:

Assumption 3.1. The sheaves $R^k \pi_*(\mathcal{L}^i)$ are locally free for all $i \geq 1$ and $k \geq 0$. For a generic $t \in T$, the line bundle $\mathcal{L}_t$ is very ample on the fiber X_t and embeds X_t as a projectively normal subscheme of $\mathbb{P}\mathrm{H}^0(X_t, \mathcal{L}_t)^\vee$.

Our main applications will be to the examples where $\pi \colon \mathcal{X} \to T$ is an exhausting family of a (often Deligne-Mumford) stack of polarized varieties of a certain type (e.g., canonical curves or K3 surfaces), and so the reader can safely specialize to that case in what follows.

Set $r := \mathrm{rank}(\pi_* \mathcal{L})$. Possibly after passing to a finite cover of T, we can assume that the line bundle $\det(\pi_* \mathcal{L})$ is divisible by r in $\mathrm{Pic}(T)$. We then choose a line bundle $\mathcal{Q} \in \mathrm{Pic}(T)$ such that $\mathcal{Q}^r \simeq \det(\pi_* \mathcal{L})$ and let $\mathbb{L} := \mathcal{L} \otimes \pi^*(\mathcal{Q}^{-1})$. Then

$$\det(\pi_* \mathbb{L}) \simeq \mathcal{O}_T.$$

Set now $\mathbb{E} = \mathbb{E}_1 := \pi_* \mathbb{L}$ and $\mathbb{E}_i := \pi_* \mathbb{L}^i$, for $i \geq 2$. By Assumption 3.1, we have that $\mathbb{E}_i$ is a locally free sheaf for every $i \geq 1$. We note for the future use that

$$c_1(\mathbb{E}_i) = c_1(\pi_* \mathcal{L}^i) - i \, \mathrm{rank}(\mathbb{E}_i) c_1(\mathcal{Q}).$$

Consider next the following commutative diagram of locally free sheaves on T:

$$
\begin{array}{ccccccc}
\bigwedge^{p+1} \mathbb{E} \otimes \mathrm{Sym}^{q-1} \mathbb{E} & \longrightarrow & \bigwedge^p \mathbb{E} \otimes \mathrm{Sym}^q \mathbb{E} & \longrightarrow & \bigwedge^{p-1} \mathbb{E} \otimes \mathrm{Sym}^{q+1} \mathbb{E} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigwedge^{p+1} \mathbb{E} \otimes \mathbb{E}_{q-1} & \longrightarrow & \bigwedge^p \mathbb{E} \otimes \mathbb{E}_q & \longrightarrow & \bigwedge^{p-1} \mathbb{E} \otimes \mathbb{E}_{q+1} & \longrightarrow & \cdots .
\end{array}
$$

Here, the Koszul differential $\bigwedge^p \mathbb{E} \otimes \mathrm{Sym}^q \mathbb{E} \to \bigwedge^{p-1} \mathbb{E} \otimes \mathrm{Sym}^{q+1} \mathbb{E}$ (resp., $\bigwedge^p \mathbb{E} \otimes \mathbb{E}_q \to \bigwedge^{p-1} \mathbb{E} \otimes \mathbb{E}_{q+1}$) is defined by

$$x_0 \wedge x_1 \wedge \cdots \wedge x_{p-1} \otimes y \mapsto \sum_{i=0}^{p-1} (-1)^i x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{p-1} \otimes x_i y,$$

where $\{x_i\}_{i=0}^p$ are local sections of $\mathbb{E}$, y is a local section of $\mathrm{Sym}^q \mathbb{E}$ (resp., $\mathbb{E}_q$), and where $x_i y$ is the local section of $\mathrm{Sym}^{q+1} \mathbb{E}$ (resp., $\mathbb{E}_{q+1}$) given by a natural multiplication map. By cohomology and base change [Gro63, Corollaire 6.9.9], on fibers at $t \in T$, this diagram is nothing but the two bottom rows of Diagram (2.1) for $R = R(X_t, \mathcal{L}_t)$.

Definition 3.2. We set $\mathcal{S}_{p,q} := \ker\left(\bigwedge^p \mathbb{E} \otimes \mathbb{E}_q \to \bigwedge^{p-1} \mathbb{E} \otimes \mathbb{E}_{q+1}\right)$. Note that $\mathcal{S}_{p,q}$ is a coherent sheaf on T such that, by cohomology and base change, its fiber at $t \in T$ is precisely $\ker(d_{p,q}^{R(X_t, \mathcal{L}_t)})$.

Let $\mathcal{U}_{p,q} \subset T$ be the open subset of points $t \in T$ such that $K_{p-i,q+i}(X_t) = 0$ for all $0 \leq i \leq p$, and $\mathrm{Sym}^{q-1}\,\mathbb{E} \to \mathbb{E}_{q-1}$ is surjective. Then on the open $\mathcal{U}_{p-1,q+1} \subset T$, we have a long exact sequence of coherent sheaves

$$0 \to \mathcal{S}_{p,q} \to \bigwedge^p \mathbb{E} \otimes \mathbb{E}_q \to \bigwedge^{p-1} \mathbb{E} \otimes \mathbb{E}_{q+1} \to \cdots \to \mathbb{E}_{p+q} \to 0.$$

Since $\mathbb{E}_i$'s are locally free by assumption, and T is connected and reduced, we conclude that $\mathcal{S}_{p,q}$ is a locally free sheaf on $\mathcal{U}_{p-1,q+1}$. Moreover, on the open $\mathcal{U}_{p,q} \subset \mathcal{U}_{p-1,q+1}$, we have a surjection

$$\mathbb{S}^{\lambda_{p,q}}(\mathbb{E}) \to \mathcal{S}_{p,q} \to 0,$$

so that $\mathrm{Syz}_{(p,q)}(X_t)$ is well-defined for all $t \in \mathcal{U}_{p,q}$.

Theorem 3.3. *Suppose that for a generic $t \in \mathcal{U}_{p,q}$, the (p,q)-syzygy point of $X_t \subset \mathbb{P}\mathrm{H}^0(X_t, \mathcal{L}_t)^\vee$ is semistable with respect to the $\mathrm{SL}(r)$-action. Then a positive multiple of the following divisor class*

$$(3.4) \qquad c_1(\mathcal{S}_{p,q}) = \sum_{i=0}^{p} (-1)^i \binom{r}{p-i} c_1(\mathbb{E}_{q+i})$$

is effective on $\mathcal{U}_{p-1,q+1}$. Moreover, the stable base locus of $c_1(\mathcal{S}_{p,q})$ on $\mathcal{U}_{p-1,q+1}$ is contained inside

$$\{t \mid \mathrm{Syz}_{(p,q)}(X_t) \text{ is either undefined, or unstable with respect to the } \mathrm{SL}(r)\text{-action}\}.$$

Proof. This follows by the original argument of Cornalba and Harris [CH88], with only cosmetic modifications. Suppose X_t has a semistable well-defined (p,q)-syzygy point, as given by Definition 2.4. Set $V = \mathrm{H}^0(X_t, \mathcal{L}_t)$, and let $W = \bigwedge^N \mathbb{S}^{\lambda_{p,q}}(V)$, where $N = \dim \ker(d_{p,q}^{X_t})$. By definition, semistability of $\mathrm{Syz}_{(p,q)}(X_t)$ is equivalent to the existence of an $\mathrm{SL}(r)$-invariant polynomial $f \in \mathrm{Sym}^d W$ that maps to a non-zero element of $\mathrm{Sym}^d \left(\bigwedge^N \mathcal{S}_{p,q} \right) = \left(\det \mathcal{S}_{p,q} \right)^d$.

Since $\det \mathbb{E} \simeq \mathcal{O}_T$, the $\mathrm{SL}(r)$-invariance of f implies that f is invariant under the transition matrices of the vector bundle $\mathrm{Sym}^d \left(\wedge^N \mathbb{S}^{\lambda_{p,q}}(\mathbb{E}) \right)$. We conclude that f gives rise to a global section of $\mathrm{Sym}^d \left(\wedge^N \mathbb{S}^{\lambda_{p,q}}(\mathbb{E}) \right)$ that maps to a section of $\left(\det \mathcal{S}_{p,q} \right)^d$ not vanishing at $t \in T$. The claim follows. $\square$

3.1. **Summation formulae.** We give a combinatorial summation formula that will be used in this paper to evaluate expressions arising from Equation (3.4).

Lemma 3.5. *We have*

$$\sum_{i=0}^{p} (-1)^i \binom{r}{p-i} \binom{i}{a} = (-1)^a \binom{r-1-a}{p-a}.$$

Proof. Indeed, the sum on the left multiplied by $(-1)^p a!$ is the coefficient of x^{p-a} in

$$(1-x)^r \frac{d^a \left(\sum_{i=0}^{\infty} x^i \right)}{dx^a} = (1-x)^r \frac{(-1)^a a!}{(1-x)^{a+1}} = (-1)^a a! (1-x)^{r-a-1}.$$

$\square$

4. K3 SURFACES

Somewhat surprisingly, we can prove GIT stability for more syzygy points of generic K3 surfaces than for syzygy points of any other non-homogeneous variety. Namely, we have the following:

Theorem 4.1. *Suppose $g = 2k + 1 \geq 3$. Let $X \subset \mathbb{P}^g$ be a generic polarized K3 surface of degree $2g - 2$. Then the syzygy points $\mathrm{Syz}_{(p,q)}(X)$ are semistable with respect to the $\mathrm{SL}(g+1)$-action for all (p, q) such $K_{p,q}(X) = 0$. In particular, the qth Hilbert points $\mathrm{Syz}_{(0,q)}(X)$ of X are semistable for every $q \geq 2$, and the pth-order linear syzygies among quadrics $\mathrm{Syz}_{(p,2)}(X)$ are semistable for every $1 \leq p \leq k - 1$.*

Note that for a generic polarized K3 surface of genus g, we have $K_{p,q}(X) = 0$ for $q \geq 3$, except when $(p, q) = (2k - 1, 3)$, by the self-duality of the Betti table of X, and we have $K_{p,2}(X) = 0$ for $p \leq k - 1$ by Voisin's proof of the generic Green's conjecture [Voi05].

