AMPLE DIVISORS ON MODULI SPACES OF POINTED RATIONAL CURVES

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Abstract. We introduce a new technique for proving positivity of certain divisor classes on $\overline{M}_{0,n}$ and its weighted variants $\overline{M}_{0,\mathcal{A}}$. Our methods give a complete description of the models arising in the Hassett's log minimal model program for $\overline{M}_{0,n}$.

Contents

1. Introduction

In [5], Brendan Hassett initiated the problem of studying certain log canonical models of moduli spaces of curves. For any rational number α such that $K_{\overline{\mathcal{M}}_{g,n}}+\alpha\delta$ is an effective divisor on the moduli stack of n-pointed genus g curves, we may define

$$\overline{M}_{g,n}(\alpha) = \operatorname{Proj} \bigoplus_{m\geq 0} H^0(\overline{\mathcal{M}}_{g,n}, m(K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta)),$$

where the sum is taken over m sufficiently divisible, and ask whether the space $\overline{M}_{g,n}(\alpha)$ admits a modular description. In the case $g = 0$, it is easy to see that $K_{\overline{M}_{0,n}} + \alpha\Delta$ is effective if and only if $\alpha > \frac{2}{n-1}$. Hassett conjectured a description of the associated birational models, and Matthew Simpson proved the conjecture, assuming the S_n-equivariant F-conjecture [11].

Theorem 1.1 (Simpson)**.** *Assume that the S_n-equivariant F-conjecture holds.*

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1. *If* $\alpha \in \mathbb{Q} \cap (\frac{2}{k+2}, \frac{2}{k+1}]$ *for some* $k = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$, *then* $\overline{M}_{0,n}(\alpha) \simeq \overline{M}_{0,\mathcal{A}}$, *the moduli space of* $\mathcal{A}$*-stable curves, with* $\mathcal{A} = \{\underbrace{1/k, \dots, 1/k}_{n}\}$.
2. *If* $\alpha \in \mathbb{Q} \cap (\frac{2}{n-1}, \frac{2}{\lfloor n/2 \rfloor + 1}]$, *then* $\overline{M}_{0,n}(\alpha) = (\mathbb{P}^1)^n /\!\!/ SL_2$.

In addition, when $k > \lceil \frac{n}{3} \rceil$, Simpson has given an unconditional proof of Theorem 1.1 by constructing the corresponding spaces $\overline{M}_{0,\mathcal{A}}$ as inverse limits of GIT quotients.

Theorem 1.2 (Simpson)**.** *If* $k > \lceil \frac{n}{3} \rceil$, *the conclusion of Theorem 1.1 holds without assuming the* S_n*-equivariant* F*-conjecture.*

The purpose of this paper is to give an unconditional proof of Theorem 1.1 which is valid for all k, thus completing Hassett's proposed log minimal model program for $\overline{M}_{0,n}$. Our methods are quite different from Simpson's in that they produce ample divisors independent of any input from geometric invariant theory or Kollár's results on positivity of push-forwards of dualizing sheaves [9]. Our methods are applicable without knowing *a priori* that $\overline{M}_{0,\mathcal{A}}$ is projective, and can thus be viewed as an elementary, characteristic-independent proof of the projectivity of $\overline{M}_{0,\mathcal{A}}$, as well as the finite-generation of all log-canonical section rings

$$R(\overline{M}_{0,n}, K_{\overline{M}_{0,n}} + \alpha\Delta) := \bigoplus_{m \geq 0} H^0(\overline{M}_{0,n}, m(K_{\overline{M}_{0,n}} + \alpha\Delta)).$$

The key ingredient in our argument is a new method for verifying the positivity of certain linear combinations of tautological divisor classes on $\overline{M}_{0,\mathcal{A}}$. In order to get a feel for the method, let us consider the problem: For which values of $c \in \mathbb{Q}$ is the divisor $c\psi - \Delta$ nef on $\overline{M}_{0,n}$? (The divisor class ψ is defined in Section 2.)

First, consider the problem of showing that $c\psi - \Delta$ has non-negative degree on curve $B \subset \overline{M}_{0,n}$ meeting the interior. Given a generically smooth n-pointed stable curve $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n)$ over a smooth curve B, there exists a sequence of elementary blow-downs:

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_{N-1} \to \mathcal{C}_N,$$

where $\mathcal{C}_0$ is the minimal desingularization of the total space $\mathcal{C}$, and $\mathcal{C}_N$ is a smooth $\mathbb{P}^1$-bundle over B. Let $\{\sigma_j^i\}_{j=1}^n$ denote the sections of $\pi_i \colon \mathcal{C}_i \to B$ obtained as the images of $\{\sigma_j\}_{j=1}^n$ on $\mathcal{C}_i$, and let $\mathrm{Sing}\,(\pi_i) := c_2(\Omega^1_{\mathcal{C}_i/B})$ be the class of the locus of nodes in the fibers of π_i. Consider the function $G\colon [0, N] \to \mathbb{Q}$ defined by

$$G_c(i) := -\frac{c}{n-1} \sum_{1 \leq j < k \leq n} (\sigma_j^i - \sigma_k^i)^2 - \deg\,(\pi_{i\,*}(\mathrm{Sing}\,(\pi_i))).$$

By expanding out the product, one verifies immediately that $G_c(0) = c(\psi.B) - (\Delta.B)$. On the other hand, $G_c(N) = 0$ since π_N is a smooth $\mathbb{P}^1$-bundle, and the difference of any two sections on a $\mathbb{P}^1$-bundle is numerically equivalent to a collection of fibers. Thus, if we choose $c \in \mathbb{Q}$ so that $G_c(i)$ is a decreasing function of i, we conclude that $c(\psi.B) - (\Delta.B) \geq 0$.

It is not hard to see that if the exceptional divisor of $\mathcal{C}_i \to \mathcal{C}_{i+1}$ intersects r sections, then

$$G_c(i) - G_c(i+1) = \frac{cr(n-r)}{n-1}$$

The hypothesis that $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n)$ is stable implies that the exceptional divisor of each blow-down intersects $r \geq 2$ sections. We conclude that

$$c \geq \frac{(n-1)}{2(n-2)} \implies (c\psi - \Delta).B \geq 0.$$

Next, we wish to use induction to analyze the positivity of $c\psi - \Delta$ on curves contained in the boundary. One difficulty with this approach is that $c\psi - \Delta$ is not functorial with respect to the boundary stratification. In fact, if we consider a boundary divisor $\Delta_{S_1,S_2} \subset \overline{M}_{0,n}$, and the natural gluing map

$$i\colon \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \Delta_{S_1,S_2} \subset \overline{M}_{0,n}$$

we find that

$$i^*(c\psi - \Delta) = \pi_1^*(c\psi_{[n_1]} + \psi_{n_1+1} - \Delta) + \pi_2^*(c\psi_{[n_2]} + \psi_{n_2+1} - \Delta)$$

where $\psi_{[n_i]} := \prod_{j=1}^{n_i} \psi_j$ and ψ_{n_i+1} is the ψ-class corresponding to the attaching section. Thus, we are led to consider the more general problem: For which values of $c \in \mathbb{Q}$, is $c\psi_{[n]} + \psi_{[m]} - \Delta$ nef on $\overline{M}_{0,n+m}$, where $\psi_{[n]} := \prod_{i=1}^{n} \psi_i$ and $\psi_{[m]} := \prod_{i=n+1}^{n+m} \psi_i$? Since a divisor of the form $c\psi_{[n]} + \psi_{[m]} - \Delta$ pulls back to a divisor of the same form on any boundary stratum, we may use induction to conclude: $c\psi_{[n]} + \psi_{[m]} - \Delta$ is nef on $\overline{M}_{0,n+m}$ (for all n, m satisfying $n + m \geq 4$) if and only if it has non-negative degree on any generically smooth family of $(n+m)$-pointed stable curves (for all n, m satisfying $n+m \geq 4$). The positivity of $c\psi_{[n]} + \psi_{[m]} - \Delta$ on generically smooth families may be analyzed using the same technique as in the previous paragraph to conclude that $c\psi - \Delta$ is nef on $\overline{M}_{0,n}$, for all $n \geq 4$, if and only if $c \geq 3/4$. Furthermore, these ideas admit a straightforward generalization to the moduli spaces $\overline{M}_{0,\mathcal{A}}$. In Section 2, we will identify a set of divisors which is functorial with respect to the boundary stratification of $\overline{M}_{0,\mathcal{A}}$, and in Section 3, we will analyze the positivity of linear combinations of these divisors on generically smooth families by relating them to various 'sums-of-squares functions' on a smooth $\mathbb{P}^1$-bundle.

In order to understand precisely which divisors are relevant to the log minimal model program for $\overline{M}_{0,n}$, let us recall Simpson's proof of Theorem 1.1: Fixing $\alpha \in \mathbb{Q} \cap (\frac{2}{k+2}, \frac{2}{k+1}]$, he considers the divisor $K_{\overline{M}_{0,n}} + \alpha\Delta$ and the birational contraction

$$\phi\colon \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}, \quad \mathcal{A} = \{\underbrace{1/k, \dots, 1/k}_{n}\}.$$

He shows that

(1) $(K_{\overline{M}_{0,n}} + \alpha\Delta) - \phi^*\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ is effective.
(2) If the S_n-equivariant F-conjecture holds, then $\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ is ample.

Together, (1) and (2) immediately imply the statement of the theorem. The ampleness of $\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ is verified by pulling this divisor back to $\overline{M}_{0,n}$, and then using the F-conjecture to check that it is nef and contracts only ϕ-exceptional curves. To obtain an unconditional proof of Simpson's theorem, we will give a direct proof of the ampleness of the divisor $\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ by showing that it lies in the interior of the nef cone of $\overline{M}_{0,\mathcal{A}}$.

In Lemma 2.5, we will see that

$$\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta) \equiv K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta \equiv \psi - 2\Delta_n + \alpha\Delta,$$

where $\psi, \Delta_n, \Delta \in N^1(\overline{M}_{0,\mathcal{A}})$ are certain tautological divisor classes on $\overline{M}_{0,\mathcal{A}}$. (See Section 2 for our notational conventions for divisor classes on $\overline{M}_{0,\mathcal{A}}$.) Using the technique sketched above, we will show that this linear combination of tautological divisor classes has positive intersection on any curve in $\overline{M}_{0,\mathcal{A}}$ (Corollary 4.4).

Main Result. *If $\alpha \in \mathbb{Q} \cap (\frac{2}{k+2}, \frac{2}{k+1}]$, the divisor class $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ has positive intersection on any 1-parameter family of $\mathcal{A}$-stable curves, $\mathcal{A} = \{1/k, \dots, 1/k\}$.*

Minor variations on the proof of this result show that $\psi - 2\Delta_n + \alpha\Delta$ remains nef when we perturb by a small linear combination of boundary divisors of $\overline{M}_{0,\mathcal{A}}$. Since the boundary divisors generate $\mathrm{Pic}(\overline{M}_{0,\mathcal{A}})$, we conclude (Corollary 4.11)

Corollary. *If $\alpha \in \mathbb{Q} \cap (\frac{2}{k+2}, \frac{2}{k+1}]$, the divisor class $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ lies in the interior of the nef cone of $\overline{M}_{0,\mathcal{A}}$, and is therefore ample. In particular, Theorem 1.1 holds without assuming the S_n-equivariant F-conjecture.*

Let us give a brief outline of the contents of this paper. In Section 2, we define a number of tautological divisor classes on the weighted spaces $\overline{M}_{0,\mathcal{A}}$, and describe how they push-forward and pull-back under the natural reduction and gluing maps. In particular, we identify a subgroup of tautological divisors functorial with respect to the boundary stratification, and show that the log-canonical divisors $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ lie in this distinguished subspace. In Section 3, we focus on proving positivity statements for certain divisor classes on a 1-parameter family of $\mathcal{A}$-stable curves with smooth general fiber. It seems likely that the methods described here could be used to produce a number of new nef divisors on $\overline{M}_{0,n}$ or $\overline{M}_{0,\mathcal{A}}$, but in this paper we will simply focus on the divisor classes which are relevant for the log minimal model program. In Section 4, we assemble the results of Section 3 to describe a two-dimensional polytope of nef divisors on $\overline{M}_{0,\mathcal{A}}$. In addition, we show that divisors in the interior of this polytope remain nef when we perturb by any small linear combination of boundary divisors. Since the boundary divisors generate $\mathrm{Pic}\,(\overline{M}_{0,\mathcal{A}})$, Kleiman's criterion allows us to conclude that divisors on the interior of this polytope are ample, and this gives our main result.

