ASSOCIATED FORM MORPHISM

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Abstract. We study the geometry of the morphism that sends a smooth hypersurface of degree $d+1$ in $\mathbb{P}^{n-1}$ to its associated hypersurface of degree $n(d-1)$ in the dual space $\left(\mathbb{P}^{n-1}\right)^{\vee}$.

Contents

1. Introduction

One of the first applications of Geometric Invariant Theory is a construction of the moduli space of smooth degree m hypersurfaces in a fixed projective space $\mathbb{P}^{n-1}$ [15]. This moduli space is an affine GIT quotient

$$U_{m,n} := \left(\mathbb{P}\mathrm{H}^0\left(\mathbb{P}^{n-1}, \mathcal{O}(m)\right) \setminus \Delta\right) /\!\!/ \mathrm{PGL}(n),$$

where Δ is the discriminant divisor parameterizing singular hypersurfaces. The GIT construction produces a natural compactification

$$U_{m,n} \subset V_{m,n} := \left(\mathbb{P}\mathrm{H}^0\left(\mathbb{P}^{n-1}, \mathcal{O}(m)\right)\right)^{ss} /\!\!/ \mathrm{PGL}(n),$$

given by a categorical quotient of the locus of GIT semistable hypersurfaces. We call $V_{m,n}$ the GIT compactification of $U_{m,n}$.

The subject of this paper is a certain rational map $V_{m,n} \dashrightarrow V_{n(m-2),n}$, where $n \geq 2$, $m \geq 3$ and where we exclude the (trivial) case $(n,m) = (2,3)$. While this map has a purely algebraic construction, which we shall recall soon, it has several surprising geometric properties that we establish in this paper. In particular, this rational map restricts to a locally closed immersion $\bar{A}\colon U_{m,n} \to V_{n(m-2),n}$, and often

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contracts the discriminant divisor in $V_{m,n}$. Consequently, the closure of the image of $\bar{A}$ in $V_{n(m-2),n}$ is a compactification of the GIT moduli space $U_{m,n}$ that is different from the GIT compactification $V_{m,n}$.

To define $\bar{A}$, we consider the *associated form morphism* defined on the space of smooth homogeneous forms $f \in \mathbb{C}[x_1, \dots, x_n]$ of fixed degree $m \geq 3$. Given such an f, its associated form $A(f)$ is a degree $n(m-2)$ homogeneous form in the graded dual polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. In our recent paper [10], we proved that the associated form $A(f)$ is always polystable in the sense of GIT. Consequently, we obtain a morphism $\bar{A}$ from $U_{m,n}$ to $V_{n(m-2),n}$ sending the image of f in $U_{m,n}$ to the image of $A(f)$ in $V_{n(m-2),n}$.

Our first result is that the morphism $\bar{A}$ is an isomorphism onto its image, a locally closed subvariety in the target.

Theorem 1.1. *The morphism*

$$\bar{A}\colon U_{m,n} \to V_{n(m-2),n}$$

is a locally closed immersion.

In the process of establishing Theorem 1.1, we generalize results of [2] to the case of an arbitrary number of variables, and, in particular, prove that the auxiliary gradient morphism sending a semistable form to the span of its partial derivatives gives rise to a closed immersion on the level of quotients (see Theorem 2.1).

Our second main result is Theorem 2.2, which describes the rational map $\bar{A}\colon V_{m,n} \dashrightarrow V_{n(m-2),n}$ in codimension one. Namely, we study how $\bar{A}$ extends to the generic point of the discriminant divisor in the GIT compactification (see Corollary 5.8), and prove that for $n = 2, 3$ and $m \geq 4$, as well as for $n \geq 4$, $m \gg 0$, the morphism $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety in the target (see Corollary 5.9). In the process, we prove that the image of $\bar{A}$ contains the orbit of the Fermat hypersurface in its closure and as a result obtain a new proof of the generic smoothness of associated forms (see Corollary 5.10).