Theorem 4.1 is a generalization (in odd genus only) of a very recent result of Farkas and Rimányi:

Theorem 4.2 ([FR17, Theorem 10.2])**.** *Let X be a polarized K3 surface of degree $2g - 2$ with $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Then the 2nd Hilbert point $\mathrm{Syz}_{(0,2)}(X)$ is semistable.*

Farkas and Rimányi obtain their result by considering the locus of genus g polarized K3 surfaces whose embedding in $\mathbb{P}^g$ lies on a rank 4 quadric. This locus is divisorial on the moduli space, and in fact is induced by an $\mathrm{SL}(g+1)$-invariant divisor on the Grassmannian parameterizing 2nd Hilbert points of such surfaces. It is an interesting problem to find similar geometric divisorial conditions on higher $(p, 2)$-syzygy points of K3 surfaces that would lead to a different proof of their semistability.

Another instance of Theorem 4.1 that was previously known is due to Morrison who proved Hilbert stability of every polarized K3 surface $X \subset \mathbb{P}^g$ with $\mathrm{Pic}(X) \simeq \mathbb{Z}$ [Mor09b]. Morrison's result covers the case of $p = 0$ and $q \gg 0$, but his methods are asymptotic in nature and so cannot be used for small values of q or for positive values of p.

Proof of Theorem 4.1. Define S_{2k+1} to be the closure in $\mathbb{P}^{2k+1}$ of the image of a non-reduced affine scheme $\mathrm{Spec}\,\mathbb{C}[s, t, \varepsilon]/(\varepsilon^2)$ given by the following morphism

$$x_0 = 1, \qquad x_{k+1} = s,$$

$$\vdots \qquad\qquad \vdots$$

(4.3) $$x_i = t^i, \quad x_{k+i+1} = st^i + it^{i-1}\varepsilon,$$

$$\vdots \qquad\qquad \vdots$$

$$x_k = t^k, \quad x_{2k+1} = st^k + kt^{k-1}\varepsilon.$$

One verifies that S_{2k+1} is a projective subscheme of $\mathbb{P}^{2k+1}$ that is covered by four affine charts isomorphic to $\mathrm{Spec}\,\mathbb{C}[s, t, \varepsilon]/(\varepsilon^2)$, and has a trivial dualizing line bundle. More precisely, S_{2k+1} is a *K3 carpet*, as defined by Bayer and Eisenbud in [BE95, Section 8]. Moreover, S_{2k+1} is a flat degeneration of smooth K3 surfaces of degree $4k$ in $\mathbb{P}^{2k+1}$, as

shown by Gallego and Purnaprajna in [GP97]. For example, $S_3 \subset \mathbb{P}^3$ is a double quadric given by the equation $(x_0 x_4 - x_1 x_3)^2 = 0$, and S_5 is a $(2, 2, 2)$ complete intersection in $\mathbb{P}^5$ cut out by the quadrics

$$x_0 x_2 - x_1^2 = x_3 x_5 - x_4^2 = x_0 x_5 + x_2 x_3 - 2 x_1 x_4 = 0.$$

Notice that $(S_{2k+1})_{\mathrm{red}}$ is a balanced rational normal scroll embedded in $\mathbb{P}^{2k+1}$ by the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k)|$. In [GP97, Theorem 1.3], Gallego and Purnaprajna also prove that the rational normal scroll supports a unique double structure that is numerically $K3$. This uniqueness result implies that the natural action of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ extends to the $\mathrm{SL}(2) \times \mathrm{SL}(2)$-action on S_{2k+1}, which can also be seen from our explicit parameterization of S_{2k+1}. It follows by the Borel-Weil theorem that $\mathrm{H}^0\big(X, \mathcal{O}_X(1)\big) \simeq \mathrm{H}^0\big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k)\big)$ is an irreducible representation of $\mathrm{Stab}_{\mathrm{SL}(2k+2)}(S_{2k+1})$. Applying Theorem 2.6, we conclude that all well-defined syzygy points $\mathrm{Syz}_{(p,q)}(S_{2k+1})$ are semistable.

A straightforward computation shows that S_{2k+1} is projectively normal, which implies that the syzygy point $\mathrm{Syz}_{(p,q)}(S_{2k+1})$ is well-defined for all (p, q) such that $K_{p,q}(S_{2k+1}) = 0$. It remains to note that $K_{p,q}(S_{2k+1}) = 0$ if and only if $K_{p,q}(X) = 0$ for a generic polarized K3 surface of genus $2k + 1$. This follows from a result of Deopurkar [Deo17], who deduced the vanishing of $K_{k-1,2}(S_{2k+1})$ from Voisin's proof of the generic Green's conjecture [Voi05]. The statement of the theorem for a generic K3 surface now follows by openness of semistability. $\qquad\square$

4.1. **Effective divisors on the moduli space of K3 surfaces.** To apply the generic syzygy stability results of Theorem 4.1 using the framework of Section 3, we take $T = \mathcal{F}_g$ to be the moduli stack of quasi-polarized K3 surfaces of genus g, parameterizing pairs (X, L), where X is a smooth K3 surface, and L is a big and nef line bundle with $L^2 = 2g - 2$. We then let $\pi \colon \mathcal{X} \to T$ be the universal family, and take $\mathbb{L}$ to be the universal polarization on $\mathcal{X}$ normalized so that $c_1(\pi_* \mathbb{L}) = 0$. Having introduced $\pi \colon \mathcal{X} \to T$ and $\mathbb{L}$, we follow notation of Section 3, and in particular consider the sheaves $\mathcal{S}_{p,q}$ on $\mathcal{F}_g$ introduced in Definition 3.2.

Before proceeding, we need to recall some standard results on tautological classes on $\mathcal{F}_g$ from [FR17] and [MOP17], whose notation we follow. First, the tautological sheaves $\mathbb{E}_i := \pi_*(\mathbb{L}^i)$ are locally free, of rank $2 + i^2(g - 1)$. Second, by [FR17, Proposition 9.2], we have

$$(4.4) \qquad c_1(\mathbb{E}_i) = \frac{i}{12}\kappa_{1,1} + \frac{i^3}{6}\kappa_{3,0} - \left(\frac{(g-1)i^2}{2} + 1\right)\lambda,$$

where $\kappa_{a,b} = \pi_*\big(c_1(\mathbb{L})^a \cdot c_2(\mathcal{T}_\pi)^b\big)$, and $\mathcal{T}_\pi$ is the relative tangent sheaf $\pi \colon \mathcal{X} \to T$. By our choice of normalization, we have $c_1(\pi_* \mathbb{L}) = 0$. Thus $\kappa_{1,1} = (6g + 6)\lambda - 2\kappa_{3,0}$, and so in terms of the canonical tautological divisors λ and $\gamma = \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1}$, we have

$$(4.5) \qquad \kappa_{1,1} = 12\lambda - \frac{4}{g+1}\gamma,$$

$$(4.6) \qquad \kappa_{3,0} = 3(g-1)\lambda + \frac{2}{g+1}\gamma.$$

We begin with an effectivity result that holds uniformly over the whole moduli space.

Theorem 4.7. *Suppose $g \geq 3$ is odd. For every $q \geq 2$, the divisor class*

$$c_1(\mathbb{E}_q) = \frac{q}{12}\kappa_{1,1} + \frac{q^3}{6}\kappa_{3,0} - \left(\frac{(g-1)q^2}{2} + 1\right)\lambda$$

is effective on $\mathcal{F}_g$. In particular, $\kappa_{3,0}$ is pseudoeffective.

Proof. Consider the vector bundle $\mathcal{S}_{0,q} = \mathbb{E}_q$ on $\mathcal{F}_g$. By Theorem 4.1, a generic deformation of the K3 carpet S_{2k+1} has a semistable $(0,q)$-syzygy point. Theorem 3.3 now implies that $c_1(\mathcal{S}_{0,q})$ is an effective $\mathbb{Q}$-line-bundle on $\mathcal{F}_g$. Moreover, the closure of the stable base locus of $c_1(\mathcal{S}_{0,q})$ inside the stack of all smoothable numerically K3 surfaces does not contain $[S_{2k+1}]$.

By taking $q \to \infty$, we see that $\kappa_{3,0}$ lies in the closure of the effective cone of $\mathcal{F}_g$. $\qquad\square$

When $q = 2$, we find that $c_1(\mathbb{E}_2)$ is proportional to $[D_g^{\mathrm{rk}\,4}]$, the class of the Farkas-Rimányi divisor of K3 surfaces lying on rank 4 quadrics; see [FR17, Theorem 3.1] and the surrounding discussion for the definition and properties of this geometric divisor. Theorem 4.7 however produces a different effective divisor spanning the same ray as $[D_g^{\mathrm{rk}\,4}]$ in the effective cone of $\mathcal{F}_g$, as the following result illustrates.

Corollary 4.8. *Suppose $g \geq 5$ is odd. The Kodaira-Itaka dimension of the divisor class*

$$c_1(\mathbb{E}_2) = (2g-1)\lambda + \frac{2}{g+1}\gamma$$

on $\mathcal{F}_g$ is at least 1.