Remark. Valery Alexeev and David Swinarski [2] have given an alternate proof of Simpson's Theorem 1.1 that is also fully independent of F-conjecture. They produce nef divisors on $\overline{M}_{0,n}$ by pulling back distinguished polarizations on various GIT quotients $(\mathbb{P}^1)^n /\!\!/ \mathrm{SL}(2)$, obtained by varying the linearization on $(\mathbb{P}^1)^n$. Although our approaches are quite different, it seems likely that the Alexeev-Swinarski symmetrized GIT cone of nef divisors ([2, Section 2.2]) coincides with the cone of nef divisors produced by our Theorem 4.3.

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2. Tautological divisor classes on $\overline{M}_{0,\mathcal{A}}$

In this section, we study tautological divisor classes on $\overline{M}_{0,\mathcal{A}}$. In Section 2.1, we identify a distinguished subgroup of tautological classes which are functorial with respect to the boundary stratification of $\overline{M}_{0,\mathcal{A}}$. In Section 2.2, we show that the log canonical divisors $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ all lie within this distinguished subgroup. Finally, in Section 2.3, we prove several pull-back formulae for tautological classes which will be needed for our inductive arguments in Section 4.

We work over a fixed algebraically closed field k (the characteristic of k plays no role in our arguments whatsoever). For any weight vector $\mathcal{A} = (a_1, \ldots, a_n) \in [0,1]^n \cap \mathbb{Q}$ satisfying $\sum_{i=1}^n a_i > 2$, there exists a smooth projective variety $\overline{M}_{0,\mathcal{A}}$, which is a fine moduli space for the moduli problem of $\mathcal{A}$-stable curves of genus zero [4]. Recall that a complete connected reduced nodal curve with n smooth marked points $(C, p_1, \ldots, p_n)$ is $\mathcal{A}$-stable provided that:

1. If $p_{i_1} = \ldots = p_{i_k} \in C$, then $\sum_{j=1}^k a_{i_j} \leq 1$.
2. $\omega_C(a_1p_1 + \ldots + a_np_n)$ is ample.

The boundary $\overline{M}_{0,\mathcal{A}} \backslash M_{0,\mathcal{A}}$ consists of divisors whose generic points parameterize curves where two sections collide, the union of these is denoted Δ_s, and of divisors whose generic point is a rational curve with two irreducible components and marked points $S_1 \subset \{p_1, \ldots, p_n\}$ on one component and marked points $S_2 := \{p_1, \ldots, p_n\} \setminus S_1$ on the other component; these divisors are denoted Δ_{S_1,S_2}.

For any pair of weight vectors $\mathcal{A}, \mathcal{A}'$ satisfying $a_i' \leq a_i$ for each $i = 1, \ldots, n$, there is a birational reduction morphism

$$\phi_{\mathcal{A},\mathcal{A}'} \colon \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{A}'}.$$

The reduction morphism $\overline{M}_{0,n} \to \overline{M}_{0,\{1/2,\ldots,1/2\}}$ is an isomorphism, while for $2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor - 1$, the morphism

$$\overline{M}_{0,\{1/k,\ldots,1/k\}} \to \overline{M}_{0,\{1/(k+1),\ldots,1/(k+1)\}}$$

contracts all boundary divisors Δ_{S_1,S_2} satisfying $|S_1| \leq k+1$ or $|S_2| \leq k+1$.

We shall be concerned exclusively with weight vectors of the form

$$\mathcal{A}^k_{n,m} := \{\underbrace{1/k, \ldots, 1/k}_{n}, \underbrace{1, \ldots, 1}_{m}\},$$

where $m \geq 0$, $k \geq 1$, and $m + n/k > 2$. (These are the conditions under which one obtains a non-empty moduli problem.) Indeed, from now on, whenever we speak of a *weight vector* $\mathcal{A}$, we mean that $\mathcal{A} = \mathcal{A}^k_{n,m}$ with n, m, k satisfying these conditions. When we speak of a *symmetric weight vector*, we mean a weight vector with $m = 0$. We will sometimes use the abbreviation $\mathcal{A}^k_n := \mathcal{A}^k_{n,0}$.

The necessity of considering $\mathcal{A}^k_{i,j}$-weight vectors stems from the inductive description of the boundary of $\overline{M}_{0,\mathcal{A}^k_n}$. For example, if $[n] = S_1 \cup S_2$ is a partition of $[n]$ into two subsets, with $|S_1| = n_1$ and $|S_2| = n_2$, then one has a natural isomorphism

$$\overline{M}_{0,\mathcal{A}^k_{n_1,1}} \times \overline{M}_{0,\mathcal{A}^k_{n_2,1}} \to \Delta_{S_1,S_2} \subset \overline{M}_{0,\mathcal{A}^k_n}.$$

Since $\overline{M}_{0,\mathcal{A}}$ is a smooth, there is a canonical isomorphism between $\mathrm{Pic}\,(\overline{M}_{0,\mathcal{A}})$ and the group of Weil divisors modulo linear equivalence. Furthermore, linear equivalence and numerical equivalence on $\overline{M}_{0,\mathcal{A}}$ coincide, so we will typically consider all line-bundles and divisors on $\overline{M}_{0,\mathcal{A}}$ as divisor classes in $N^1(\overline{M}_{0,\mathcal{A}})$, defined up to numerical equivalence. Each reduction morphism $\phi_{\mathcal{A},\mathcal{A}'}$ gives rise to well-defined push-forward and pull-back maps on the space of divisors modulo numerical equivalence, induced by push-forward of cycles and pull-back of line bundles, respectively.

For $\mathcal{A} := \mathcal{A}^k_{n,m}$, we let $(\pi\colon \mathcal{C} \to \overline{M}_{0,\mathcal{A}}, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ denote the universal curve. Let $\mathrm{Sing}\,(\pi) := c_2(\Omega^1_\pi)$ be the class of the locus of nodes in the fibers of π.

We define the following tautological divisor classes on $\overline{M}_{0,\mathcal{A}}$:

$$\begin{aligned}\psi_\sigma &:= -\sum_{i=1}^{n} \pi_*(\sigma_i^2),\\ \psi_\tau &:= -\sum_{i=1}^{m} \pi_*(\tau_i^2),\\ \Delta_s &:= \sum_{1\le i<j\le n} \pi_*(\sigma_i.\sigma_j),\\ \Delta_n &:= \pi_*(\mathrm{Sing}\,(\pi)).\\ \Delta &:= \Delta_s + \Delta_n\end{aligned}$$

Note that the boundary divisor Δ_n parameterizes nodal curves, whereas the boundary divisor Δ_s parameterizes curves where sections of weight $1/k$ collide. Also, note that $\psi_\tau = 0$ when $m = 0$.

2.1. **Functorial divisor classes.** In the following lemma, we identify a subgroup of tautological divisors that are functorial with respect to the boundary stratification.

Lemma 2.1. *Consider a boundary divisor $\Delta_{S_1,S_2} \subset \overline{M}_{0,\mathcal{A}^k_{n,m}}$. Let*

$$\phi\colon \overline{M}_{0,\mathcal{A}^k_{n_1,m_1+1}} \times \overline{M}_{0,\mathcal{A}^k_{n_2,m_2+1}} \to \Delta_{S_1,S_2} \subset \overline{M}_{0,\mathcal{A}^k_{n,m}},$$

and let π_1 and π_2 denote the two projections from the product. We have

$$\begin{aligned}\phi^*(\psi_\tau - \Delta_n) &= \pi_1^*(\psi_\tau - \Delta_n) + \pi_2^*(\psi_\tau - \Delta_n),\\ \phi^*(\psi_\sigma) &= \pi_1^*(\psi_\sigma) + \pi_2^*(\psi_\sigma),\\ \phi^*(\Delta_s) &= \pi_1^*\Delta_s + \pi_2^*\Delta_s.\end{aligned}$$

Proof. The formulae are immediate from the definition of the tautological divisors and the well-known formula for the normal bundle of Δ_{S_1,S_2}. □

The following corollary allows us to pass from positivity results on families with smooth general fiber to positivity results on arbitrary families.

Corollary 2.2. *Fix $a, b, c \in \mathbb{Q}$ and suppose that, for all weight vectors $\mathcal{A}$, the divisor $D := a\psi_\sigma + b\Delta_s + c(\psi_\tau - \Delta_n)$ has non-negative (resp. positive) degree on any complete 1-parameter family of $\mathcal{A}$-stable curves with smooth general fiber. Then, for all weight vectors $\mathcal{A}$, D has non-negative (resp. positive) degree on any complete 1-parameter family of $\mathcal{A}$-stable curves.*

Proof. By Lemma 2.1, we have $\phi^*D = \pi_1^*D + \ldots + \pi_l^*D$ for an arbitrary boundary stratum $\phi\colon \prod_{j=1}^{l} \overline{M}_{0,\mathcal{A}_j} \to \overline{M}_{0,\mathcal{A}}$. Any curve $B \subset \overline{M}_{0,\mathcal{A}}$, whose general point passes through the interior of this stratum is numerically equivalent to $B_1 + \ldots + B_l$, where B_i lies in a fiber of $\prod_{j=1}^{l} \overline{M}_{0,\mathcal{A}_j} \to \prod_{j\neq i} \overline{M}_{0,\mathcal{A}_j}$, and the general point of B_i maps into the interior of $\overline{M}_{0,\mathcal{A}_i}$. Thus,

$$D.B = \sum_{i=1}^{l} (\pi_i^* D).B_i = \sum_{i=1}^{l} D.\pi_i(B_i) > 0.$$

Since every curve $B \subset \overline{M}_{0,\mathcal{A}}$ meets the interior of some boundary stratum, we are done. □

On account of this corollary, the subgroup of divisors generated by ψ_σ, Δ_s and $\psi_\tau - \Delta_n$ will play a special role in this paper. For this reason we make the following definition.

Definition 2.3. For any weight vector $\mathcal{A}$ and $a, b \in \mathbb{Q}$, we define the divisor $D_k(a,b)$ on $\overline{M}_{0,\mathcal{A}}$ by the formula

$$D_k(a,b) := a\psi_\sigma + b\Delta_s + \psi_\tau - \Delta_n.$$

2.2. **Functoriality of log canonical divisors.** In this section, we study how tautological divisors push-forward under the natural reduction morphism $\overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$. In particular, we will show that the divisors $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta = \phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ take the form $D_k(a,b)$ for suitable $a, b \in \mathbb{Q}$.