1.1. **Notation and conventions.** Let $S := \operatorname{Sym} V \simeq \mathbb{C}[x_1, \dots, x_n]$ be a symmetric algebra of an n-dimensional vector space V, with its standard grading. Let $\mathcal{D} := \operatorname{Sym} V^{\vee} \simeq \mathbb{C}[z_1, \dots, z_n]$ be the graded dual of S, with the structure of the S-module given by the *polar pairing* $S \times \mathcal{D} \to \mathcal{D}$, which is defined by

$$g(x_1, \dots, x_n) \circ F(z_1, \dots, z_n) := g(\partial/\partial z_1, \dots, \partial/\partial z_1) F(z_1, \dots, z_n). \tag{1.1}$$

A homogeneous polynomial $f \in S_m$ is called a *direct sum* if, after a linear change of variables, it can be written as the sum of two non-zero polynomials in disjoint sets of variables:

$$f = f_1(x_1, \dots, x_a) + f_2(x_{a+1}, \dots, x_n).$$

We will use the recognition criteria for direct sums established in [8], and so we keep the pertinent terminology of that paper. We will say that $f \in S_m$ is a k-partial

Fermat form for some $k \leq n$, if, after a linear change of variables, it can be written as follows:

$$f = x_1^m + \cdots + x_k^m + g(x_{k+1}, \dots, x_n).$$

Clearly, any n-partial Fermat form is linearly equivalent to the standard Fermat form. Furthermore, all k-partial Fermat forms are direct sums. We denote by $\mathfrak{DS}_m$ the locus of direct sums in S_m.

2. Associated form of a balanced complete intersection

Fix $d \geq 2$. In what follows the trivial case $(n, d) = (2, 2)$ will be excluded. A length n regular sequence $g_1, \dots, g_n$ of elements of S_d will be called a balanced complete intersection of type $(d)^n$. It defines a graded Gorenstein Artin $\mathbb{C}$-algebra

$$\mathcal{A}(g_1, \dots, g_n) := S/(g_1, \dots, g_n),$$

whose socle lies in degree $n(d-1)$. In [2] an element $\mathbf{A}(g_1, \dots, g_n) \in \mathcal{D}_{n(d-1)}$, called *the associated form of* $g_1, \dots, g_n$, was introduced. The form $\mathbf{A}(g_1, \dots, g_n)$ is a homogeneous Macaulay inverse system, or a dual socle generator, of the algebra $\mathcal{A}(g_1, \dots, g_n)$. It follows that $[\mathbf{A}(g_1, \dots, g_n)] \in \mathbb{P}\mathcal{D}_{n(d-1)}$ depends only on the linear span $\langle g_1, \dots, g_n \rangle$, which we regard as a point in $\mathrm{Grass}(n, S_d)$.

Recall that $g_1, \dots, g_n$ is a regular sequence in S_d if and only if $\langle g_1, \dots, g_n \rangle$ does not in lie in the resultant divisor $\mathfrak{Res} \subset \mathrm{Grass}(n, S_d)$. Setting $\mathrm{Grass}(n, S_d)_{\mathrm{Res}} := \mathrm{Grass}(n, S_d) \setminus \mathfrak{Res}$, we obtain a morphism

$$\mathbf{A}\colon\ \mathrm{Grass}(n, S_d)_{\mathrm{Res}} \to \mathbb{P}\mathcal{D}_{n(d-1)}.$$

Given $f \in S_{d+1}$, the partial derivatives $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ form a regular sequence if and only if f is non-degenerate. For a non-degenerate $f \in S_{d+1}$, in [1, 3] the *associated form of* f was defined to be

$$A(f) := \mathbf{A}(\partial f/\partial x_1, \dots, \partial f/\partial x_n) \in \mathcal{D}_{n(d-1)}.$$