Proof. Since the K3 carpet S_{2k+1} lies on many rank 4 quadrics as soon as $k \geq 2$, there exists a smooth deformation X of S_{2k+1} such that $\mathrm{Syz}_{(0,2)}(X)$ is semistable and X lies on a rank 4 quadric. It then follows by Theorem 3.3 that there is an effective multiple of the divisor class $c_1(\mathbb{E}_2)$ that avoids $[X] \in \mathcal{F}_g$. The claim follows. $\qquad\square$

More generally, the generic semistability result of Theorem 4.1 shows that $c_1(\mathcal{S}_{p,q})$ is an effective divisor class on the open locus $U_{p-1,q+1} \subset \mathcal{F}_g$. While we will not pursue a detailed discussion of these effectivity results, we at least compute the resulting divisor class. By Equations (3.4) and (4.4), we have

$$c_1(\mathcal{S}_{p,q}) = \sum_{i=0}^{p}(-1)^i\binom{g+1}{p-i}\left(\frac{(q+i)}{12}\kappa_{1,1} + \frac{(q+i)^3}{6}\kappa_{3,0} - \left(\frac{(g-1)(q+i)^2}{2} + 1\right)\lambda\right).$$

Using summation formulae of Lemma 3.5, this evaluates to an unwieldy

$$
\begin{aligned}
(4.9) \quad c_1(\mathcal{S}_{p,q}) = {} & \frac{1}{12}\left(q\binom{g}{p} - \binom{g-1}{p-1}\right)\kappa_{1,1} \\
& + \frac{1}{6}\left(q^3\binom{g}{p} - (3q^2+3q+1)\binom{g-1}{p-1} + (6q+6)\binom{g-2}{p-2} - 6\binom{g-3}{p-3}\right)\kappa_{3,0} \\
& - \frac{g-1}{2}\left(q^2\binom{g}{p} - (2q+1)\binom{g-1}{p-1} + 2\binom{g-2}{p-2}\right)\lambda - \binom{g}{p}\lambda.
\end{aligned}
$$

For $q = 2$, the above formula simplifies significantly yielding

$$(4.10) \qquad c_1(\mathcal{S}_{p,2}) = \binom{g-2}{p}\left(\left(1 - \frac{p}{g-2}\right)\kappa_{3,0} - \left(g - 1 - \frac{g-1}{g-p-1}\right)\lambda\right).$$

We remark that $\mathcal{S}_{p,2}$ coincides with the sheaf $\mathcal{G}_{p,2}$ from [FR17, Section 9]. Recall also from [FR17, Section 9.2] that there exists a Noether-Lefschetz divisor $D_{1,1} \subset \mathcal{F}_g$ such that for every $(X, L) \in \mathcal{F}_g \setminus D_{1,1}$ the line bundle L is base-point-free on X.

Corollary 4.11. *Suppose $g = 2k + 1$. Then for every $0 \leq p \leq k - 1$, the following divisor class is effective on $\mathcal{F}_g \setminus D_{1,1}$:*

$$(4.12) \qquad \left(\frac{3(g-1)(g-3)(g-p-2)}{(g-2)(g+1)} - \frac{g-p}{g-p-1}\right)\lambda + \frac{2(g-p-2)}{(g+1)(g-2)}\gamma$$

Proof. We only need to observe that by the self-duality of the Betti table, every $(X, L) \in \mathcal{F}_g \setminus D_{1,1}$ satisfies $K_{p-1,3}(X) = \cdots = K_{0,p+2}(X) = 0$. It follows that $\mathcal{S}_{p,2}$ is a locally free sheaf on $\mathcal{F}_g \setminus D_{1,1}$. The claim now follows from Theorem 4.1 and Theorem 3.3 by substituting (4.6) into (4.10). $\qquad \square$

For $p = 0$, the divisor class of Corollary 4.11 reduces to that of Corollary 4.8 for $q = 2$. For $p = (g-3)/2$, the divisor class of Corollary 4.11 is proportional to the Koszul divisor $\mathfrak{Kosz}_g$ of Farkas and Rimányi [FR17, Theorem 9.5]; this of course is immediate from the fact the $(k-1, 2)$-syzyzy point of the generic K3 surface X of genus $2k + 1$ lives in a 0-dimensional Grassmannian, and so the locus where this point is not semistable is precisely $\mathfrak{Kosz}_g$, a degeneracy locus of a map between two vector bundles of the same rank over $\mathcal{F}_g$.

Remark 4.13. While it is likely that none of the effective divisors we produce on $\mathcal{F}_g$ are new, we note that our proof of the effectivity of the divisor classes given by Theorem 4.7 does not rely on the knowledge of the Betti tables for generic K3 surfaces and so is completely independent of the Green's conjecture (Voisin's theorem).

5. Canonical curves

Having established GIT semistability of syzygy points of generic K3 surfaces (of odd genus), it is natural to expect that we can prove an analogous result for their hyperplane sections, namely, canonical curves. Unfortunately, this is outside of our reach at the moment, so we will be content with stating several conjectures and drawing some consequences out of them. We begin with the following:

Conjecture 5.1. *Suppose $C \subset \mathbb{P}^{g-1}$ is a generic canonical curve. Then the $(p, 2)$-syzygy point $\mathrm{Syz}_{(p,2)}(C)$ is GIT semistable for all $p \geq \lfloor (g-3)/2 \rfloor$.*

Aside from some low genus cases discussed below, Conjecture 5.1 is known only in two instances. The first is that of the 2nd Hilbert point ($p = 0$), which is established in two different ways in [AFS13] and [FJ13]. The second is that of the first linear syzygies among quadrics ($p = 1$) which is proved in [DFS16] for generic curves of odd genus. The proof in [DFS16] proceeds by considering syzygies of a special canonically embedded non-reduced Gorenstein curve R_{2k+1} of genus $g = 2k+1$, called a *balanced canonical ribbon.* Considered

earlier in [AFS13], R_{2k+1} is the unique canonical ribbon in the sense of [BE95] with a $\mathbb{G}_m$-action and maximal Clifford index. Explicitly, R_{2k+1} is the section of the K3 carpet S_{2k+1} (defined by Equation (4.3)) by the hyperplane $x_k - x_{x+1} = 0$. Computer computations for small values of k give a hope that in fact the following more precise conjecture is true:

Conjecture 5.2. *The balanced canonical ribbon R_{2k+1} has a semistable syzygy point* $\mathrm{Syz}_{(p,2)}(R_{2k+1})$ *for all $2 \leq p \leq k-1$.*

We note that $K_{k-1,2}(R_{2k+1}) = 0$ by [Deo17], and so all of the above syzygy points are well-defined.

5.1. Hassett-Keel program for $\overline{M}_g$ and syzygies.

Assuming that Conjecture 5.1 is true, what could we possibly deduce from it? To answer this question using the framework of Section 3, whose notation we keep, we take $T = \overline{\mathcal{M}}_g$ and $\pi \colon \mathcal{X} \to \overline{\mathcal{M}}_g$ to be the universal curve. Then the line bundle $\mathcal{L} := \omega_\pi$ given by the relative dualizing sheaf satisfies Assumption 3.1. In particular, $\pi_* \mathcal{L}$ is the usual Hodge bundle over $\overline{\mathcal{M}}_g$, and so $c_1(\pi_* \mathcal{L}) = \lambda$. It follows that $c_1(\mathcal{Q}) = -\frac{1}{g}\lambda$. Recall that by the Grothendieck-Riemann-Roch, we have

$$\mathrm{rank}(\pi_* \omega_\pi^n) = \begin{cases} g & \text{for } n = 1, \\ (2n-1)(g-1) & \text{for } n \geq 2, \end{cases}$$

and

$$c_1(\pi_* \omega_\pi^n) = \binom{n}{2}\kappa + \lambda,$$

where $\kappa = \pi_*\big(c_1(\omega_\pi)^2\big) = 12\lambda - \delta$. Using this, we compute that

$$c_1(\mathbb{E}_n) = \binom{n}{2}\kappa + \lambda - n(2n-1)\frac{(g-1)}{g}\lambda.$$

We therefore compute, using Lemma 3.5, that

$$c_1(\mathcal{S}_{p,q}) = \sum_{i=0}^{p}(-1)^i \binom{g}{p-i}\big(c_1(\mathbb{E}_{q+i}) - (q+i)(2(q+i)-1)(g-1)c_1(\mathcal{Q})\big)$$

$$= \left(\binom{g-3}{p-2} - q\binom{g-2}{p-1} + \binom{q}{2}\binom{g-1}{p}\right)\left(\left(8 + \frac{4}{g}\right)\lambda - \delta\right)$$

$$- \frac{(g-1)}{g}\left(q\binom{g-1}{p} - \binom{g-2}{p-1}\right)\lambda + \binom{g-1}{p}\lambda.$$

We consider the two extreme cases of $p = 0$ and $q = 2$. When $p = 0$, we are of course in the realm of Hilbert points, and so $\mathcal{S}_{0,q}$ is simply a (twisted) q^{th} Hodge bundle over $\overline{\mathcal{M}}_g$. We see that

$$c_1(\mathcal{S}_{0,q}) = \binom{q}{2}\left(\left(8 + \frac{4}{g}\right)\lambda - \delta\right) - \left(q - 1 - \frac{q}{g}\right)\lambda.$$

When $q = 2$, we have that $\mathcal{S}_{p,2}$ is the bundle of pth-order linear syzygies among quadrics, and we calculate that

$$(5.3) \qquad c_1(\mathcal{S}_{p,2}) = \binom{g-3}{p}\left[\left(8 + \frac{4}{g} - \frac{(g-1)(g-2)}{g(g-p-1)}\right)\lambda - \delta\right].$$

Divisors of this form arise naturally in *the Hassett-Keel program* for $\overline{\mathcal{M}}_g$, whose main goal is to construct modular interpretations of the following log canonical models of $\overline{\mathcal{M}}_g$:

$$(5.4) \quad \overline{M}_g(\alpha) := \mathrm{Proj} \oplus_{m \geq 0} \mathrm{H}^0\big(\overline{\mathcal{M}}_g, m(K_{\overline{\mathcal{M}}_g} + \alpha\delta)\big)$$
$$= \mathrm{Proj} \oplus_{m \geq 0} \mathrm{H}^0\big(\overline{\mathcal{M}}_g, m(13\lambda - (2-\alpha)\delta)\big), \quad \text{where } \alpha \in [0,1] \cap \mathbb{Q}.$$

This program was initiated by Hassett and Hyeon who discovered that by studying GIT of k-canonically embedded curves for smaller values of k than in the original GIT construction of $\overline{M}_g$ by Gieseker [Gie82], one obtains moduli stacks of more and more singular curves. These new GIT quotients miraculously turn out to be isomorphic to $\overline{M}_g(\alpha)$ for $\alpha > 7/10 - \epsilon$ [HH09, HH13].