Lemma 2.4 (Push-forward formulae)**.** *Let $\phi_k\colon \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}_n^k}$ be the natural reduction morphism ($k \geq 2$). Then*

$$(\phi_k)_*\psi = \psi + 2\Delta_s \in N^1\big(\overline{M}_{0,\mathcal{A}_n^k}\big),$$
$$(\phi_k)_*\Delta = \Delta \in N^1\big(\overline{M}_{0,\mathcal{A}_n^k}\big).$$

Proof. We will prove that $\phi_2\colon \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}_n^2}$ satisfies

$$(\phi_2)_*\psi = \psi + 2\Delta_s,$$
$$(\phi_2)_*\Delta = \Delta,$$

and that $\phi_{2,k}\colon \overline{M}_{0,\mathcal{A}_n^2} \to \overline{M}_{0,\mathcal{A}_n^k}$ satisfies

$$(\phi_{2,k})_*\psi = \psi,$$
$$(\phi_{2,k})_*\Delta = \Delta.$$

The latter formulae are immediate from the fact that the locus in $\overline{M}_{0,\mathcal{A}_n^k}$ over which the universal curve fails to be $\mathcal{A}_n^2$-stable has codimension ≥ 2 (it is precisely the locus where three sections collide). The same reasoning shows that

$$(\phi_2)_*\psi = \psi + a\Delta_s,$$
$$(\phi_2)_*\Delta = \Delta_n + b\Delta_s,$$

for some $a, b \in \mathbb{Q}$. Indeed, it is equivalent to showing that $\phi_*\psi = \psi$ and $\phi_*\Delta = \Delta_n$ as cycles in $\overline{\mathcal{M}}_{0,\mathcal{A}_n^2} \backslash \Delta_s$. But this is immediate since the locus of non-stable curves in $\overline{\mathcal{M}}_{0,\mathcal{A}_n^2} \backslash \Delta_s$ has codimension ≥ 2. To establish that $a = 2$ and $b = 1$, we will use the following test curve.

Let $(\mathcal{C} \to B, \{\sigma_i\}_{i=1}^n)$ be the complete 1-parameter family of $\mathcal{A}_n^2$-stable curves, obtained by taking $\mathcal{C} \to B$ to be the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, taking $\sigma_1, \ldots, \sigma_{n-1}$ to be $n-1$ distinct constant sections, and taking σ_n to be the diagonal section. We have the following intersection numbers on $\overline{M}_{0,\mathcal{A}_n^2}$.

$$\psi.B = 2,$$
$$\Delta_n.B = 0,$$
$$\Delta_s.B = n-1.$$

Now let $(\mathcal{C}^s \to B^s, \{\sigma_i^s\}_{i=1}^n)$ be the stable curve over the same base obtained by blowing-up the intersection points $\{\sigma_i \cap \sigma_n\}_{i=1}^{n-1}$ and taking σ_i^s to be the strict transforms of σ_i. On $\overline{M}_{0,n}$, we have

$$\begin{aligned}\psi.B^s &= 4+2n,\\ \Delta_n.B^s &= n-1,\\ \Delta_s.B^s &= 0\end{aligned}$$

Since $\phi_2\colon \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}_n^2}$ is an isomorphism, mapping B^s isomorphically onto B, we must have $\psi.B^s = ((\phi_2)_*\psi).B$ and $\Delta_n.B^s = ((\phi_2)_*\Delta_n).B$. Thus,

$$\begin{aligned}4-2n &= (\psi + a\Delta_s).B = 2 + a(n-1),\\ n-1 &= (\Delta_n + b\Delta_s).B = b(n-1),\end{aligned}$$

from which we conclude that $a = 2$ and $b = 1$ as desired. □

Now we can express the divisors $\phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta)$ in terms of the tautological classes defined above.

Lemma 2.5. *Let $\phi\colon \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}_n^k}$ be the natural morphism. For any $\alpha \in \mathbb{Q} \cap [0,1]$, we have*

1. $K_{\overline{M}_{0,n}} + \alpha\Delta = \psi - 2\Delta + \alpha\Delta \in N^1(\overline{M}_{0,n})$,
2. $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta = \phi_*(K_{\overline{M}_{0,n}} + \alpha\Delta) = \psi - 2\Delta_n + \alpha\Delta \in N^1(\overline{M}_{0,\mathcal{A}_n^k})$.

Proof. Using the Grothendieck-Riemann-Roch formula as in [3], the canonical class of the stack $\overline{\mathcal{M}}_{g,n}$ is $13\lambda - 2\Delta + \psi$. Since $\overline{\mathcal{M}}_{0,n} = \overline{M}_{0,n}$ and $\lambda = 0 \in \mathrm{Pic}(\overline{M}_{0,n})$, we have $K_{\overline{M}_{0,n}} = \psi - 2\Delta$. The lemma is now an immediate consequence of the push-forward formulae

$$\begin{aligned}\phi_*\psi &\equiv \psi + 2\Delta_s,\\ \phi_*\Delta &\equiv \Delta,\end{aligned}$$

which are proved in Lemma 2.4. □

In our subsequent analysis, it will be convenient to rescale these $\mathbb{Q}$-divisors in order to ensure that the coefficient of Δ_n is -1.

Remark 2.6. For any $\alpha \in \mathbb{Q} \cap [0,1]$, set $c = 1/(2-\alpha)$. Then we have

1. $K_{\overline{M}_{0,n}} + \alpha\Delta \sim c\psi - \Delta$
2. $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta \sim\sim c\psi + (2c-1)\Delta_s - \Delta_n$,

where '$\sim$' denotes numerical proportionality.

Note that the divisor $c\psi + (2c-1)\Delta_s - \Delta_n$ is equal to $D_k(c, 2c-1)$ introduced in Definition 2.3. In particular, it is functorial with respect to the boundary stratification. Since it plays a distinguished role in our subsequent analysis, we make the following definition.

Definition 2.7. For any weight vector $\mathcal{A}$ and $c \in \mathbb{Q}$, we define the divisor class $D_k(c)$ on $\overline{M}_{0,\mathcal{A}}$ by the formula

$$D_k(c) := c\psi_\sigma + (2c-1)\Delta_s + \psi_\tau - \Delta_n.$$

2.3. **Pull-back formulae.** In this section, we study how the tautological divisor classes pull-back under certain reduction and replacement morphisms.

Lemma 2.8 (Pull-back formulae for ϕ)**.** *Consider the reduction morphism*

$$\phi\colon \overline{M}_{0,\mathcal{A}^{k-1}_{n,m}} \to \overline{M}_{0,\mathcal{A}^{k}_{n,m}},$$

and let F denote the union of the exceptional divisors of ϕ. Then we have

1. $\phi^*\psi_\sigma = \psi_\sigma - kF$,
2. $\phi^*\psi_\tau = \psi_\tau$,
3. $\phi^*\Delta_s = \Delta_s + \binom{k}{2}F$,
4. $\phi^*\Delta_n = \Delta_n - F$.

Proof. The pullback formulae for the ψ-classes are proved in Theorem 5.3 of [1]. Here, we give a quick proof which also works for the boundary divisors.

Clearly,

$$\phi^*\psi_\sigma = \psi_\sigma + aF,$$

for some constant a. To determine this coefficient, we use a test curve contracted by ϕ. Let $\mathcal{C}_1$ be the blow-up of $\mathbb{P}^2$ at a point with exceptional curve E, and regard E as a section of the natural $\mathbb{P}^1$-bundle fibration $\mathcal{C}_1 \to B = \mathbb{P}^1$. Let k sections of $\mathcal{C}_1 \to B$ be given by the strict transforms of k lines on $\mathcal{C}_1$; note that each section has self-intersection 1. Let $\mathcal{C}_2 = \mathbb{P}^1 \times \mathbb{P}^1$ and take $n+m-k$ constant sections of the second projection $\mathcal{C}_2 \to B = \mathbb{P}^1$. Let $\mathcal{C}$ be the union of $\mathcal{C}_1$ and $\mathcal{C}_2$, obtained by identifying $E \subset \mathcal{C}_1$ with a constant section of $\mathcal{C}_2 \to B$. Then we can regard $\mathcal{C} \to B$ as the family of $\mathcal{A}^{k-1}_{n,m}$-stable curves by giving all k sections of $\mathcal{C}_1$ and $n-k$ sections of $\mathcal{C}_2$ weight $1/(k-1)$, and giving the remaining m sections of $\mathcal{C}_2$ weight 1.

Clearly, $B \subset \overline{M}_{0,\mathcal{A}^{k-1}_{n,m}}$ is contracted by ϕ. We calculate

$$\begin{aligned}
\psi_\sigma.B &= -k,\\
\psi_\tau.B &= 0,\\
\Delta_s.B &= \binom{k}{2},\\
\Delta_n.B &= -1,\\
F.B &= -1.
\end{aligned}$$

By the projection formula

$$0 = B\cdot\phi^*\psi_\sigma = \psi_\sigma.B + a(F.B) = -k - a.$$

Therefore, $a = -k$. The remaining formulae are proved in the same fashion. □

We will also utilize the natural morphism

$$\chi\colon \overline{M}_{0,\mathcal{A}^{k}_{n-k,m+1}} \to \overline{M}_{0,\mathcal{A}^{k}_{n,m}},$$

obtained by replacing the $(m+1)^{\text{st}}$-section of weight 1 with k coincident sections of weight $1/k$. Let $(\pi\colon \mathcal{C} \to \overline{M}_{0,\mathcal{A}^{k}_{n-k,m+1}}, \{\sigma_i\}_{i=1}^{n-k}, \{\tau_j\}_{j=1}^{m+1})$ be the universal curve, and consider the divisor class $\psi_{\tau_{m+1}} := -\pi_*(\tau_{m+1}^2)$ on $\overline{M}_{0,\mathcal{A}^{k}_{n-k,m+1}}$.

Lemma 2.9 (Pull-back formulae for χ)**.** *Under the replacement morphism*

$$\chi : \overline{M}_{0,\mathcal{A}^{k}_{n-k,m+1}} \to \overline{M}_{0,\mathcal{A}^{k}_{n,m}},$$

the tautological divisors pull-back according to the following formulae

(1) $\chi^*\psi_\sigma = \psi_\sigma + k\psi_{\tau_{m+1}}$,
(2) $\chi^*\psi_\tau = \psi_\tau - \psi_{\tau_{m+1}}$,
(3) $\chi^*\Delta_s = \Delta_s - \binom{k}{2}\psi_{\tau_{m+1}}$,
(4) $\chi^*\Delta_n = \Delta_n$.

Proof. Observe that $\pi\colon \mathcal{C} \to \overline{M}_{0,\mathcal{A}^k_{n-k,m+1}}$ together with sections $\{\sigma'_i\}_{i=1}^n$ and $\{\tau_j\}_{j=1}^m$, where $\sigma'_i = \sigma_i$ for $1 \leq i \leq n-k$ and $\sigma'_i = \tau_{m+1}$ for $n-k+1 \leq i \leq n$, is a family of $\mathcal{A}^k_{n,m}$-stable curves. The morphism χ is induced by this family and is a closed embedding. The lemma now follows from the projection formula and from the following list of equalities

$$\chi^*\psi_\sigma = -\sum_{i=1}^n \pi_*(\sigma'_i)^2 = -\sum_{i=1}^{n-k} \pi_*(\sigma_i^2) - k\pi_*(\tau_{m+1}^2) = \psi_\sigma + k\psi_{\tau_{m+1}},$$

$$\chi^*\psi_\tau = -\sum_{j=1}^m \pi_*(\tau_j^2) = -\sum_{j=1}^{m+1} \pi_*(\tau_j^2) + \pi_*(\tau_{m+1}^2) = \psi_\tau - \psi_{\tau_{m+1}},$$

$$\chi^*\Delta_s = \sum_{1\leq i<j\leq n} \pi_*(\sigma'_i \cdot \sigma'_j) = \sum_{1\leq i<j\leq n-k} \pi_*(\sigma_i \cdot \sigma_j) + \binom{k}{2}\pi_*(\tau_{m+1}^2) = \Delta_s - \binom{k}{2}\psi_{\tau_{m+1}},$$

$$\chi^*\Delta_n = \pi_*(\mathrm{Sing}\,(\pi)) = \Delta_n.$$

In the third equality, we use the fact that τ_{m+1} does not intersect any of the sections $\{\sigma_i\}_{i=1}^{n-k}$.