Summarizing, we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{P}(S_{d+1})_\Delta \arrow[rr, "A"] \arrow[dr, "\nabla"'] & & \mathbb{P}(\mathcal{D}_{n(d-1)}) \\
& \mathrm{Grass}(n, S_d)_{\mathrm{Res}}, \arrow[ur, "\mathbf{A}"'] &
\end{tikzcd}$$

where $\mathbb{P}(S_{d+1})_\Delta$ denotes the complement to the discriminant divisor in $\mathbb{P}(S_{d+1})$ and ∇ is the morphism sending a form into the linear span of its first partial derivatives. The above diagram is equivariant with respect to the standard $\mathrm{SL}(n)$-actions on S and $\mathcal{D}$. By [2], the morphism $\mathbf{A}$ is a locally closed immersion, and it was proved in [10] that $\mathbf{A}$ sends polystable orbits to polystable orbits. Passing to the GIT quotients, we thus obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{P}(S_{d+1})_{\Delta} /\!/ \mathrm{SL}(n) \arrow[rr, "\bar{A}"] \arrow[dr, "\widetilde{\nabla}"'] & & \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!/ \mathrm{SL}(n) \\
& \mathrm{Grass}(n, S_d)_{\mathrm{Res}} /\!/ \mathrm{SL}(n), \arrow[ur, "\bar{\mathbf{A}}"'] &
\end{tikzcd} \tag{2.1}$$

where $\widetilde{\nabla} := \nabla /\!/ \mathrm{SL}(n)$ is a finite injective morphism (see [9]) and $\bar{\mathbf{A}} := \mathbf{A} /\!/ \mathrm{SL}(n)$ is a locally closed immersion. The main focus of this paper is the geometry of diagram (2.1).

Noting that by [9] the map ∇ extends to a morphism from $\mathbb{P}(S_{d+1})^{ss}$ to $\mathrm{Grass}(n, S_d)^{ss}$ and thus induces a map $\overline{\nabla}$ of the corresponding GIT quotients, we will now state our two main results as follows:

Theorem 2.1. *The morphism $\overline{\nabla}\colon \mathbb{P}(S_{d+1})^{ss} /\!/ \mathrm{SL}(n) \to \mathrm{Grass}(n, S_d)^{ss} /\!/ \mathrm{SL}(n)$ is a closed immersion.*

Theorem 2.2. *The rational map*

$$\bar{A}\colon \mathbb{P}(S_{d+1})^{ss} /\!/ \mathrm{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!/ \mathrm{SL}(n)$$

extends to the generic point of the discriminant divisor $\Delta /\!/ \mathrm{SL}(n)$ in the GIT compactification and contracts the discriminant divisor to a lower-dimensional variety for all sufficiently large d as described in Corollaries 5.8 and 5.9.

3. Preliminaries on dualities

In this section we collect results on Macaulay inverse systems of graded Gorenstein Artin $\mathbb{C}$-algebras. We also recall the duality between the Hilbert points of such algebras and the gradient points of their inverse systems.

Recall that we regard $S = \mathbb{C}[x_1, \dots, x_n]$ as a ring of polynomial differential operators on the graded dual ring $\mathcal{D} := \mathbb{C}[z_1, \dots, z_n]$ via polar pairing (1.1). For every positive m, the restricted pairing

$$S_m \times \mathcal{D}_m \to \mathbb{C}$$

is perfect and so defines an isomorphism

$$\mathcal{D}_m \simeq S_m^{\vee}, \tag{3.1}$$

where, as usual, $V^{\vee}$ stands for the dual of a vector space V.

Given $W \subset \mathcal{D}$, we define

$$W^{\perp} := \{f \in S \mid f \circ g = 0, \text{ for all } g \in W\} \subset S.$$

Similarly given $U \subset S$, we define

$$U^{\perp} := \{g \in \mathcal{D} \mid f \circ g = 0, \text{ for all } f \in U\} \subset \mathcal{D}.$$

Claim 3.1. *Isomorphism* (3.1) *sends an element* $\omega \in S_m^\vee$ *to the element*

$$\mathfrak{D}_\omega := \sum_{i_1+\cdots+i_n=m} \frac{\omega\big(x_1^{i_1}\cdots x_n^{i_n}\big)}{i_1!\cdots i_n!} z_1^{i_1}\cdots z_n^{i_n} \in \mathcal{D}_m.$$

Conversely, an element $g \in \mathcal{D}_m$ *is mapped by isomorphism* (3.1) *to the projection*

$$S_m \twoheadrightarrow S_m/(g^\perp)_m \simeq \mathbb{C},$$

where the isomorphism with $\mathbb{C}$ *is chosen so that* $1 \in \mathbb{C}$ *pairs to* 1 *with* g.