While the GIT approach was successful in constructing the first two steps in the Hassett-Keel program for $\overline{\mathcal{M}}_g$, it quickly became clear that classical Hilbert stability constructions do not produce the next step in the program. On the other hand, heuristic computations predict that already the next step in the Hassett-Keel program, if constructed via GIT, necessitates a stability analysis of 6th Hilbert points of bicanonical curves [Mor09a]. A divisor class computation of Equation (5.3) similarly suggests the following (perhaps overly optimistic) conjecture:

Conjecture 5.5. *Let* $\mathrm{Hilb}(\mathbb{P}^{g-1})^{\mathrm{ss}}_{p,2}$ *be the locus inside the Hilbert scheme of genus g canonically embedded curves consisting of those curves with a semistable $(p,2)$-syzygy point. Then*

$$\mathrm{Hilb}(\mathbb{P}^{g-1})^{\mathrm{ss}}_{p,2} /\!\!/ \mathrm{SL}(g) \simeq \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0 \left(\overline{\mathcal{M}}_g, m \left(\left(8 + \frac{4}{g} - \frac{(g-1)(g-2)}{g(g-p-1)} \right) \lambda - \delta \right) \right).$$

It goes without saying that to prove this conjecture in any particular case, one would need to have a good understanding of GIT stability of syzygy points of canonical curves. Aside from some generic stability results discussed above, our knowledge here is very limited. Whatever understanding we have, it does suffice to work out the first non-trivial variant of Conjecture 5.5 for genus six canonical curves. This result is described in more detail in Section 6 below.

Remark 5.6. A recent work of Aprodu, Bruno, and Sernesi shows that in genus greater than 10 and gonality greater than 3, the $(1,2)$-syzygy point of a canonical curve determines the curve uniquely, unless the curve is bielliptic, in which case the $(1,2)$-syzygy point of the curve coincides with that of a cone over an elliptic curve [ABS17, Theorem 1]. Thus, it is natural to expect that for $g \geq 11$, the GIT quotient $\mathrm{Hilb}(\mathbb{P}^{g-1})^{\mathrm{ss}}_{1,2} /\!\!/ \mathrm{SL}(g)$ will be a birational model of $\mathcal{M}_g$ in which the hyperelliptic, trigonal, and bielliptic loci are contracted (or flipped).

6. GIT FOR SYZYGIES OF CANONICAL GENUS SIX CURVES

The first instance where the consideration of syzygy points leads to a genuinely new moduli space is the case of genus 6 curves, which is the smallest genus for which the $(1,2)$-syzygy point of a canonical curve is well-defined and non-trivial. What aids the

GIT stability analysis here is the beautiful geometry of canonical genus 6 curves, given by a well-known story, which we now recall. A smooth genus 6 curve can be exactly one of the following: hyperelliptic, trigonal, bielliptic, a plane quintic, or a quadric section of an anti-canonically embedded degree 5 (possibly singular) del Pezzo in $\mathbb{P}^5$. A generic curve appears only in the last case, and only on a smooth del Pezzo. Quadric sections of singular del Pezzos form a divisor in the moduli space called the Gieseker-Petri divisor $D_{6,4}$ (we follow the taxonomy of [FR17] for the Gieseker-Petri divisors; in particular $D_{6,4}$ is the divisor in $\mathcal{M}_6$ of curves with a base-point-free g_4^1 for which the Petri map is not injective). Since a smooth del Pezzo Σ of degree 5 is unique up to an isomorphism, and has a group of automorphisms isomorphic to S_5, there is a distinguished birational model of $\overline{M}_6$ given by

$$(6.1) \qquad\qquad X_6 := \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)/S_5.$$

This was the model used by Shepherd-Barron to prove rationality of $\overline{M}_6$ [SB89]. It has also reappeared recently in the context of the Hassett-Keel program of $\overline{M}_6$, as the ultimate non-trivial log canonical model of $\overline{M}_6$ [Mül14]. In this section, we reinterpret X_6 using GIT of $(0,2)$ and $(1,2)$-syzygy points of canonical genus 6 curves. This allows us to also construct the penultimate log canonical model of $\overline{M}_6$, and to realize the contraction of the Gieseker-Petri divisor $D_{6,4}$ as a VGIT two-ray game.

Consider a smooth non-hyperelliptic curve C of genus 6. By Max Noether's theorem, the canonical embedding of C is a projectively normal degree 10 curve in $\mathbb{P}^5$. We set $V := \mathrm{H}^0(C, K_C)$, and identify C with its canonical model in $\mathbb{P}V^\vee \simeq \mathbb{P}^5$. According to Schreyer [Sch86], there are exactly two possible graded Betti tables of C, depending on the Clifford index of C.

6.1. **Clifford index** 1. We have $\mathrm{Cliff}(C) = 1$ if and only if C has either g_3^1 (i.e., C is trigonal) or g_5^2 (i.e., C is a plane quintic). In this case the Betti table is:

$$
\begin{array}{cccc}
1 & & & \\
& 6 & 8 & 3 \\
& 3 & 8 & 6 \\
& & & 1
\end{array}
$$

Since $\dim K_{1,2}(C) = 3$, the $(1,2)$-syzygy point of C is not defined, and so we need to analyze only Hilbert points of C. In fact, already the stability of 2nd Hilbert point detects finer aspects of the curve's geometry.

Proposition 6.2. *Suppose C is a canonically embedded smooth genus six curve with $\mathrm{Cliff}(C) = 1$. Then, with respect to the $\mathrm{SL}(V)$-action, the 2nd Hilbert point*

$$\mathrm{Syz}_{(0,2)}(C) \in \mathrm{Grass}\big(6, \mathrm{Sym}^2 V\big)$$

(1) *is strictly semistable with $\dim\mathrm{Stab} = 8$ if and only if C is a plane quintic.*
(2) *is strictly semistable with $\dim\mathrm{Stab} = 6$ if and only if C is trigonal with Maroni invariant 0.*
(3) *is unstable if and only if C is trigonal with positive Maroni invariant.*

Proof. The key observation is that the quadrics containing C cut out a surface S of minimal degree in $\mathbb{P}^5$ such that $\mathrm{Syz}_{(0,2)}(C) = \mathrm{Syz}_{(0,2)}(S)$. The stability analysis of $\mathrm{Syz}_{(0,2)}(S)$ is greatly simplified by the fact that S is a rational surface with a large automorphism group.

If C is trigonal, then the canonical embedding of C lies on a rational normal surface scroll $S_{a,4-a}$ in $\mathbb{P}^5$, where $a \in \{1, 2\}$ (see [Cop86, pp.12-13] for a modern exposition of the classical work of Maroni [Mar46] and for a discussion of surface scrolls). The homogeneous ideal of $S_{a,4-a}$ is generated by the 2×2 minors of the following matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{a-1} & x_{a+1} & x_{a+2} & \cdots & x_4 \\ x_1 & x_2 & \cdots & x_a & x_{a+2} & x_{a+3} & \cdots & x_5 \end{array} \right)$$

The Maroni invariant of C is $|4 - 2a|$, and equals to 0 if and only if the scroll is balanced. In the latter case, $S_{2,2} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, embedded by the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)|$. Since $\mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2))$ is an irreducible representation of $\mathrm{SL}(2) \times \mathrm{SL}(2) \subset \mathrm{Stab}(S_{2,2})$, we conclude that $\mathrm{Syz}_{(0,2)}(S_{2,2}) = \mathrm{Syz}_{(0,2)}(C)$ is strictly semistable by Theorem 2.6, and has $\dim \mathrm{Stab} = 6$. Suppose now C is trigonal with a positive Maroni invariant. Since $\mathrm{Syz}_{(0,2)}(C) = \mathrm{Syz}_{(0,2)}(S_{a,4-a})$, it suffices to destabilize $\mathrm{Syz}_{(0,2)}(S_{a,4-a})$, which is easily done using the one-parameter subgroup acting with weight $a - 5$ on $x_0, \ldots, x_a$, and weight $a + 1$ on $x_{a+1}, \ldots, x_5$ (cf. [FJ13]).

If C is a plane quintic, then it lies on the Veronese surface $S = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$. The homogeneous ideal of $v_2(\mathbb{P}^2)$ is generated by the 2×2 minors of the following symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

Since the Veronese is embedded into $\mathbb{P}^5$ by an irreducible representation of $\mathrm{SL}(3)$, we conclude that $\mathrm{Syz}_{(0,2)}(v_2(\mathbb{P}^2)) = \mathrm{Syz}_{(0,2)}(C)$ is strictly semistable by Theorem 2.6, with $\dim \mathrm{Stab} = 8$.