□

3. Positivity on 1-parameter families

Throughout this section, we suppose that $(\pi\colon \mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ is a *generically smooth* family of $\mathcal{A} := \mathcal{A}^k_{n,m}$-stable pointed curves over a smooth curve B. In particular, up to k of the sections $\{\sigma_j\}_{j=1}^n$ can collide, while the sections $\{\tau_j\}_{j=1}^m$ are each disjoint from each other and the $\{\sigma_j\}_{j=1}^n$. Note that we allow the case $m = 0$, i.e., $\mathcal{A} := \mathcal{A}^k_n$. Under these conditions, we will explain how to derive inequalities among the intersection numbers:

$$\psi_\sigma.B := -\sum_{i=1}^n \sigma_i^2\,,$$

$$\psi_\tau.B := -\sum_{i=1}^m \tau_i^2\,,$$

$$\Delta_s.B := \sum_{1\leq j<k\leq n} \sigma_i.\sigma_j\,,$$

$$\Delta_n.B := \deg\left(\pi_*(\mathrm{Sing}\,(\pi))\right).$$

To start, we let $\mathcal{C}_0$ be a minimal resolution of singularities of $\mathcal{C}$. By successively blowing-down (-1)-curves contained in the fibers, we obtain a sequence of birational morphisms over B

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \ldots \to \mathcal{C}_N,$$

such that $\mathcal{C}_N$ is a smooth $\mathbb{P}^1$-bundle over B. If we let $\{\sigma^i_j\}_{j=1}^n$ and $\{\tau^i_j\}_{j=1}^m$ denote the sections of $\pi_i\colon \mathcal{C}_i \to B$ obtained as the images of $\{\sigma_j\}_{j=1}^n$ and $\{\tau_j\}_{j=1}^m$ on $\mathcal{C}_i$, then each family $(\pi_i\colon \mathcal{C}_i \to B, \{\sigma^i_j\}_{j=1}^n, \{\tau^i_j\}_{j=1}^m)$ satisfies the following conditions:

1. The geometric fibers of π are reduced connected nodal curves of arithmetic genus zero.
2. The generic fiber of π is smooth.
3. The sections $\{\sigma_j\}_{j=1}^n$ and $\{\tau_j\}_{j=1}^m$ lie in the smooth locus of π.
4. If the exceptional divisor of $\mathcal{C}_i \to \mathcal{C}_{i+1}$ meets r_1 sections of weight $1/k$ and r_2 sections of weight 1, then $r_1/k + r_2 > 1$.

With notation as above, we define functions $F_\Delta, F_\sigma, F_\tau, F_{\sigma,\tau}\colon [0,N] \to \mathbb{Q}$ by the formulae

$$\begin{aligned}
F_\Delta(i) &:= \deg\left(\operatorname{Sing}\left(\pi_i\right)\right),\\
F_\sigma(i) &:= -\frac{1}{n-1}\sum_{1\le j<k\le n}(\sigma_j^i - \sigma_k^i)^2,\\
F_\tau(i) &:= -\frac{1}{m-1}\sum_{1\le j<k\le m}(\tau_j^i - \tau_k^i)^2,\\
F_{\sigma,\tau}(i) &:= -\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m}(\sigma_j^i - \tau_k^i)^2.
\end{aligned}$$

The following lemma is the key ingredient in all our subsequent arguments.

Lemma 3.1 (Key Lemma)**.**

(a) *The numerical functions $F_\Delta(i), F_\sigma(i), F_\tau(i), F_{\sigma,\tau}(i)$ satisfy*

$$\begin{aligned}
F_\Delta(0) &= \Delta_n.B, & F_\Delta(N) &= 0,\\
F_\sigma(0) &= \psi_\sigma.B + \frac{2}{n-1}\Delta_s.B, & F_\sigma(N) &= 0,\\
F_\tau(0) &= \psi_\tau.B, & F_\tau(N) &= 0,\\
F_{\sigma,\tau}(0) &= \frac{1}{n}\psi_\sigma.B + \frac{1}{m}\psi_\tau.B, & F_{\sigma,\tau}(N) &= 0.
\end{aligned}$$

(b) *Suppose that the exceptional divisor of the birational map $\mathcal{C}_i \to \mathcal{C}_{i+1}$ meets r_1 of the sections $\{\sigma_j\}_{j=1}^n$ and r_2 of the sections $\{\tau_j\}_{j=1}^m$. Then we have*

$$\begin{aligned}
F_\Delta(i) - F_\Delta(i+1) &= 1,\\
F_\sigma(i) - F_\sigma(i+1) &= \frac{r_1(n-r_1)}{n-1},\\
F_\tau(i) - F_\tau(i+1) &= \frac{r_2(m-r_2)}{m-1},\\
F_{\sigma,\tau}(i) - F_{\sigma,\tau}(i+1) &= \frac{r_1(m-r_2)+r_2(n-r_1)}{nm}.
\end{aligned}$$

Proof. We first prove part (a). To see that

$$F_\sigma(0) = F_\tau(0) = F_{\sigma,\tau}(0) = 0,$$

simply observe that on a $\mathbb{P}^1$-bundle, the difference of two sections is numerically equivalent to a multiple of the fiber class, and the fiber class has self-intersection zero. Also, $F_\Delta(0) = 0$ is clear, since a $\mathbb{P}^1$-bundle has no singular fibers.

To see that $F_\Delta(0)=\Delta_n.B$, it is sufficient to observe that the minimal desingularization $\mathcal{C}^0\to\mathcal{C}$ has the effect of replacing nodes where the total space has an A_k-singularity (thus contributing k to $\deg(\mathrm{Sing}(\pi))$) by k nodes with smooth total space, each of which contributes 1 to $\deg(\mathrm{Sing}(\pi_0))$.

To see that $F_\sigma(0)=\psi_\sigma+\frac{2}{n-1}\Delta_s$, note that since the sections $\{\sigma_j\}_{j=1}^n,\{\tau_j\}_{j=1}^m$ are disjoint from the singular locus of π, their self-intersections are not changed by taking a minimal desingularization. Thus,

$$\begin{aligned}F_\sigma(0)&=-\frac{1}{n-1}\sum_{1\le i<j\le n}(\sigma_j^0-\sigma_k^0)^2=-\frac{1}{n-1}\sum_{1\le i<j\le n}(\sigma_j-\sigma_k)^2\\&=-\sum_{i=1}^n\sigma_i^2+\frac{2}{n-1}\sum_{1\le i<j\le n}\sigma_i.\sigma_j=(\psi_\sigma.B)+\frac{2}{n-1}(\Delta_s.B).\end{aligned}$$

The computations for $F_\sigma(0)$ and $F_{\sigma,\tau}(0)$ are similar, bearing in mind the fact that all intersections $\tau_i.\sigma_j$ and $\tau_i.\tau_j$ $(i\neq j)$ are zero. We have

$$\begin{aligned}F_\tau(0)&=-\frac{1}{n-1}\sum_{1\le i<j\le n}(\tau_j^0-\tau_k^0)^2=-\frac{1}{n-1}\sum_{1\le i<j\le n}(\tau_j-\tau_k)^2\\&=-\sum_{i=1}^n\tau_i^2=(\psi_\tau.B)\\F_{\sigma,\tau}(0)&=-\frac{1}{nm}\sum_{j=1}^n\sum_{k=1}^m(\sigma_j^0-\tau_k^0)^2=-\frac{1}{nm}\sum_{j=1}^n\sum_{k=1}^m(\sigma_j-\tau_k)^2\\&=-\frac{1}{nm}\sum_{i=1}^n m\sigma_i^2-\frac{1}{nm}\sum_{i=1}^m n\tau_i^2=\frac{1}{n}(\psi_\sigma.B)+\frac{1}{m}(\psi_\tau.B)\end{aligned}$$

It remains to prove part (b) of the lemma. Let $\phi\colon\mathcal{C}_i\to\mathcal{C}_{i+1}$ denote the i^{th} blow-down morphism. Since ϕ contracts a single (-1)-curve, it is clear that $\mathcal{C}_{i+1}$ has one less node in the union of its singular fibers than $\mathcal{C}_i$, i.e. $F_\Delta(i)-F_\Delta(i+1)=1$.

We may assume without loss of generality that the exceptional divisor E of ϕ meets $\{\sigma_j\}_{j=1}^{r_1}$ and $\{\tau_j\}_{j=1}^{r_2}$, and is disjoint from $\{\sigma_j\}_{j=r_1+1}^n$ and $\{\tau_j\}_{j=r_2+1}^m$. Then we have

$$\begin{aligned}\sigma_j^i-\sigma_k^i&=\phi^*(\sigma_j^{i+1}-\sigma_k^{i+1}), &&\text{if } j,k\le r_1,\ \text{or } j,k\ge r_1+1,\\\sigma_j^i-\sigma_k^i&=\phi^*(\sigma_j^{i+1}-\sigma_k^{i+1})+E, &&\text{otherwise.}\end{aligned}$$

It follows that

$$\begin{aligned}(\sigma_j^i-\sigma_k^i)^2&=(\sigma_j^{i+1}-\sigma_k^{i+1})^2, &&\text{if } j,k\le r_1,\ \text{or } j,k\ge r_1+1,\\(\sigma_j^i-\sigma_k^i)^2&=(\sigma_j^{i+1}-\sigma_k^{i+1})^2-1, &&\text{otherwise.}\end{aligned}$$

Thus,

$$F_\sigma(i)-F_\sigma(i+1)=\frac{1}{n-1}\sum_{1\le j<k\le n}\left((\sigma_j^{i+1}-\sigma_k^{i+1})^2-(\sigma_j^i-\sigma_k^i)^2\right)=\frac{r_1(n-r_1)}{n-1}.$$

The remaining statements in part (b) are proved in the same fashion. □

How can we use this lemma to prove positivity of certain linear combinations $\alpha(\psi_\sigma.B)+\beta(\psi_\tau.B)+\gamma(\Delta_s.B)+\eta(\Delta_n.B)$? Well, part (a) implies that we can find coefficients $a_\sigma, a_\tau, a_{\sigma,\tau}, a_\delta \in \mathbb{Q}$, such that

$$a_\sigma F_\sigma(0)+a_\tau F_\tau(0)+a_{\sigma,\tau}F_{\sigma,\tau}(0)+a_\delta F_\Delta(0) = \alpha(\psi_\sigma.B)+\beta(\psi_\tau.B)+\gamma(\Delta_s.B)+\eta(\Delta_n.B)$$

On the other hand, we have

$$a_\sigma F_\sigma(N)+a_\tau F_\tau(N)+a_{\sigma,\tau}F_{\sigma,\tau}(N)+a_\delta F_\Delta(N)=0.$$

Thus, if we can show that $a_\sigma F_\sigma(i)+a_\tau F_\tau(i)+a_{\sigma,\tau}F_{\sigma,\tau}(i)+a_\delta F_\Delta(i)$ is a decreasing function of i, we may conclude that

$$\alpha(\psi_\sigma.B)+\beta(\psi_\tau.B)+\gamma(\Delta_s.B)+\eta(\Delta_n.B)\geq 0,$$

with equality holding only if $N=0$.

Thus, the key point is to determine what conditions on the coefficients $a_\sigma, a_\tau, a_{\sigma,\tau}, a_\delta$ make the function $a_\sigma F_\sigma(i)+a_\tau F_\tau(i)+a_{\sigma,\tau}F_{\sigma,\tau}(i)+a_\delta F_\Delta(i)$ decreasing.