Proof. One observes that $f \circ \mathfrak{D}_\omega = \omega(f)$ for every $f \in S_m$, and the first part of the claim follows. The second part is immediate from definitions. $\square$

Corollary 3.2. *Given* $\omega \in S_m^\vee$, *for every* $(a_1,\dots,a_n) \in \mathbb{C}^n$ *we have*

$$\mathfrak{D}_\omega(a_1,\dots,a_n) = \omega\big((a_1x_1+\cdots+a_nx_n)^m/m!\big). \tag{3.2}$$

Proof.

$$\begin{aligned}\omega\big((a_1x_1+\cdots+a_nx_n)^m/m!\big) &= \frac{(a_1x_1+\cdots+a_nx_n)^m}{m!}\circ\mathfrak{D}_\omega \\ &= \frac{(a_1\partial/\partial z_1+\cdots+a_n\partial/\partial z_n)^m}{m!}\mathfrak{D}_\omega = \mathfrak{D}_\omega(a_1,\dots,a_n),\end{aligned}$$

where the last equality is easily checked, say on monomials. $\square$

Remark 3.3. It follows from Corollary 3.2 that all forms in a subset $W \subset \mathcal{D}_m$ vanish at a given point $(a_1,\dots,a_n) \in \mathbb{C}^n$ if and only if $(a_1x_1+\cdots+a_nx_n)^m \in W^\perp$.

Notice that the maps

$$\big[\langle\mathfrak{D}_\omega\rangle \subset \mathcal{D}_m\big] \mapsto \big[(\mathfrak{D}_\omega^\perp)_m \subset S_m\big] = \big[\ker(\omega) \subset S_m\big]$$

define isomorphisms

$$\mathrm{Grass}(1,\mathcal{D}_m) \simeq \mathrm{Grass}\left(\dim_{\mathbb{C}} S_m - 1, S_m\right).$$

More generally, for any $1 \le m \le \binom{m+n-1}{n-1} - 1$ the correspondence

$$\big[W \subset \mathcal{D}_m\big] \mapsto \big[(W^\perp)_m \subset S_m\big]$$

yields an isomorphism

$$\mathrm{Grass}(k,\mathcal{D}_m) \simeq \mathrm{Grass}\left(\dim_{\mathbb{C}} S_m - k, S_m\right). \tag{3.3}$$

Let $I \subset S$ be a Gorenstein ideal and ν the socle degree of the algebra $\mathcal{A} = S/I$. Recall that a *(homogeneous) Macaulay inverse system* of $\mathcal{A}$ is an element $f_{\mathcal{A}} \in \mathcal{D}_\nu$ such that

$$f_{\mathcal{A}}^\perp = I$$

(see [11, Lemma 2.12] or [6, Exercise 21.7]). As $(f_{\mathcal{A}}^\perp)_\nu = I_\nu$, we see that all Macaulay inverse systems are mutually proportional and $\langle f_{\mathcal{A}}\rangle = \big((I_\nu)^\perp\big)_\nu$. Clearly, the line $\langle f_{\mathcal{A}}\rangle \in \mathrm{Grass}(1,\mathcal{D}_\nu)$ maps to the ν^{th} Hilbert point $H_\nu \in \mathrm{Grass}(\dim_{\mathbb{C}} S_\nu - 1, S_\nu)$ of $\mathcal{A}$ under isomorphism (3.3) with $k = 1$.