$\square$

6.2. **Clifford index** 2. In this case the Betti table of C is:

```
1
    6  5
       5  6
          1
```

We see that the canonical embedding of C satisfies $K_{1,2}(C) = 0$, and so has both a well-defined syzygy point $\mathrm{Syz}_{(0,2)}(C) \in \mathrm{Grass}(6, \mathrm{Sym}^2 V)$ and a well-defined syzygy point $\mathrm{Syz}_{(1,2)}(C) \in \mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))$. Given such a curve C, we can thus associate to it a point

$$(6.3) \quad h(C) := (\mathrm{Syz}_{(0,2)}(C), \mathrm{Syz}_{(1,2)}(C)) \in \mathbb{H} := \mathrm{Grass}(6, \mathrm{Sym}^2 V) \times \mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V)).$$

The $\mathrm{SL}(V)$-linearized semiample cone of $\mathbb{H}$ is spanned by the pullbacks $p_i^* \mathcal{O}(1)$ of the Plücker line bundles from the two factors. This gives rise to a two-ray VGIT game, whose endpoints correspond to GIT for $(0, 2)$ and $(1, 2)$-syzygy points, respectively.

The $(1,2)$-syzygy point of C coincides with the $(1,2)$-syzygy point of the unique degree 5 surface S containing C, called *the second syzygy scheme* of C; see [ABS17]. There are two different possibilities:

(1) S is a degree 5 del Pezzo surface, namely, a blow-up of $\mathbb{P}^2$ at four possibly infinitely near points, embedded anti-canonically into $\mathbb{P}^5$.

(2) S is a cone over an elliptic normal curve of degree 5 in $\mathbb{P}^4$.

In both cases $h^0(\mathbb{P}^5, \mathcal{I}_S(2)) = 5$, and C is a quadric section of S. This shows that the space of quadrics cutting out C is a span of five quadrics cutting out S and another free quadric:

$$H^0(\mathbb{P}^5, \mathcal{I}_C(2)) = H^0(\mathbb{P}^5, \mathcal{I}_S(2)) + \mathbb{C}\langle Q \rangle \subset \mathrm{Sym}^2 V,$$

and the only linear syzygies among the quadrics are those coming from S.

6.2.1. *Bielliptic curves and elliptic ribbons.* In genus 6, bielliptic curves arise as quadric sections of a cone over a genus one smooth curve E embedded into $\mathbb{P}^4$ by a complete linear system $|\mathcal{L}|$ of a line bundle $\mathcal{L} \in \mathrm{Pic}^5(E)$. A related construction that also produces a genus 6 curve out of the same datum is given by ribbons of Bayer and Eisenbud [BE95]. Given an elliptic curve E and $\mathcal{L} \in \mathrm{Pic}^5(E)$, we can consider an infinitesimal thickening $\mathcal{O}_R$ of $\mathcal{O}_E$ by $\mathcal{L}$. More precisely, we consider a short exact sequence of sheaves on E of the form

$$0 \to \mathcal{I} \to \mathcal{O}_R \to \mathcal{O}_E \to 0,$$

where $\mathcal{O}_R$ is a sheaf of $\mathbb{C}$-algebras, and $\mathcal{I}$ is a sheaf of principal ideals such that $\mathcal{I}^2 = 0$ and $\mathcal{I} \simeq \mathcal{L}^{-1}$ as an $\mathcal{O}_E$-module. We denote by R the non-reduced curve with structure sheaf $\mathcal{O}_R$ supported on E, and call R an *elliptic ribbon*. By construction, R can be specified as an element of $\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \mathcal{L}^{-1}) \simeq H^0(E, \mathcal{L})^\vee$. Via the natural $\mathbb{G}_m$-action given by scaling the extension class, every genus 6 elliptic ribbon on E can be isotrivially degenerated to the split ribbon

$$(6.4) \qquad R_E^{\mathcal{L}} := Spec_{\mathcal{O}_E}\left(\left(\mathcal{O}_E \oplus \epsilon \mathcal{L}^{-1} \right) / (\epsilon^2) \right).$$

The connection between elliptic ribbons and bielliptic curves is the following. If E is embedded into $\mathbb{P}^4$ by the complete linear system $|\mathcal{L}|$ and $\mathrm{Cone}(E)$ is a cone over it in $\mathbb{P}^5$, then $R_E^{\mathcal{L}}$ is isomorphic to the double hyperplane section of $\mathrm{Cone}(E)$. In particular, $R_E^{\mathcal{L}}$ is an isotrivial specialization of every smooth bielliptic genus 6 quadric section of $\mathrm{Cone}(E)$.

We are now ready to state the following result:

Proposition 6.5.

(a) *Suppose C is a smooth bielliptic curve. Then $\mathrm{Syz}_{(0,2)}(C) \in \mathrm{Grass}(6, \mathrm{Sym}^2 V)$ is strictly semistable, $\mathrm{Syz}_{(1,2)}(C) \in \mathrm{Grass}(5, \mathbb{S}^{(2,1)}V)$ is unstable, and $h(C)$ is unstable with respect to any ample linearization on $\mathrm{Grass}(6, \mathrm{Sym}^2 V) \times \mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))$.*

(b) *Suppose C is a genus 6 elliptic ribbon in $\mathbb{P}^5$. Then $\mathrm{Syz}_{(0,2)}(C) \in \mathrm{Grass}(6, \mathrm{Sym}^2 V)$ is strictly semistable.*

Proof. Suppose C is a smooth bielliptic curve. As we have discussed, C is a quadric section of a cone $S = \mathrm{Cone}(E)$ over a normal elliptic curve $E \subset \mathbb{P}^4$, where E is embedded by a complete linear system $|\mathcal{L}|$ of degree 5. Choose a basis $x_0, x_1, \ldots, x_5$ of $H^0(C, K_C)^\vee$ such

that the vertex of the cone S is given by $x_1 = \cdots = x_5 = 0$. Let ρ be the one-parameter subgroup of $\mathrm{SL}(V)$ acting with weight -5 on x_0 and weight 1 on each of $x_1, \ldots, x_5$. Then all five quadrics in $\mathrm{H}^0(\mathbb{P}^5, \mathcal{I}_S(2))$ are homogeneous of weight 2, while the smallest weight term of the free quadric Q has weight -10. At the same time, all six syzygies in $\mathrm{Syz}_{(1,2)}(C) = \mathrm{Syz}_{(1,2)}(S)$ are homogeneous of weight 3. This proves all instability claims.

It remains to show that $\mathrm{Syz}_{(0,2)}(C)$ is actually semistable. Note that

$$\lim_{t \to 0} \rho(t) \cdot \mathrm{Syz}_{(0,2)}(C) = \mathrm{Syz}_{(0,2)}(R),$$

where $R = R_E^{\mathcal{L}}$ is the canonically embedded genus 6 split ribbon supported on E and given by Equation (6.4). Thus it suffices to prove that R is semistable.

Explicitly, $I_R = (I_S, x_0^2)$ and the one-parameter subgroup ρ introduced above is a subgroup of $\mathrm{Stab}(R)$. By Theorem 2.6, it suffices to verify semistability of $\mathrm{Syz}_{(0,2)}(R)$ with respect to the centralizer of ρ in $\mathrm{SL}(6)$. Let μ be a one-parameter subgroup of this centralizer. Suppose μ acts on x_0 by $x_0 \mapsto t^{w_0} x_0$. Since $\mathrm{Syz}_{(0,2)}(R)$ is strictly semistable with respect to ρ, it will be μ-semistable if and only if it is semistable with respect to the one-parameter subgroup $\mu' := \rho^{-w_0} \mu^5$. Since μ' acts trivially on x_0 and acts via a one-parameter subgroup of $\mathrm{SL}(5)$ on $\mathbb{C}\langle x_1, \ldots, x_5 \rangle$, we have reduced to verifying μ'-semistability of the 2nd Hilbert point of $E \subset \mathbb{P}^4$. However, E is an abelian variety, embedded by a complete linear series, and so has a semistable 2nd Hilbert point by [Kem78, Corollary 5.2].

Having established semistability of all canonical split elliptic ribbons of genus 6, we immediately obtain Part (b), as every elliptic ribbon isotrivially degenerates to a split ribbon. $\qquad\square$

6.2.2. *Singular degree* 5 *del Pezzo.* Consider now a degree 5 del Pezzo surface Σ_0 with exactly one ordinary double point and no other singularities, which is constructed as follows. Let x, y, z be the standard coordinates on $\mathbb{P}^2$, and let X be the blow-up of $\mathbb{P}^2$ at the points $p_1 := [1 : 0 : 0]$, $p_2 := [1 : 1 : 0]$, $p_3 := [0 : 1 : 0]$ and $p_4 := [0 : 0 : 1]$. The first three of these points lie on the line $z = 0$, and so the strict transform of this line in X is a (-2)-curve. The anti-canonical line bundle of X is globally generated, and defines a morphism to $\mathbb{P}^5$ whose image is Σ_0, with the A_1-singularity of Σ_0 being precisely the contraction of the (-2)-curve on X.

The automorphism group of Σ_0 will play an important role in our GIT stability analysis, so we note first that Σ_0 admits a $\mathbb{G}_m$-action, induced by the scaling action on $\mathbb{P}^2$ that fixes p_4 and the line $z = 0$. We then note that there is also an action of the symmetric group S_3, induced by the action of S_3 on $\mathbb{P}^2$ permuting the three points p_1, p_2, p_3, and leaving p_4 fixed.