Since $\psi_\tau - \Delta_n$ is functorial with respect to the boundary stratification (see Lemma 2.1), whereas ψ_τ and Δ_n individually are not, we are mainly interested in the subspace of divisors for which $\eta=-\beta$. After scaling, we can restrict our attention to divisors of the form

$$D_k(a,b)=a\psi_\sigma+b\Delta_s+\psi_\tau-\Delta_n.$$

Thus, we are left with the problem of describing a two-dimensional polytope in the (a,b)-space for which the divisor $D_k(a,b)$ is positive, and this is what we do beginning with the following proposition and finishing in Section 4.1.

In order to make the numerics tractable, we divide the analysis into several cases. Note that the one case in which we get a sharp inequality is case (2.), where we have one section of weight 1 and $k+1$ sections of weight $1/k$ on a $\mathbb{P}^1$-bundle. This corresponds to the fact that the moving components of any curve contracted by the reduction morphism $\overline{M}_{0,\mathcal{A}_n^k} \to \overline{M}_{0,\mathcal{A}_n^{k+1}}$ take precisely this form. Eventually, we will see that this inequality determines the precise value of α at which $\overline{M}_{0,n}(\alpha)$ transforms from $\overline{M}_{0,\mathcal{A}_n^k}$ into $\overline{M}_{0,\mathcal{A}_n^{k+1}}$.

Proposition 3.2. *Let $(\mathcal{C}\to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ be an arbitrary complete 1-parameter family of $\mathcal{A}$-stable curves with smooth general fiber. Then the intersection numbers $\psi_\sigma.B, \psi_\tau.B, \Delta_s.B, \Delta_n.B$ satisfy the following inequalities.*

1. *If $m=0$, then for $\alpha > \frac{n-1}{(n-k-1)(k+1)}$ we have*

$$\alpha(\psi_\sigma.B)+\frac{2\alpha}{n-1}(\Delta_s.B)-(\Delta_n.B)>0.$$

2. *If $m=1$, then for $\alpha>\frac{n-1}{n(k+1)}$ we have*

$$\left(\alpha+\frac{1}{n}\right)(\psi_\sigma.B)+\frac{2\alpha}{n-1}(\Delta_s.B)+(\psi_\tau.B)-(\Delta_n.B)\geq 0.$$

   Furthermore, equality holds if and only if $n=k+1$.

3. *If $m\geq 2$, $2\leq n\leq k$, then for $\alpha>0$ and $\beta>0$ we have*

$$\left(\alpha+\frac{\beta}{n}\right)(\psi_\sigma.B)+\frac{2\alpha}{n-1}(\Delta_s.B)+(\psi_\tau.B)-(\Delta_n.B)>0.$$

4. *If* $m \geq 2$, $n \geq k+1$, *then for* $\frac{(k+1)(n-k-1)}{n-1}\alpha + \frac{k+1}{n}\beta > 1$ *and* $\beta > 1$ *we have*

$$\left(\alpha + \frac{\beta}{n}\right)(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B) > 0.$$

Proof.

(1.) We have

$$\alpha F_\sigma(0) - F_\Delta(0) = \alpha(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) - (\Delta_n.B).$$

Thus, it suffices to show that $G(i) := \alpha F_\sigma(i) - F_\Delta(i)$ is a decreasing function of i. The exceptional divisor of $\mathcal{C}_i \to \mathcal{C}_{i+1}$ meets at least $k+1$ sections, so Lemma 3.1 (b) implies

$$\begin{aligned}\alpha F_\sigma(i) - F_\Delta(i) - (\alpha F_\sigma(i+1) - F_\Delta(i+1)) &= \frac{r_1(n-r_1)}{n-1}\alpha - 1 \\ &\geq \frac{(k+1)(n-k-1)}{n-1}\alpha - 1 > 0,\end{aligned}$$

when $\alpha > \frac{n-1}{(n-k-1)(k+1)}$.

(2.) We have

$$\alpha F_\sigma(0) + F_{\sigma,\tau}(0) - F_\Delta(0) = (\alpha + \frac{1}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B).$$

Thus, it suffices to show that $G(i) := \alpha F_\sigma(i) + F_{\sigma,\tau}(i) - F_\Delta(i)$ is a decreasing function of i. If $n = k+1$ and $m = 1$, then the hypothesis of $\mathcal{A}$-stability implies that the original family $\mathcal{C} \to B$ has no singular fibers, i.e. $\mathcal{C} = \mathcal{C}_0$ is already a $\mathbb{P}^1$-bundle. In this case,

$$G(0) = G(N) = 0.$$

Thus, we may assume that $n \geq k+2$, so the exceptional divisor of $\mathcal{C}_{i+1} \to \mathcal{C}_i$ meets $n-1 \geq r_1 \geq k+1$ sections of weight $1/k$. In addition, since there is only one section of weight 1, and we may always choose to blow-down a (-1)-curve disjoint from this section, we may assume that the exceptional divisor $\mathcal{C}_{i+1} \to \mathcal{C}_i$ meets no sections of weight 1. Thus,

$$\begin{aligned}G(i) - G(i+1) &= \frac{r_1(n-r_1)}{n-1}\alpha + \frac{r_1}{n} - 1 \\ &\geq \min\left\{\frac{(k+1)(n-k-1)}{n-1}\alpha + \frac{k+1}{n} - 1, \alpha + \frac{n-1}{n} - 1\right\} > 0,\end{aligned}$$

when $\alpha > \frac{n-1}{n(k+1)}$.

(3.) We have

$$\alpha F_\sigma(0) + \beta F_{\sigma,\tau}(0) + \frac{m-\beta}{m}F_\tau(0) - F_\Delta(0) = \left(\alpha + \frac{\beta}{n}\right)(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B).$$

Thus, it suffices to show that the function $G(i) := \alpha F_\sigma(i) + \beta F_{\sigma,\tau} + \frac{m-\beta}{m}F_\tau(i) - F_\Delta(i)$ is a decreasing function of i. Suppose that the exceptional divisor of $\mathcal{C}_i \to \mathcal{C}_{i+1}$ meets r_1 sections of weight k and r_2 sections of weight 1. We have

$$G(i) - G(i+1) = \frac{r_1(n-r_1)}{n-1}\alpha + \frac{r_1(m-r_2) + r_2(n-r_1)}{mn}\beta + \frac{r_2(m-r_2)(m-\beta)}{m(m-1)} - 1.$$

Denote the right-hand side by $\mathbf{H}(r_1, r_2)$. Then $\mathbf{H}$ is a convex function in each variable r_1 and r_2 (but not necessarily in both) and is symmetric about the point $(n/2, m/2)$. The pair of integers (r_1, r_2) satisfies the inequalities

(i) $1 \leq r_1 \leq n$,
(ii) $1 \leq r_2 \leq m$,
(iii) $r_1/k + r_2 > 1$,
(iv) $(n - r_1)/k + (m - r_2) > 1$.

For $m \geq 3$, a pair (r_1, r_2) satisfying (i)-(iv) is easily seen to lie in the convex hull of the points $(0, 2), (1, 1), (n, 1)$ and $(n, m - 2), (n - 1, m - 1), (0, m - 1)$ (the second triple is the reflection of the first triple about $(n/2, m/2)$). For $m = 2$, a pair (r_1, r_2) satisfying (i)-(iv) is in the convex hull of $(1, 1)$ and $(n - 1, 1)$.

By convexity of $\mathbf{H}$ in each argument we have

$$\mathbf{H}(r_1, r_2) \geq \min\{\mathbf{H}(0, 2), \mathbf{H}(1, 1), \mathbf{H}(n, 1)\}$$

for $m \geq 3$, and

$$\mathbf{H}(r_1, r_2) \geq \mathbf{H}(1, 1)$$

for $m = 2$. To prove the statement of the proposition, we calculate

$$\begin{aligned}\mathbf{H}(1, 1) &= \alpha + \frac{m-2}{mn}\beta > 0,\\ \mathbf{H}(0, 2) &= \frac{2\beta}{m(m-1)} + \frac{m-3}{m-1} > 0 \ \text{ when } m \geq 3,\\ \mathbf{H}(n, 1) &= \frac{m-2}{m}\beta > 0 \ \text{ when } m \geq 3.\end{aligned}$$

(4.) As in Part (3.), we have

$$\alpha F_\sigma(0) + \beta F_{\sigma,\tau}(0) + \frac{m-\beta}{m} F_\tau(0) - F_\Delta(0) = (\alpha + \frac{\beta}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B).$$

To show that the function $G(i) := \alpha F_\sigma(i) + \beta F_{\sigma,\tau} + \frac{m-\beta}{m} F_\tau(i) - F_\Delta(i)$ is a decreasing function of i, we recall that if the exceptional divisor of $\mathcal{C}_i \to \mathcal{C}_{i+1}$ meets r_1 sections of weight $1/k$ and r_2 sections of weight 1, then

$$G(i) - G(i + 1) = \mathbf{H}(r_1, r_2),$$

where $\mathbf{H}(r_1, r_2)$ is as defined in the proof of part (3.). Integers (r_1, r_2) satisfy the same inequalities (i)-(iv) as in part (3.). By the convexity of $\mathbf{H}$ in each factor we have

$$\begin{aligned}\mathbf{H}(r_1, r_2) \geq \min\big\{&\mathbf{H}(0, m), \mathbf{H}(0, 2), \mathbf{H}(1, 1), \mathbf{H}(k + 1, 0), \mathbf{H}(n, 0), \mathbf{H}(n, m - 2),\\ &\mathbf{H}(n - 1, m - 1), \mathbf{H}(n - k - 1, m)\big\}.\end{aligned}$$

The statement of the proposition now follows from the following computations

$$\begin{aligned}\mathbf{H}(0, m) = \mathbf{H}(n, 0) &= \beta - 1\\ \mathbf{H}(0, 2) = \mathbf{H}(n, m - 2) &= \frac{2\beta}{m(m-1)} + \frac{m-3}{m-1}\\ \mathbf{H}(1, 1) = \mathbf{H}(n - 1, m - 1) &= \alpha + \frac{m-2}{mn}\beta\\ \mathbf{H}(k + 1, 0) = \mathbf{H}(n - k - 1, m) &= \frac{(k+1)(n-k-1)}{n-1}\alpha + \frac{k+1}{n}\beta - 1.\end{aligned}$$

□

Each part of Proposition 3.2 concerns the positivity of a certain ray of divisors of the form $D_k(a,b) := a\psi_\sigma + b\Delta_s + \psi_\tau - \Delta_n$. In order to plot out the position of these rays in (a,b)-space, it is useful to introduce the following notation.

Definition 3.3. Let $L_n(b,x)$ be the linear function defined by the formula

$$L_n(b,x) := \frac{n-1}{2}b + \frac{x}{n},$$

With this notation, we may restate Proposition 3.2 as follows.

Corollary 3.4. *Suppose $\mathcal{A} := \mathcal{A}^k_{n,m}$ is a weight vector with $k \geq 2$. If $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ is any complete 1-parameter family of $\mathcal{A}$-stable curves with smooth general fiber, we have*

$$D_k(a,b).B = a(\psi_\sigma.B) + b(\Delta_s.B) + \psi_\tau.B - \Delta_n.B > 0,$$

for all $(a,b) \in \mathbb{Q}^2$ that lie in the following open rays:

1. $\left\{a = L_n(b,0) : b > \dfrac{2}{(n-k-1)(k+1)}\right\}$, *when* $m = 0$;
2. $\left\{a = L_n(b,1) : b > \dfrac{2}{n(k+1)}\right\}$, *when* $m = 1$, $n \geq k+2$;
3. $\left\{a = L_n(b,1) : b > \dfrac{2}{n(k+1)}\right\}$, *when* $m \geq 2$, $2 \leq n \leq k$;
4. $\left\{a = L_n(b,x) : b > \dfrac{2}{(k+1)(n-k-1)} - \dfrac{2x}{n(n-k-1)}\right\}$, *for all* $x > 1$, *when* $m \geq 2$, $n \geq k+1$.