Proposition 6.6. *Let C be a quadric section of Σ_0 that does not pass through the singular point of Σ_0. Consider an ample linearization of the $\mathrm{SL}(V)$-action on $\mathbb{H}$ given by $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$. Then $h(C) = (\mathrm{Syz}_{(0,2)}(C), \mathrm{Syz}_{(1,2)}(C))$ is:*

    (1) *unstable if $\beta > 4$.*
    (2) *strictly semistable if $\beta = 4$.*

Proof. The natural scaling action of $\mathbb{G}_m$ on $\mathbb{P}^2$, given by $t \cdot [x : y : z] = [t^{-1}x : t^{-1}y : z]$, extends to X, and gives rise to a one-parameter subgroup ρ of $\mathrm{Stab}(\Sigma_0) \subset \mathrm{SL}(V)$. To understand this subgroup, we begin by choosing a $\mathbb{G}_m$-semi-invariant basis of $\mathrm{H}^0(X, -K_X)$:

$$(6.7) \qquad a := xy(x - y), b_1 := zx^2, b_2 := zxy, b_3 := zy^2, c_1 := z^2x, c_2 := z^2y.$$

Re-normalizing the weights of the action so as to obtain a one-parameter subgroup of $\mathrm{SL}(6)$, we see that ρ acts via

$$t \cdot (a, b_1, b_2, b_3, c_1, c_2) = (t^{-7}a, t^{-1}b_1, t^{-1}b_2, t^{-1}b_3, t^5c_1, t^5c_2).$$

Recall that S_3 acts on Σ_0 and note that $U_1 := \langle c_1, c_2 \rangle$ is the standard representation of S_3, while $U_2 := \langle b_1, b_2, b_3 \rangle$ is its second symmetric power. It follows that $\mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(1)) \simeq \mathrm{H}^0(X, -K_X)$ is a multiplicity-free representation of $\mathrm{Aut}(\Sigma_0)$.

The quadrics cutting out Σ_0 are given by the Pfaffians of the following anti-symmetric matrix:

$$(6.8) \qquad \begin{pmatrix} 0 & z_1 & z_1 & z_2 & -z_0 \\ -z_1 & 0 & z_4 & 0 & z_2 \\ -z_1 & -z_4 & 0 & z_5 & z_3 \\ -z_2 & 0 & -z_5 & 0 & z_3 \\ z_0 & -z_2 & -z_3 & -z_3 & 0 \end{pmatrix}$$

A routine computation now shows that ρ acts on $\det \mathrm{H}^0\big(\mathbb{P}^5, \mathcal{I}_{\Sigma_0}(2)\big)$ with weight 2, and on $\det \mathrm{H}^0\big(\mathbb{P}^5, \mathcal{I}_{\Sigma_0}(3)\big)$ with weight 9. We also record that $\mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(2))$ is a homomorphic image of

$$(6.9) \qquad \langle a^2 \rangle \oplus aU_2 \oplus \mathrm{Sym}^2 U_1 \oplus \langle b_1^2, b_1b_2, b_2^2, b_2b_3, b_3^2 \rangle \oplus \langle b_1c_1, b_2c_1, b_2c_2, b_3c_2 \rangle$$

which also shows that ρ acts on $\mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(2))$ with weights

$$(-14, -8, -8, -8, 10, 10, 10, -2, -2, -2, -2, -2, 4, 4, 4, 4),$$

and on $\det \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(2))$ with weight -2. From this, we conclude that the ρ-weight of $\mathrm{Syz}_{(1,2)}(\Sigma_0)$ is -3, which shows in particular that $\mathrm{Syz}_{(1,2)}(\Sigma_0)$ is unstable.

Suppose now $C \subset \Sigma_0$ is a quadric section of Σ_0, given by an equation $Q(z_0, \ldots, z_6) = 0$. Since $\mathrm{H}^0(C, \mathcal{I}_C(2)) = \mathrm{H}^0(\Sigma_0, \mathcal{I}_{\Sigma_0}(2)) \bigoplus \mathbb{C}\langle Q \rangle$, we see that ρ acts on $\mathrm{Syz}_{(0,2)}(C)$ with total weight

$$w_\rho(\mathrm{Syz}_{(0,2)}(C)) = -w_\rho(Q) + w_\rho(\mathrm{Syz}_{(0,2)}(\Sigma_0)) \leq 14 - 2 = 12,$$

with equality holding if and only if the lowest ρ-weight term of Q is z_0^2 if and only if C does not pass through the singularity of Σ_0.

Continue with the assumption that C does not pass through the singularity of Σ_0. Set

$$C_0 := \lim_{t \to 0} \rho(t) \cdot C,$$

where we are taking a flat limit in $\mathbb{P}^5$. Then it is easy to see that

$$(6.10) \quad I_{C_0} = (z_0^2, I_{\Sigma_0})$$
$$= (z_0^2, z_3z_4 - z_2z_5, z_2z_4 - z_1z_5, z_2^2 - z_1z_3, z_1z_3 - z_2z_3 - z_0z_5, z_1z_2 - z_0z_4 + z_1z_3).$$

The curve C_0 will play a prominent role in what follows. We note some of its properties. To begin, C_0 is a canonically embedded Gorenstein curve. Scheme-theoretically, it is a union of a fourfold line $L_0 = \mathbb{V}(z_4 - z_5, z_2 - z_3, z_1 - z_3, z_0)$ and three double lines $L_1 = \mathbb{V}(z_4, z_2, z_1, z_0)$, $L_2 = \mathbb{V}(z_3, z_2, z_1, z_0)$, and $L_3 = \mathbb{V}(z_5, z_3, z_2, z_0)$. By construction, C_0 is also the unique $\mathbb{G}_m$-invariant double hyperplane section of Σ_0 that does not pass through the singular point of Σ_0. The statement of the proposition now follows from the following lemma:

Lemma 6.11. *Consider the linearization $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$ on $\mathbb{H}$. Then C_0 is polystable for $\beta = 4$. Furthermore, C_0 is destablized by ρ when $\beta > 4$, and is destabilized by ρ^{-1} when $\beta < 4$.*

Proof. By the above computation, we have

$$w_\rho(\mathrm{Syz}_{(0,2)}(C_0)) = w_\rho(z_0^2) + w_\rho(\mathrm{Syz}_{(0,2)}(\Sigma_0)) = 14 - 2 = 12,$$
$$w_\rho(\mathrm{Syz}_{(1,2)}(C_0)) = w_\rho(\mathrm{Syz}_{(1,2)}(\Sigma_0)) = -3.$$

The instability claims follow.

It remains to show that C_0 is polystable for $\beta = 4$. For this we note that C_0 is fixed by $\mathrm{Aut}(\Sigma_0)$, and so in particular $\mathrm{Aut}(C_0) = \mathrm{Aut}(\Sigma_0) \simeq \mathbb{G}_m \times S_3$. Hence, by Theorem 2.6, it suffices to verify polystability of C_0 with respect to those one-parameter subgroup of $\mathrm{SL}(V)$ that commute with $\mathbb{G}_m \times S_3 \subset \mathrm{Aut}(C_0)$. Any such one-parameter subgroup acts diagonally on the distinguished basis of V given by Equation (6.7) as follows

$$t \cdot (z_0, \ldots, z_5) = (t^{w_0} z_0, t^{w_1} z_1, t^{w_1} z_2, t^{w_1} z_3, t^{w_2} z_4, t^{w_2} z_5),$$

where $w_0 + 3w_1 + 2w_2 = 0$. Since the ρ-action stabilizes C_0, it suffices to check the polystability of C_0 only with respect to the one-parameter subgroup given by weights $(-2, 0, 0, 0, 1, 1)$ and $(2, 0, 0, 0, -1, -1)$. This is a routine calculation using the fact that the quadrics and syzygies of Σ_0 are those satisfied by the Pfaffians of the matrix (6.8). $\square$

6.3. **The Hassett-Keel program for $\overline{M}_6$.** In this subsection, we bootstrap the stability results of §6.2 in order to show that the last two steps in the Hassett-Keel program for $\overline{M}_6$ are constructed via GIT for syzygy points of canonically embedded curves.

Let $V = \mathbb{C}^6$. Recall that $\mathbb{H} = \mathrm{Grass}(6, \mathrm{Sym}^2 V) \times \mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))$ is equipped with a two-dimensional $\mathrm{SL}(V)$-ample cone, whose elements we denote by $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$. For every smooth canonically embedded curve $C \subset \mathbb{P}V^\vee$ of genus six and Clifford index 2, we have a well-defined point $h(C) = (\mathrm{Syz}_{(0,2)}(C), \mathrm{Syz}_{(1,2)}(C)) \in \mathbb{H}$. Denote by $\mathcal{Q} \subset \mathbb{H}$ the Zariski closure of the locus of all such $h(C)$ in $\mathbb{H}$.

We have a natural $\mathrm{SL}(V)$-action on $\mathcal{Q}$, and we denote by $\mathcal{Q}^{\mathrm{ss}}(\beta)$ the semistable locus in $\mathcal{Q}$ with respect to the linearization $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$. We also let $\mathcal{G}(\beta) := [\mathcal{Q}^{\mathrm{ss}}(\beta)/\mathrm{SL}(V)]$ be the corresponding GIT quotient stack and denote by $\overline{G}(\beta)$ its moduli space. Namely, we have that

$$\overline{G}(\beta) = \mathcal{Q}^{\mathrm{ss}}(\beta) /\!\!/ \mathrm{SL}(V).$$

We recall that Σ is a smooth degree 5 del Pezzo and Σ_0 is a degree 5 del Pezzo with a unique A_1-singularity. Let $\mathcal{U} \subset \mathbb{P}\mathrm{H}^0(\Sigma_0, -2K_{\Sigma_0})$ be the open locus of quadric sections of Σ_0 not passing through the singularity. We recall that C_0, given by Equation (6.10), is the unique curve with $\mathbb{G}_m$-action in $\mathcal{U}$. We note that, on a smooth del Pezzo, there exists a unique $C_0' \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)$ such that C_0' isotrivially specializes to C_0. Explicitly, we have

$$C_0' = 4F_1 + 2F_2 + 2F_3 + 2F_4,$$

where F_i's are (-1)-curves on Σ such that F_2, F_3, F_4 meet F_1, and have no other pairwise intersections; C_0' is the same curve as described in [Mül14, Proposition 2.6].

We will also need the following counterpart of Proposition 6.6:

Lemma 6.12. *For every $C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)$, the syzygy point $\mathrm{Syz}_{(1,2)}(C)$ is semistable.*

Proof. The second syzygy scheme of C is Σ, and so $\mathrm{Syz}_{(1,2)}(C) = \mathrm{Syz}_{(1,2)}(\Sigma)$. Since Σ is embedded into $\mathbb{P}^5$ by an irreducible representation of $\mathrm{Aut}(\Sigma) = S_5$ by [SB89], the semistability of $\mathrm{Syz}_{(1,2)}(\Sigma)$ follows from Theorem 2.6. $\square$

Having fixed all the notation, we are ready to state the main result of this section:

Theorem 6.13 (Contraction of the Gieseker-Petri divisor in $\overline{M}_6$ via VGIT)**.**
(1) *For $\beta > 4$, $\mathcal{Q}^{\mathrm{ss}}(\beta) = \{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)\}$. Moreover,*

$$\mathcal{G}(\beta) \simeq [\mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)/S_5]$$

is a Deligne-Mumford stack and $\overline{G}(\beta) \simeq X_6 \simeq \overline{M}_6(\alpha)$, where $\alpha \in (16/47, 35/102)$.
(2) *For $\beta = 4$, $\mathcal{Q}^{\mathrm{ss}}(4) = \mathcal{Q}^{\mathrm{ss}}(4 + \epsilon) \cup \{h(C) \mid C \in \mathcal{U}\}$. Moreover,*

$$\overline{G}(4) \simeq \overline{M}_6(35/102).$$

(3) *For $\beta \in (4 - \epsilon, 4)$,*

$$\mathcal{Q}^{\mathrm{ss}}(\beta) = \{h(C) \mid C \in \mathcal{U}, \ C \neq C_0\} \cup \{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma), \ C \neq C_0'\}.$$

The stack $\mathcal{G}(\beta)$ is Deligne-Mumford, and we have

$$\overline{G}(\beta) \simeq \overline{M}_6\left(\frac{35}{102} + \epsilon\right).$$

(4) *We have a commutative diagram*

$$\begin{array}{ccccc}
\mathcal{G}(4 - \epsilon) & \hookrightarrow & \mathcal{G}(4) & \hookleftarrow & \mathcal{G}(4 + \epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{M}_6(\frac{35}{102} + \epsilon) & & & & \\
& \searrow & & & \downarrow \\
& & \overline{M}_6(\frac{35}{102}) & \xleftarrow{\sim} & \overline{M}_6(\frac{35}{102} - \epsilon)
\end{array}$$

(6.14)

Moreover, $\overline{M}_6(\frac{35}{102} + \epsilon)$ is isomorphic to $\overline{M}_6$ at the generic point of the Gieseker-Petri divisor $D_{6,4}$ and $\overline{M}_6(\frac{35}{102} + \epsilon) \to \overline{M}_6(\frac{35}{102})$ is the contraction of this divisor to $[C_0] \in \overline{M}_6(\frac{35}{102})$.

Proof. A starting point for us is the fact that the generic point of $\mathcal{Q}$ lies on a smooth del Pezzo of degree 5. Thus the projection of $\mathcal{Q}$ to $\mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))$ consists of the closure of a single orbit $\mathrm{SL}(V) \cdot \mathrm{Syz}_{(1,2)}(\Sigma)$. We have seen in Propositions 6.5 and 6.6 that for $\beta > 4$ any point of $\mathcal{Q}$ lying over the boundary of this orbit's closure is unstable, and for $\beta = 4$ only the point lying over $\mathrm{SL}(V) \cdot \mathrm{Syz}_{(1,2)}(\Sigma_0)$ becomes strictly semistable. It follows that $\beta = 4$ is the first wall, as β decreases, where the semistability locus changes, and so for $\beta > 4 - \epsilon$, a point in $\mathcal{Q}$ is semistable only if its projection to $\mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))$ lies in the union of the two orbits $\mathrm{SL}(V) \cdot \mathrm{Syz}_{(1,2)}(\Sigma) \cup \mathrm{SL}(V) \cdot \mathrm{Syz}_{(1,2)}(\Sigma_0)$.

(1) For $\beta > 4$, we have that $\mathcal{Q}^{\mathrm{ss}}(\beta) \subset \{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)\}$. The non-emptiness of $\mathcal{Q}^{\mathrm{ss}}(\beta)$ follows from the semistability of $\mathrm{Syz}_{(1,2)}(C)$ given by Lemma 6.12 and the generic semistability of $\mathrm{Syz}_{(0,2)}(C)$. Since Σ, and hence its every quadric section, has no infinitesimal automorphisms, every semistable point is stable. Using the properness of the GIT quotient stack $\mathcal{G}(\beta)$, we conclude that in fact we must have an equality

$$\mathcal{Q}^{\mathrm{ss}}(\beta) = \{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)\}.$$

It follows that

$$\mathcal{G}(\beta) \simeq \{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)\} / \mathrm{SL}(V) \simeq \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma) / \mathrm{Aut}(\Sigma) = X_6.$$

(2) Since $\beta = 4$ is the first wall (as β decreases) where the semistability changes, we have $\{h(C) \mid C \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)\} \subset \mathcal{Q}^{\mathrm{ss}}(4)$. The inclusion $\{h(C) \mid C \in \mathcal{U}\} \subset \mathcal{Q}^{\mathrm{ss}}(4)$ follows by Proposition 6.6. As we have already discussed, there can be no other semistable points for $\beta = 4$.

Next we note that C_0 is the only point in $\mathcal{G}(4) \setminus (\mathcal{G}(4 - \epsilon) \cup \mathcal{G}(4 + \epsilon))$. The basin of attraction of C_0 (that is the locus of points isotrivially degenerating to C_0) in $\mathcal{G}(4 - \epsilon)$ consists of all of $\mathcal{U}$, and the basin of attraction of C_0 in $\mathcal{G}(4 + \epsilon)$ consists of a single point, $C_0' \in \mathbb{P}\mathrm{H}^0(\Sigma, -2K_\Sigma)$. It follows (by normality of the GIT quotients) that the open immersion of stacks $\mathcal{G}(4 + \epsilon) \subset \mathcal{G}(4)$ induces an isomorphism on the level of their moduli spaces. Namely, $\overline{G}(4 + \epsilon) \simeq \overline{G}(4)$.

(3) Let C be a *generic* quadric section of Σ_0 that does not pass through the singular point of Σ_0. We are going to prove that $h(C) = (\mathrm{Syz}_{(0,2)}(C), \mathrm{Syz}_{(1,2)}(C))$ is stable for $0 < \beta < 4$. To this end, consider a double smooth cubic passing through the points $p_1 = [1:0:0]$, $p_2 = [1:1:0]$, $p_3 = [0:1:0]$ and $p_4 = [0:0:1]$ in $\mathbb{P}^2$. Its strict transform in Σ_0 is an elliptic ribbon R of genus 6 that does not pass through the singular point of Σ_0. By Proposition 6.6, $h(R) = (\mathrm{Syz}_{(0,2)}(R), \mathrm{Syz}_{(1,2)}(R))$ is semistable with respect to the linearization $\mathcal{O}(1) \boxtimes \mathcal{O}(4)$. By Proposition 6.5, $\mathrm{Syz}_{(0,2)}(R)$ is strictly semistable. It follows that $h(R)$ is semistable with respect to all linearizations $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$, where $\beta \in (0, 4)$.

To prove that $\mathcal{Q}^{\mathrm{ss}}(\beta)$ has no strictly semistable points, we note that for $\beta \in (4 - \epsilon, 4)$ no semistable point admits a $\mathbb{G}_m$-action. Indeed, the smooth del Pezzo Σ has no quadric sections with $\mathbb{G}_m$-action, and the only quadric section of the singular del Pezzo Σ_0 is C_0, which is unstable for $\beta < 4$. This proves that $\mathcal{G}(\beta)$ is Deligne-Mumford for $\beta \in (4 - \epsilon, 4)$.

It remains to show that *every* quadric section C of Σ_0 that does not pass through the singular point of Σ_0 and such that $C \neq C_0$ is in fact semistable for $\beta \in (4 - \epsilon, 4)$. This follows from the properness of the GIT quotient stack. Indeed, the open inclusion

$\{h(C) \mid C \in \mathcal{U} \cap \mathcal{Q}^{\mathrm{ss}}(\beta)\} \subset \{h(C) \mid C \in \mathcal{U}, \ C \neq C_0\}$ must induce a birational morphism of projective quotients

$$\{h(C) \mid C \in \mathcal{U} \cap \mathcal{Q}^{\mathrm{ss}}(\beta)\} /\!\!/ \operatorname{SL}(V) \to (\mathcal{U} \setminus [C_0]) /\!\!/ \operatorname{Aut}(\Sigma_0) \simeq \mathbb{P}(6^3, 12^5, 24^3, 28^4)/S_3,$$

where the last identification follows from the explicit description of the $\mathbb{G}_m$-action on $\mathcal{U} \subset \mathbb{P}\mathrm{H}^0(\Sigma_0, -2K_{\Sigma_0})$ given by Equation (6.9). Since $\mathcal{Q}^{\mathrm{ss}}(\beta)$ has no strictly semistable points, this morphism must be an isomorphism and so every element of $\mathcal{U} \setminus [C_0]$ is stable in $\mathcal{Q}^{\mathrm{ss}}(4 - \epsilon)$. In particular, $\mathcal{G}(4 - \epsilon)$ is isomorphic to $\overline{\mathcal{M}}_6$ at the generic point of the Gieseker-Petri divisor $D_{6,4}$.