In addition, when $m = 1$ and $n = k+1$, we have $D_k(a,b).B = 0$ for all $(a,b) \in \left\{a = L_n(b,1) : b \geq \frac{2}{n(k+1)}\right\}$.

4. Ample divisors on $\overline{M}_{0,\mathcal{A}}$

In this section, we will use the positivity results of Section 3 to deduce the existence of certain ample divisors on $\overline{M}_{0,\mathcal{A}}$. In Section 4.1, we describe a polytope in $\mathbb{Q}^2$ on which our fundamental divisor class

$$D_k(a,b) := a\psi_\sigma + b\Delta_s + \psi_\tau - \Delta_n,$$

is nef for all weight vectors $\mathcal{A}$. The intersection of this polytope with the ray $b = 2a-1$ will yield the interval $\left[\frac{k+2}{2k+2}, \frac{k+1}{2k}\right]$, so we conclude that $D_k(c)$ is nef for all $c \in \left[\frac{k+2}{2k+2}, \frac{k+1}{2k}\right]$. By Remark 2.6, this implies that for any symmetric weight vector $\mathcal{A} := \mathcal{A}^k_n$, we have

$$K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta \text{ is nef on } \overline{M}_{0,\mathcal{A}} \text{ for } \alpha \in \mathbb{Q} \cap \left[\frac{2}{k+2}, \frac{2}{k+1}\right].$$

In Section 4.2, we strengthen this statement to show that $D_k(c)$ is ample on $\overline{M}_{0,\mathcal{A}}$ if $c \in \left(\frac{k+2}{2k+2}, \frac{k+1}{2k}\right]$. In particular, for symmetric weight vectors $\mathcal{A} := \mathcal{A}^k_n$, we have

$$K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta \text{ is ample on } \overline{M}_{0,\mathcal{A}} \text{ for } \alpha \in \mathbb{Q} \cap \left(\frac{2}{k+1}, \frac{2}{k+1}\right].$$

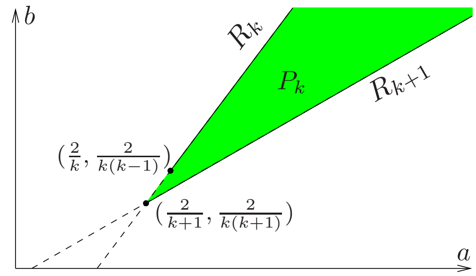

FIGURE 1. The polytope P_k

To achieve this strengthening, we show that $D_k(c)$ remains nef it we perturb the divisor by a small linear combination of boundary divisors of $\overline{M}_{0,\mathcal{A}}$. Since the boundary divisors of $\overline{M}_{0,\mathcal{A}}$ generate its Picard group, this implies that $D_k(c)$ lies on the interior of the nef cone of $\overline{M}_{0,\mathcal{A}}$. On smooth proper schemes, Kleiman's criterion implies that any divisor which lies on the interior of the nef cone is ample. Since Hassett has shown that $\overline{M}_{g,\mathcal{A}}$ is a projective scheme using Kollár's semipositivity techiques [4], we could stop here. In order to make our argument independent of Kollár's results, however, we explain in Section 4.3 how to apply Kleiman's criterion without assuming *a priori* that $\overline{M}_{0,\mathcal{A}}$ is a scheme. Since it is often easier to construct moduli spaces as algebraic spaces rather than projective schemes, we wish to emphasize the point that it is actually possible to prove projectivity using our explicit intersection theory.

4.1. **Nef divisors on $\overline{M}_{0,\mathcal{A}}$.** In this section, we prove that $D_k(a,b)$ is nef for all (a,b) contained in the following polytope.

Definition 4.1. Let P_k be the convex hull of the rays R_k and R_{k+1}, where

$$R_k := \left\{ a = L_k(b,1) = \frac{k-1}{2}b + \frac{1}{k} \ : \ b \geq \frac{2}{k(k-1)} \right\}.$$

The proof will proceed by induction on k. The induction hypothesis will show that the ray of divisors R_k is nef, and to show that the ray R_{k+1} is nef, we will need to see that, in each of the four cases listed in Corollary 3.4, R_{k+1} lies in the convex hull of R_k and the positive divisors given by the corollary. We record this statement of elementary convex geometry in the following lemma, whose proof is left to the reader.

Lemma 4.2. *Let $\mathcal{A} := \mathcal{A}^k_{n,m}$ be a weight vector with $k \geq 2$.*

1. *If $m = 0$, then R_{k+1} lies in the convex hull of R_k and*
$$\left\{ a = L_n(b,0) : b > \frac{2}{(n-k-1)(k+1)} \right\}.$$
2. *If $m = 1$,then R_{k+1} lies in the convex hull of R_k and*
$$\left\{ a = L_n(b,1) : b > \frac{2}{n(k+1)} \right\}.$$
3. *If $m \geq 2, 2 \leq n \leq k$, then R_{k+1} lies in the convex hull of R_k and*
$$\left\{ a = L_n(b,1) : b > \frac{2}{n(k+1)} \right\}.$$

4. *If $m \geq 2, n \geq k+1$, then R_{k+1} lies in the convex hull of R_k and the union of the rays*
$$\left\{ a = L_n(b,x) : b > \frac{2}{(k+1)(n-k-1)} - \frac{2}{n(n-k-1)} x \right\},$$
where $x \in (1,\infty) \cap \mathbb{Q}$.

In addition, the containment is strict in all the cases except for $(m,n) = (1,k+1)$.

We are ready to prove our main theorem.

Theorem 4.3 (Main Theorem)**.** *Let $\mathcal{A} := \mathcal{A}^k_{n,m}$ be an arbitrary weight vector.*
- (a) *For any $(a,b) \in P_k - R_{k+1}$, $D_k(a,b)$ has positive intersection with every curve in $\overline{M}_{0,\mathcal{A}}$.*
- (b) *For any $(a,b) \in R_{k+1}$, $D_k(a,b)$ is nef on $\overline{M}_{0,\mathcal{A}}$. Furthermore, it has degree zero precisely on those curves contracted by the reduction morphism $\overline{M}_{0,\mathcal{A}^k_{n,m}} \to \overline{M}_{0,\mathcal{A}^{k+1}_{n,m}}$.*

Proof. We proceed by induction on k. When $k=1$, we must show that $c\psi_\sigma + \psi_\tau - \Delta$ has positive intersection on all curves $\overline{M}_{0,n+m}$ for all $c \geq 1$. By Lemma 2.5,
$$c\psi_\sigma + \psi_\tau - \Delta = K_{\overline{M}_{0.n+m}} + \Delta + (c-1)\psi_\sigma.$$
Since $K_{\overline{M}_{0,n+m}} + \Delta$ is ample on $\overline{M}_{0,n+m}$ [8, Lemma 3.6] and ψ_σ is nef, the desired statement holds.

From now on, we assume $k \geq 2$. To prove part (a), it suffices to show that $D_k(a,b)$ has positive intersection with every curve in $\overline{M}_{0,\mathcal{A}}$ for $(a,b) \in R_k$ and is nef for $(a,b) \in R_{k+1}$. First, suppose that $(a,b) \in R_k$, and consider the reduction morphism
$$\phi\colon \overline{M}_{0,\mathcal{A}^{k-1}_{n,m}} \to \overline{M}_{0,\mathcal{A}^k_{n,m}}.$$
Using Lemma 2.8, one easily checks that if $(a,b) \in R_k$ then
$$\phi^*\left(D_k(a,b)\right) = D_{k-1}(a,b).$$
Part (b) of the induction hypothesis implies that $D_{k-1}(a,b)$ is nef and has degree zero only on curves contracted by ϕ. It follows that $D_k(a,b)$ has positive degree on all curves in $\overline{M}_{0,\mathcal{A}}$ as desired.

To complete the proof of part (a), it remains to show that $D_k(a,b)$ is nef for $(a,b) \in R_{k+1}$. By Lemma 2.2, it suffices to check that $D_k(a,b)$ has non-negative intersection with any generically smooth complete 1-parameter family of $\mathcal{A}$-stable curves $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$. Lemma 4.2 shows that for each choice of (n,m), the ray R_{k+1} is in the convex hull of R_k and a divisor whose non-negativity is guaranteed by Corollary 3.4. Thus, $D_k(a,b).B \geq 0$ for all $(a,b) \in R_{k+1}$ as desired. Furthermore, note that the last statement in Lemma 4.2 implies that $D_k(a,b).B = 0$ only if $(n,m) = (k+1,1)$.

For part (b), we have already seen that $D_k(a,b)$ is nef for all $(a,b) \in R_{k+1}$. Furthermore, the last sentence in the previous paragraph implies that if $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ is any 1-parameter family of $\mathcal{A}$-stable curves on which $D_k(a,b)$ has degree zero, then every moving component of the generic fiber of $\mathcal{C} \to B$ must have $k+1$ marked points of weight $1/k$ and must be attached to the rest of the fiber in a single point. Equivalently, B is contained in the fiber of the reduction morphism $\overline{M}_{0,\mathcal{A}^k_{n,m}} \to \overline{M}_{0,\mathcal{A}^{k+1}_{n,m}}$. $\square$

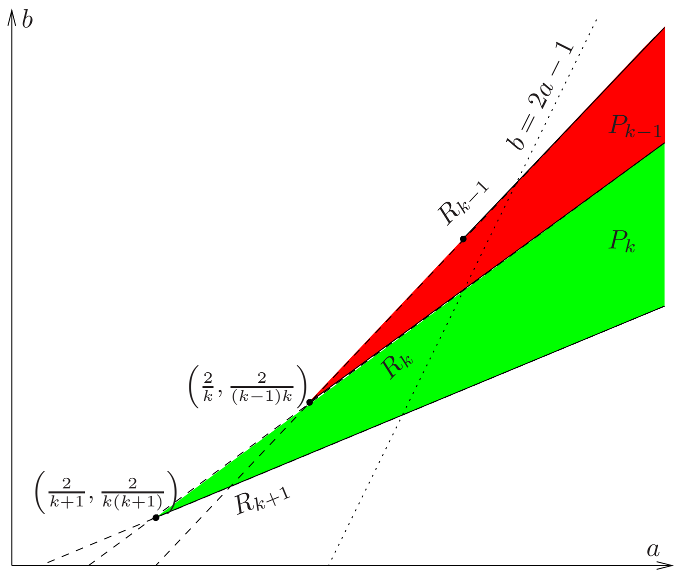

FIGURE 2. Running the induction

Let us note two corollaries of Theorem 4.3. First, we obtain a positivity result for the fundamental divisors $D_k(c) = c\psi_\sigma + (2c-1)\Delta_s + \psi_\tau - \Delta_n$.

Corollary 4.4.
(a) *For any weight vector $\mathcal{A}$, $D_k(c)$ has positive intersection with any 1-parameter family of $\mathcal{A}$-stable curves if $c \in \left(\frac{k+2}{2k+2}, \frac{k+1}{2k}\right]$.*
(b) *For symmetric weight vectors $\mathcal{A}$, $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ has positive intersection with any 1-parameter family of $\mathcal{A}$-stable curves if $\alpha \in \left(\frac{2}{k+2}, \frac{2}{k+1}\right]$.*

Proof. For part (a), note that the ray $b = 2a - 1$ intersects the boundary of P_k at the points $\left(\frac{1}{k}, \frac{k+1}{2k}\right)$ and $\left(\frac{1}{k+1}, \frac{k+2}{2k+2}\right)$. Part (b) follows from part (a) by Remark 2.6. □

For a second consequence, consider the natural line bundle

$$\mathcal{L} := \omega_\pi \left(\frac{1}{k}(\sigma_1 + \cdots + \sigma_n) + \tau_1 + \cdots + \tau_m \right)$$

on the universal curve $(\pi\colon \mathcal{C} \to \overline{M}_{0,\mathcal{A}}, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$. By the definition of $\mathcal{A}$-stability, the line bundle $\mathcal{L}$ is π-ample. In analogy with the case of $\overline{M}_g$, we can ask what positivity properties does $\kappa := \pi_*(c_1(\mathcal{L})^2)$ enjoy? When $k = 1$, $\kappa = \psi - \Delta$ and is therefore ample by [8, Lemma 3.6]. For general weight vectors, we recover the following unpublished result of Keel.

Corollary 4.5. *For any weight vector $\mathcal{A}$, the divisor κ is nef on $\overline{M}_{0,\mathcal{A}}$.*

Proof. Using the standard formulae

$$\sigma_j \cdot \omega_\pi = -\sigma_j^2,$$
$$\tau_j \cdot \omega_\pi = -\tau_j^2,$$
$$\pi_*(\omega_\pi^2) = -\Delta_n,$$

we compute

$$\begin{aligned}
\pi_*(c_1(\mathcal{L})^2) &= \pi_* \left(\omega_\pi^2 + \frac{2}{k}\omega_\pi(\sum_{j=1}^n \sigma_j) + \frac{1}{k^2}(\sum_{j=1}^n \sigma_j)^2 + 2\omega_\pi(\sum_{j=1}^m \tau_j) + (\sum_{j=1}^m \tau_j)^2 \right) \\
&= \pi_* \left(\omega_\pi^2 - \left(\frac{2}{k} - \frac{1}{k^2}\right) \sum_{j=1}^n \sigma_j^2 + \frac{2}{k^2} \sum_{1\le i<j\le n} \sigma_i \cdot \sigma_j - \sum_{j=1}^m \tau_j^2 \right) \\
&= -\Delta_n + \frac{2k-1}{k^2}\psi_\sigma + \frac{2}{k^2}\Delta_s + \psi_\tau \\
&= D_k\left(\frac{2k-1}{k^2}, \frac{2}{k^2}\right).
\end{aligned}$$

We observe that $\frac{2}{k^2} > \frac{2}{k(k+1)}$ and $L_k(\frac{2}{k^2}) = \frac{2k-1}{k^2}$. Hence by Theorem 4.3 (a), the divisor κ is nef. Moreover, by Part (b) of Theorem 4.3, κ has positive degree on any 1-parameter family of $\mathcal{A}$-stable curves. It will follow from Proposition 4.10 that it is actually ample. □

4.2. **Perturbations of the fundamental divisor class** $D_k(c)$**.** Corollary 4.4 nearly implies that $D_k(c)$ is ample on $\overline{M}_{0,\mathcal{A}}$, but of course one cannot always check ampleness simply by testing positivity on curves. In order to prove that $D_k(c)$ is ample, we will show that the divisor remains nef when perturbed by a small linear combination of boundary divisors. In fact, it is enough to consider perturbations by $S_n \times S_m$-equivariant divisors, as we explain in Lemma 4.6. The same methods used in the proof of Theorem 4.3 are easily adapted to prove this stronger statement. Thoughout this section, we write $\Delta_{i,j} \subset \overline{M}_{0,\mathcal{A}}$ to denote the sum of all irreducible components of the boundary of the form Δ_{S_1,S_2} where S_1 is a subset of i weight $1/k$ sections and j weight 1 sections of the universal curve, so that $\Delta_{i,j}$ is $S_n \times S_m$-equivariant.

Lemma 4.6. *Given a weight vector $\mathcal{A}$ and $c_0 \in (\frac{k+2}{2k+2}, \frac{k+1}{2k})$, suppose there exists $\epsilon = \epsilon(k,n,m,c_0) > 0$ such that for all $\epsilon_{i,j} \in \mathbb{Q} \cap [-\epsilon, \epsilon]$ and all $c \in \mathbb{Q} \cap [c_0 - \epsilon, c_0 + \epsilon]$, the divisor*

$$D_k(c) + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j} = D_k(c_0) + (c - c_0)(\psi_\sigma + 2\Delta_s) + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}$$

has positive degree on any complete 1-parameter family of $\mathcal{A}$-stable curves. Then $D_k(c_0)$ is ample on $\overline{M}_{0,\mathcal{A}}$.

Proof. Let $\pi\colon \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{A}}/S_n \times S_m$ be the quotient morphism for the natural action of $S_n \times S_m$ on $\overline{M}_{0,\mathcal{A}}$. Since $D_k(c_0)$ is $S_n \times S_m$-equivariant, we have

$$D_k(c_0) = \pi^* D_k'(c_0)$$

for some divisor class $D_k'(c_0)$ on $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$, and it suffices to prove that $D_k'(c_0)$ is ample. Since the boundary divisors of $\overline{M}_{0,\mathcal{A}}$ generate $\mathrm{Pic}_{\mathbb{Q}}(\overline{M}_{0,\mathcal{A}})$,

$\operatorname{Pic}_{\mathbb{Q}}(\overline{M}_{0,\mathcal{A}}/S_n \times S_m)$ is generated by the images of the equivariant boundary divisors, i.e. by the images of Δ_s and the various divisors $\Delta_{i,j}$.

Now our assumption implies that $D'_k(c_0)$ lies on the interior of the nef cone of $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$. Since $\overline{M}_{0,\mathcal{A}}$ is smooth projective, $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$ is projective with $\mathbb{Q}$-factorial singularities, and we may apply Kleiman's criterion to conclude that $D'_k(c_0)$ is ample.

□

In order to apply Lemma 4.6, we must check that the statements of Proposition 3.2 remain valid when we replace $D_k(c)$ by a small perturbation $D_k(c) + \sum_{i,j} \epsilon_{i,j} \Delta_{i,j}$.

Proposition 4.7. *For any weight vector $\mathcal{A}$ and rational number $\delta > 0$, there exists $\epsilon = \epsilon(\delta, k, n, m)$ such that, for any generically smooth 1-parameter family of $\mathcal{A}$-stable curves $(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$ and any $\epsilon_i \in \mathbb{Q} \cap [-\epsilon, \epsilon]$, the following inequalities are satisfied.*

1. *If $m = 0$, then for $\alpha > \frac{n-1}{(n-k-1)(k+1)} + \delta$ we have*

$$\alpha(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0.$$

2. *If $m = 1$ and $n \geq k+2$, then for $\alpha > \frac{n-1}{n(k+1)} + \delta$ we have*

$$(\alpha + \frac{1}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0.$$

3. *If $m \geq 2$, $2 \leq n \leq k$, then for $\alpha > \delta$ and $\beta > \delta$ we have*

$$(\alpha + \frac{\beta}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0.$$

4. *If $m \geq 2$, $n \geq k+1$, then for $\frac{(k+1)(n-k-1)}{n-1}\alpha + \frac{k+1}{n}\beta > 1 + \delta$ and $\beta > 1 + \delta$ we have*

$$(\alpha + \frac{\beta}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) + (\psi_\tau.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0.$$

Proof. The proof is essentially identical to the proof of Proposition 3.2, so we provide details only for the case $m = 0$. We consider a sequence of birational contractions $\mathcal{C}_0 \to \mathcal{C}_1 \to \ldots \to \mathcal{C}_N$, where $\mathcal{C}_0$ is a minimal desingularization of $\mathcal{C}$ and $\mathcal{C}_N$ is a $\mathbb{P}^1$-bundle over B. We then have have

$$\alpha(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) = \alpha F_\sigma(0) - F_\Delta(0) + \sum_{i,j} \epsilon_{i,j} F_{i,j}(0),$$

and it suffices to show that the function $G(r) := \alpha F_\sigma(r) - F_\Delta(r) + \sum_{i,j} \epsilon_{i,j} F_{i,j}(r)$ is a decreasing function of r. Here $F_{i,j} \colon [0, N] \to \mathbb{Z}$ is the function defined by setting $F_{i,j}(r)$ equal to the number of disconnecting nodes in fibers of the family $\mathcal{C}_r \to B$ which separate i sections of weight $1/k$ and j sections of weight 1 from the rest of the sections. Suppose that the exceptional divisor of $\mathcal{C}_r \to \mathcal{C}_{r+1}$ meets $r_1 \geq k+1$

sections weight $1/k$ so that we are eliminating a node corresponding to the boundary component $\Delta_{r_1,0}$. Then, using Lemma 3.1 (b), we have for $0 < \epsilon < \frac{(n-k-1)(k+1)}{n-1}\delta$:

$$G(r) - G(r+1) = \frac{r_1(n-r_1)}{n-1}\alpha - 1 + \epsilon_{r_1,0} > 0$$

for $\alpha > \frac{(n-1)}{(n-k-1)(k+1)} + \delta$. □

Corollary 4.8. *Let $\mathcal{A}$ be a weight vector with $k \geq 2$. Suppose $(m,n) \neq (1,k+1)$. Then there exists $\epsilon := \epsilon(k,n,m) > 0$ and $c_0 \leq \frac{k+2}{2k+2}$ such that, for any generically smooth 1-parameter family of $\mathcal{A}$-stable curves*

$$(\mathcal{C} \to B, \{\sigma_j\}_{j=1}^n, \{\tau_j\}_{j=1}^m)$$

and any $\epsilon_{i,j} \in \mathbb{Q} \cap [-\epsilon, \epsilon]$, we have

$$D_k(c_0).B + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0.$$

Proof.

(1.) Suppose $m = 0$. Take $c_0 := \frac{n-1}{2(n-2)}$. Since $n \geq 2k+1$, we have $c_0 > \frac{n-1}{(n-k-1)(k+1)} + \delta$ for δ sufficiently small. By Proposition 4.7 (1.), there exists $\epsilon = \epsilon(\delta,k,n,m)$ such that

$$(D_k(c_0) + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}).B = c_0(\psi_\sigma.B) + \frac{2c_0}{n-1}(\Delta_s.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0,$$

for all $|\epsilon_{i,j}| \leq \epsilon$. It remains to observe that $c_0 \leq \frac{k+2}{2k+2}$.

(2.) Suppose $m = 1$ and $n \geq k+2$. Take $\alpha := \frac{n-1}{2n}$ and $c_0 := \alpha + 1/n = \frac{n+1}{2n}$. Then $\alpha > \frac{n-1}{n(k+1)} + \delta$ for δ sufficiently small, and so by Proposition 4.7 (2.), there exists $\epsilon = \epsilon(\delta,k,n,m)$ such that

$$(D_k(c_0) + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}).B = (\alpha + \frac{1}{n})(\psi_\sigma.B) + \frac{2\alpha}{n-1}(\Delta_s.B) - (\Delta_n.B) + \sum_{i,j} \epsilon_{i,j}(\Delta_{i,j}.B) > 0,$$

for all $|\epsilon_{i,j}| \leq \epsilon$. It remains to observe that $c_0 \leq \frac{k+2}{2k+2}$.

(3.) The case of $m \geq 2$ similarly follows from Proposition 4.7 (3.-4.). □

Following the arguments of Theorem 4.3, we obtain

Proposition 4.9. *For $c \in (\frac{k+2}{2k+2}, \frac{k+1}{2k})$, the divisor $D_k(c)$ is ample on $\overline{M}_{0,\mathcal{A}}$.*

Proof. We fix k and proceed by induction on $\dim \overline{M}_{0,\mathcal{A}}$. The case of $\dim \overline{M}_{0,\mathcal{A}} = 0$ is trivial. Suppose the statement is established for all weight vectors $\mathcal{A}' = \mathcal{A}^k_{n,m}$ with $\dim \overline{M}_{0,\mathcal{A}'} < \dim \overline{M}_{0,\mathcal{A}}$.