(4) Given our stability results, the existence of the commutative diagram (6.14) follows from the general theory of VGIT. We finally address the identifications of the GIT quotients $\overline{G}(\beta)$ with log canonical models appearing in the Hassett-Keel program for $\overline{M}_6$. Since for all $\beta > 4 - \epsilon$, the stacks $\mathcal{G}(\beta)$ parameterize Gorenstein curves with ample dualizing sheaf, and the locus of worse-than-nodal curves in $\mathcal{G}(\beta)$ has codimension at least 2, it follows from Equation (5.3) (for $g = 6$ and $p = 0, 1$) that the Plücker line bundles $p_1^* \mathcal{O}_{\mathrm{Grass}(6, \mathrm{Sym}^2 V)}(1)$ and $p_2^* \mathcal{O}_{\mathrm{Grass}(5, \mathbb{S}^{(2,1)}(V))}(1)$ descend to $8\lambda - \delta$ and $\frac{47}{2}\lambda - 3\delta$, respectively, on the GIT quotient. It follows that $\mathcal{O}(1) \boxtimes \mathcal{O}(\beta)$ descends to $\overline{G}(\beta)$ as an ample line bundle

$$(6.15) \qquad\qquad (8\lambda - \delta) + \beta\left(\frac{47}{2}\lambda - 3\delta\right).$$

The isomorphism $\overline{G}(4 \mp \epsilon) \simeq \overline{M}_6(35/102 \pm \epsilon)$ now follows from routine discrepancy computations as in [Mül14, Proposition 4.3], where the identification of X_6 with the log canonical models $\overline{M}_6(\alpha)$ for $\alpha \in (16/47, 35/102)$ was first established.

$\square$

Corollary 6.16. *The moving slope of $\overline{M}_6$ is $102/13$.*

Proof. By [Mül14, Prop. 4.1], the moving slope of $\overline{M}_6$ is at most $102/13$. It remains to find a family T of curves in $\overline{M}_6$ passing through the generic point of the Gieseker-Petri divisor $D_{6,4}$, such that $(T \cdot \delta_0)/(T \cdot \lambda) = 102/13$, and such that T avoids the boundary divisors δ_i, $i \geq 1$. The existence of such a family is immediate from the fact that the strict transform of $D_{6,4}$ in $\overline{M}_6(35/102 + \epsilon)$ is the GIT quotient

$$\mathfrak{D} := (\mathcal{U} \setminus [C_0]) /\!\!/ \operatorname{SL}(6) \simeq \mathbb{P}(6^3, 12^5, 24^3, 28^4)/S_3.$$

Indeed, since $\mathcal{U}$ is an open subset of a complete linear system on Σ_0, it follows that $\mathfrak{D}$ parameterizes at worst nodal irreducible curves away from codimension 2. Since $\mathfrak{D}$ has Picard number 1, all curves in $\mathfrak{D}$ have the same slope. The existence of a requisite family T follows from the fact that the line bundle

$$\mathcal{O}(1) \boxtimes \mathcal{O}(4) = (8\lambda - \delta) + 4\left(\frac{47}{2}\lambda - 3\delta\right) = 102\lambda - 13\delta$$

is trivial on $\mathfrak{D}$.

$\square$

6.3.1. *Future directions.* It is conceivable that one might be able to complete the VGIT analysis of $\mathcal{Q}^{\mathrm{ss}}(\beta)$ for *all* $\beta \in (0,4) \cap \mathbb{Q}$. Since genus 6 curves of Clifford index 0 and 1 have no well-defined $(1,2)$-syzyzy points, and bielliptic curves are unstable for all positive β by Proposition 6.5, we expect that the resulting GIT quotients

$$\overline{G}(\beta) = \mathcal{Q}^{\mathrm{ss}}(\beta) /\!\!/ \operatorname{SL}(V)$$

will parameterize canonical genus 6 curves lying on arbitrary del Pezzo surfaces (in the sense of [Dol12, Definition 8.1.12]) of degree 5 in $\mathbb{P}^5$. By [Dol12, §8.5.1 and Table 8.5], there are just 6 types of singular del Pezzos of degree 5, corresponding to different root sublattices in a root lattice of type A_4, with Σ_0 being the least singular of them. We expect that as β decreases, $\overline{G}(\beta)$ will contain quadric sections of del Pezzos with worse and worse singularities, until for some small β, we will see all possible del Pezzo surfaces.

If for all positive β the GIT quotients $\overline{G}(\beta)$ do indeed parameterize only quadric sections of all degree 5 del Pezzos, then, by a standard double covering construction, we will obtain a sequence of compact moduli spaces of K3 surfaces that fits into the Hassett-Keel-Looijenga program (see [LO16]) for the moduli space of K3 surfaces studied in [AK11].

References

[ABS17] Marian Aprodu, Andrea Bruno, and Edoardo Sernesi. A characterization of bielliptic curves via syzygy schemes, 2017. `arXiv:1708.08056 [math.AG]`.

[AF11] Marian Aprodu and Gavril Farkas. Koszul cohomology and applications to moduli. In *Grassmannians, moduli spaces and vector bundles*, volume 14 of *Clay Math. Proc.*, pages 25–50. Amer. Math. Soc., Providence, RI, 2011.

[AFS13] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. Finite Hilbert stability of (bi)canonical curves. *Invent. Math.*, 191(3):671–718, 2013.

[AK11] Michela Artebani and Shigeyuki Kondō. The moduli of curves of genus six and $K3$ surfaces. *Trans. Amer. Math. Soc.*, 363(3):1445–1462, 2011.

[BE95] Dave Bayer and David Eisenbud. Ribbons and their canonical embeddings. *Trans. Amer. Math. Soc.*, 347(3):719–756, 1995.

[CH88] Maurizio Cornalba and Joe Harris. Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. *Ann. Sci. École Norm. Sup. (4)*, 21(3):455–475, 1988.

[Cop86] Marc Coppens. The Weierstrass gap sequence of the ordinary ramification points of trigonal coverings of $\mathbf{P}^1$; existence of a kind of Weierstrass gap sequence. *J. Pure Appl. Algebra*, 43(1):11–25, 1986.

[Deo17] Anand Deopurkar. The canonical syzygy conjecture for ribbons. *Mathematische Zeitschrift*, 2017.

[DFS16] Anand Deopurkar, Maksym Fedorchuk, and David Swinarski. Toward GIT stability of syzygies of canonical curves. *Algebr. Geom.*, 3(1):1–22, 2016.

[DH98] Igor V. Dolgachev and Yi Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998. With an appendix by Nicolas Ressayre.

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view.

[Far17] Gavril Farkas. Progress on syzygies of algebraic curves, 2017. `arXiv:1703.08056 [math.AG]`.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[FJ13] Maksym Fedorchuk and David Jensen. Stability of 2nd Hilbert points of canonical curves. *Int. Math. Res. Not. IMRN*, 2013(22):5270–5287, 2013.

[FR17]   Gavril Farkas and Richard Rimányi. Quadric rank loci on moduli of curves and K3 surfaces, 2017. `arXiv:1707.00756 [math.AG]`.

[Gie77]   David Gieseker. Global moduli for surfaces of general type. *Invent. Math.*, 43(3):233–282, 1977.

[Gie82]   David Gieseker. *Lectures on moduli of curves*, volume 69 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay, 1982.

[Gie83]   David Gieseker. Geometric invariant theory and applications to moduli problems. In *Invariant theory. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME), Montecatini, June 10–18, 1982*, volume 996 of *Lecture Notes in Mathematics*, pages v+159. Springer-Verlag, Berlin, 1983.

[GP97]   Francisco Javier Gallego and B. P. Purnaprajna. Degenerations of $K3$ surfaces in projective space. *Trans. Amer. Math. Soc.*, 349(6):2477–2492, 1997.

[Gre84]   Mark L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.*, 19(1):125–171, 1984.

[Gro63]   A. Grothendieck. éléments de géométrie algébrique. III. étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[HH09]   Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HH13]   Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of stable curves: the first flip. *Ann. of Math. (2)*, 177(3):911–968, 2013.

[Kem78]   George R. Kempf. Instability in invariant theory. *Ann. of Math. (2)*, 108(2):299–316, 1978.

[Kem17]   Michael Kemeny. Syzygies of curves beyond Green's Conjecture, 2017. `arXiv:1711.04463 [math.AG]`.

[LO16]   Radu Laza and Kieran G. O'Grady. Birational geometry of the moduli space of quartic k3 surfaces, 2016. `arXiv:1607.01324`.

[Lun73]   Domingo Luna. Slices étales. In *Sur les groupes algébriques*, pages 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. Soc. Math. France, Paris, 1973.

[Mar46]   Arturo Maroni. Le serie lineari speciali sulle curve trigonali. *Ann. Mat. Pura Appl. (4)*, 25:343–354, 1946.

[MFK94]   David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MOP17]   Alina Marian, Dragos Oprea, and Rahul Pandharipande. Segre classes and Hilbert schemes of points. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(1):239–267, 2017.

[Mor09a] Ian Morrison. GIT constructions of moduli spaces of stable curves and maps. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 315–369. Int. Press, Somerville, MA, 2009.

[Mor09b] Ian Morrison. Stability of Hilbert points of generic K3 surfaces. *Centre de Recerca Matemàtica Publication*, 401, 2009.

[Mül14]   Fabian Müller. The final log canonical model of $\overline{\mathcal{M}}_6$. *Algebra Number Theory*, 8(5):1113–1126, 2014.

[SB89]   Nicholas I. Shepherd-Barron. Invariant theory for S_5 and the rationality of M_6. *Compositio Math.*, 70(1):13–25, 1989.

[Sch86]   Frank-Olaf Schreyer. Syzygies of canonical curves and special linear series. *Math. Ann.*, 275(1):105–137, 1986.

[Vie95]   Eckart Viehweg. *Quasi-projective moduli for polarized manifolds*, volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1995.

[Voi05]   Claire Voisin. Green's canonical syzygy conjecture for generic curves of odd genus. *Compos. Math.*, 141(5):1163–1190, 2005.

(Fedorchuk) DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, 140 COMMONWEALTH AVE, CHESTNUT HILL, MA 02467, USA

Email address: maksym.fedorchuk@bc.edu