By Lemma 4.6, it suffices to show that there exists an $\epsilon > 0$ such that $D_k(c') + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}$ has non-negative intersection on any 1-parameter family of $\mathcal{A}$-stable curves for all $\epsilon_{i,j} \in [-\epsilon, \epsilon] \cap \mathbb{Q}$ and all $c' \in \mathbb{Q} \cap [c-\epsilon, c+\epsilon]$.

By the induction hypothesis, $D_k(c') + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}$ has non-negative degree on any complete 1-parameter family with reducible generic fiber. It remains to show that $D_k(c') + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}$ has non-negative degree on any complete 1-parameter family with smooth general fiber.

First, suppose that $(m,n) \neq (1,k+1)$. By Corollary 4.8, there exists $c_0 \leq \frac{k+2}{2k+2}$ and $\epsilon' > 0$ such that $D_k(c_0) + \epsilon'_{i,j}\Delta_{i,j}$ has non-negative intersection on any generically smooth 1-parameter family of $\mathcal{A}$-stable curves for all $\epsilon'_{i,j} \in [-\epsilon', \epsilon'] \cap \mathbb{Q}$. By Theorem 4.3, $D_k(\frac{k+1}{2k})$ has positive intersection on any generically smooth 1-parameter family of $\mathcal{A}$-stable curves. Take

$$\epsilon_1 = \min\left\{\left(\frac{k+1}{2k} - c\right)/2, \left(c - \frac{k+2}{2k+2}\right)/2\right\}.$$

For any $c' \in [c-\epsilon_1, c+\epsilon_1] \subset (\frac{k+2}{2k+2}, \frac{k+1}{2k})$, we have

$$D_k(c') = \lambda D_k(c_0) + (1-\lambda) D_k\left(\frac{k+1}{2k}\right),$$

for $\lambda = \dfrac{k+1-2kc'}{k+1-2kc_0}$. Note that $\lambda \geq \dfrac{k+1-2kc}{2(k+1-2kc_0)}$.

Furthermore, for any $\epsilon_{i,j} \in [-\epsilon'\lambda, \epsilon'\lambda]$, we can write

$$D_k(c') + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j} = \lambda(D_k(c_0) + \sum_{i,j} \epsilon'_{i,j}\Delta_{i,j}) + (1-\lambda) D_k\left(\frac{k+1}{2k}\right),$$

where $\epsilon'_{i,j} \in [-\epsilon', \epsilon'] \cap \mathbb{Q}$. It follows that $D_k(c') + \sum_{i,j} \epsilon_{i,j}\Delta_{i,j}$ has non-negative intersection on any generically smooth 1-parameter family of $\mathcal{A}$-stable curves. The positive number $\epsilon = \min\{\epsilon_1, \epsilon' \cdot \frac{k+1-2kc}{2(k+1-2kc_0)}\}$ provides the desired result.

Finally, in the case $(m,n) = (1, k+1)$, there are no boundary divisors $\Delta_{i,j}$ and so by Theorem 4.3, the divisor $D_k(c')$ has non-negative degree on any complete 1-parameter family with smooth general fiber as long as $c' \in \left(\frac{k+2}{2k+2)}, \frac{k+1}{2k}\right)$. □

It remains to check ampleness at the endpoint $c = \frac{k+1}{2k}$.

Proposition 4.10. *$D_k(\frac{k+1}{2k})$ is ample on $\overline{M}_{0,\mathcal{A}}$.*

Proof. We proceed by induction on $\dim \overline{M}_{0,\mathcal{A}}$. Fix a weight vector $\mathcal{A}$, and assume that the given statement holds for all weight vectors $\mathcal{A}'$ satisfying $\dim \overline{M}_{0,\mathcal{A}'} < \dim \overline{M}_{0,\mathcal{A}}$. By Proposition 4.9, divisors $D_k(c)$ is ample for $c \in \left(\frac{k+2}{2k+2}, \frac{k+1}{2k}\right)$. To show that $D_k(\frac{k+1}{2k})$ is ample, it suffices to exhibit a rational number $\epsilon > 0$ such that $D_k(\frac{k+1}{2k} + \epsilon)$ is nef.

We will show that for small enough ϵ and any complete curve $B \subset \overline{M}_{0,\mathcal{A}}$, we have $D_k(\frac{k+1}{2k} + \epsilon).B \geq 0$. If

$$\phi\colon\ \overline{M}_{0,\mathcal{A}^{k-1}_{n,m}} \to \overline{M}_{0,\mathcal{A}^{k}_{n,m}}$$

is the natural reduction morphism, we will consider separately the cases where $B \subset \phi(\mathrm{Exc}\,(\phi))$ and $B \not\subset \phi(\mathrm{Exc}\,(\phi))$. Suppose first that $B \subset \phi(\mathrm{Exc}\,(\phi))$. In this case, k of the sections $\{\sigma_j\}_{j=1}^n$ are coincident on the corresponding family of $\mathcal{A}$-stable curves. It follows that B lies in the image of the closed immersion $\chi\colon\ \overline{M}_{0,\mathcal{A}^k_{n-k,m+1}} \to \overline{M}_{0,\mathcal{A}^k_{n,m}}$, which replaces the $(m+1)^{\text{st}}$-section of weight 1 with the k coincident sections of weight $1/k$. By Lemma 2.9, we have

$$\chi^* D_k\left(\frac{k+1}{2k} + \epsilon\right) = D_k\left(\frac{k+1}{2k}\right) - \epsilon k(k-2)\psi_{\tau_{m+1}}.$$

Since $\dim \overline{M}_{0,\mathcal{A}^k_{n-k,m+1}} < \dim \overline{M}_{0,\mathcal{A}}$, the induction hypothesis implies that $D_k(\frac{k+1}{2k})$ is ample on $\overline{M}_{0,\mathcal{A}^k_{n-k,m+1}}$. There exists $\epsilon > 0$ sufficiently small so that $D_k(\frac{k+1}{2k}) -$

$\epsilon k(k-2)\psi_{\tau_{m+1}}$ is still ample on $\overline{M}_{0,\mathcal{A}^k_{n-k,m+1}}$, and for this choice of ϵ, we have $D_k\left(\frac{k+1}{2k}+\epsilon\right).B \geq 0$ as desired.

Next, suppose that $B \not\subset \phi(\mathrm{Exc}\,(\phi))$. Let B' denote the ϕ-transform of B on $\overline{M}_{0,\mathcal{A}^{k-1}_{n,m}}$ so that

$$D_k\left(\frac{k+1}{2k}+\epsilon\right).B = \phi^* D_k\left(\frac{k+1}{2k}+\epsilon\right).B'.$$

By Lemma 2.8, we have

$$\phi^* D_k\left(\frac{k+1}{2k}+\epsilon\right) = D_{k-1}\left(\frac{k+1}{2k}+\epsilon\right) + \epsilon k(k-2)E,$$

where E is the union of the exceptional divisors of ϕ. Since $\frac{k+1}{2k}+\epsilon = \frac{(k-1)+2}{2(k-1)+2}+\epsilon$, Proposition 4.9 implies that for sufficiently small ϵ, $D_{k-1}(\frac{k+1}{2k}+\epsilon).B' > 0$. Since B' is not contained in E, we also have $E.B' > 0$. It follows that $D_k(\frac{k+1}{2k}+\epsilon).B \geq 0$, as desired. □

Putting together Proposition 4.9 and Proposition 4.10, we obtain

Corollary 4.11.

(a) *For any weight vector $\mathcal{A}$, the divisor $D_k(c)$ is ample on $\overline{M}_{0,\mathcal{A}}$ for all $c \in \mathbb{Q} \cap (\frac{k+2}{2k+2}, \frac{k+1}{2k}]$.*

(b) *For symmetric weight vectors $\mathcal{A}$, the divisor $K_{\overline{M}_{0,\mathcal{A}}} + \alpha\Delta$ is ample on $\overline{M}_{0,\mathcal{A}}$ for all $\alpha \in \mathbb{Q} \cap (\frac{2}{k+1}, \frac{2}{k+1}]$.*

Proof. Part (a) follows from Proposition 4.9 and Proposition 4.10. Part (b) follows from Part (a) by Remark 2.6. □

4.3. **Kleiman's criterion on an algebraic space.** In the proof of Proposition 4.9, we used the fact that $\overline{M}_{0,\mathcal{A}}$ (or rather the quotient $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$) is a scheme when we invoked Kleiman's criterion. In general, Kleiman's criterion may fail for algebraic spaces (see [10], VI, 2.9.13). Since one often encounters situations where one would like to prove projectivity of a moduli space without knowing *a priori* that it is a scheme, it seems worth pointing out that our method can be used to prove projectivity by checking the additional hypothesis of Lemma 4.12 below.

The proof of the following lemma is nothing more than a logical rehashing of the proof that Nakai's criterion ([10], VI, 2.18) implies Kleiman's criterion ([10], VI, 2.19).

Lemma 4.12 (Kleiman's criterion on algebraic spaces)**.** *Suppose that X is an algebraic space, proper over an algebraically closed field. Suppose X has the property that, for any subvariety $Z \subset X$, there exists an effective Cartier divisor E such that E meets Z properly. Then Kleiman's criterion holds for X, i.e. any divisor D which lies in the interior of the nef cone of X is ample.*

Proof. Suppose that $D \subset X$ is a Cartier divisor which lies in the interior of the nef cone of X. To prove that D is ample, it suffices to show that $D^k.[Z] > 0$ for an arbitrary k-dimensional subvariety $Z \subset X$.

Given a k-dimensional subvariety $Z \subset X$, our hypothesis gives an effective Cartier divisor $E \subset X$, such that

$$E.[Z] = \sum_i a_i[Z_i],$$

with each $Z_i \subset X$ a $(k-1)$-dimensional subvariety, and each $a_i > 0$. Since D lies on the interior of the nef cone, there exists a rational number $\epsilon > 0$ such that $D - \epsilon E$ is nef. Now we have

$$D^{k-1}(D - \epsilon E).[Z] \geq 0,$$

since D and $D - \epsilon E$ are nef ([10], VI, 2.18.7.3). It follows that

$$D^k.[Z] \geq \epsilon D^{k-1}[E.Z] = \epsilon \sum_i a_i D^{k-1}.[Z_i] > 0,$$

by induction on the dimension of Z. □

In order to prove that Kleiman's criterion holds for $\overline{M}_{0,\mathcal{A}}$ without knowing a priori that $\overline{M}_{0,\mathcal{A}}$ is a scheme, it suffices to check that $\overline{M}_{0,\mathcal{A}}$ satisfies the hypothesis of Lemma 4.12. (Technically, we applied Kleiman's criterion to the quotient $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$, but it is clear that if $\overline{M}_{0,\mathcal{A}}$ satisfies the hypothesis of Lemma 4.12, then so does $\overline{M}_{0,\mathcal{A}}/S_n \times S_m$.)

If $Z \subset \overline{M}_{0,\mathcal{A}}$ is an arbitrary positive-dimensional subvariety, the general point of Z lies in the interior of some boundary stratum

$$\overline{M}_{0,\mathcal{A}_1} \times \ldots \times \overline{M}_{0,\mathcal{A}_k} \subset \overline{M}_{0,\mathcal{A}},$$

Since the interior $M_{0,n_1} \times \ldots \times M_{0,n_k} \subset \overline{M}_{0,\mathcal{A}_1} \times \ldots \times \overline{M}_{0,\mathcal{A}_k}$ is affine, Z must meet some irreducible component of the boundary of $\overline{M}_{0,\mathcal{A}_1} \times \ldots \times \overline{M}_{0,\mathcal{A}_k}$. Since every irreducible component of the boundary of $\overline{M}_{0,\mathcal{A}_1} \times \ldots \times \overline{M}_{0,\mathcal{A}_k}$ is the restriction of a boundary divisor on the ambient space $\overline{M}_{0,\mathcal{A}}$, all of which are Cartier, we are done.

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