Remark 3.4. Papers [3, 4], for any $\omega \in S_\nu^\vee$ with $\ker\omega = I_\nu$, introduced the *associated form* of $\mathcal{A}$ as the element of $\mathcal{D}_\nu$ given by the right-hand side of formula (3.2) with $m = \nu$ (up to the factor $\nu!$). By Corollary 3.2, under isomorphism (3.3) with $k = 1$ the span of every associated form in $\mathcal{D}_\nu$ also maps to the ν^{th} Hilbert point $H_\nu \in \mathrm{Grass}(\dim_{\mathbb{C}} S_\nu - 1, S_\nu)$ of $\mathcal{A}$. In particular, for the algebra $\mathcal{A}$ any associated form is simply one of its Macaulay inverse systems, and equation (3.2) with $m = \nu$ and $\ker\omega = I_\nu$ is an explicit formula for a Macaulay inverse system of $\mathcal{A}$ (see [12] for more details).

3.1. **Gradient points.** Given a polynomial $F \in \mathcal{D}_m$, we define the p^{th} *gradient point of* F to be the linear span of all p^{th} partial derivatives of F in $\mathcal{D}_{m-p}$. We denote the p^{th} gradient point by $\nabla^p(F)$. Note that

$$\nabla^p(F) = \{g \circ F \mid g \in S_p\}$$

is simply the $(m-p)^{th}$ graded piece of the principal S-module SF. The 1^{st} gradient point $\nabla F := \langle \partial F/\partial z_1, \dots, \partial F/\partial z_n\rangle$ will be called simply *the gradient point of* F.

Proposition 3.5 (Duality between gradient and Hilbert points)**.** *The* p^{th} *gradient point of a Macaulay inverse system* $f_{\mathcal{A}} \in \mathcal{D}_\nu$ *maps to the* $(\nu - p)^{th}$ *Hilbert point* $H_{\nu-p}$ *of* $\mathcal{A}$ *under isomorphism* (3.3)*.*

Proof. Let G be the p^{th} gradient point of $f_{\mathcal{A}}$, that is

$$G := \left\langle \frac{\partial^p}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}} f_{\mathcal{A}} \mid i_1 + \cdots + i_n = p \right\rangle.$$

We need to verify that $I_{\nu-p} = (G^\perp)_{\nu-p}$. We have

$$\begin{aligned}
(G^\perp)_{\nu-p} &= \left\{ f \in S_{\nu-p} \mid f \circ \frac{\partial^p}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}} f_{\mathcal{A}} = 0 \text{ for all } i_1 + \cdots + i_n = p \right\} \\
&= \left\{ f \in S_{\nu-p} \mid f x_1^{i_1}\cdots x_n^{i_n} \circ f_{\mathcal{A}} = 0 \text{ for all degree } p \text{ monomials} \right\} \\
&= \left\{ f \in S_{\nu-p} \mid x_1^{i_1}\cdots x_n^{i_n} f \in f_{\mathcal{A}}^\perp \text{ for all degree } p \text{ monomials} \right\} \\
&= \left\{ f \in S_{\nu-p} \mid x_1^{i_1}\cdots x_n^{i_n} f \in I_\nu \text{ for all degree } p \text{ monomials} \right\} \\
&= I_{\nu-p},
\end{aligned}$$

where the last equality comes from the fact that I is Gorenstein. $\square$

As a corollary of the above duality result, we recall in Proposition 3.6 below a generalization of [1, Lemma 4.4]. Although this statement is well-known (it appears, for example, in [5, Proposition 4.1, p. 174]), we provide a short proof for completeness. We first recall that a non-zero homogeneous form f in n variables has multiplicity $\ell + 1$ at a point $p \in \mathbb{P}^{n-1}$ if and only if all partial derivatives of f of order ℓ (hence of all orders $\leq \ell$) vanish at p, and some partial derivative of f of order $\ell + 1$ does not vanish at p. We define the *Veronese cone* $\mathcal{C}_m$ to be the variety of all degree m powers of linear forms in S_m:

$$\mathcal{C}_m := \left\{L^m \mid L \in S_1\right\} \subset S_m.$$

Proposition 3.6. *Let $I \subset S$ be a Gorenstein ideal and ν the socle degree of the algebra $\mathcal{A} = S/I$. Then a Macaulay inverse system $f_{\mathcal{A}}$ of $\mathcal{A}$ has a point of multiplicity $\ell+1$ if and only if there exists a non-zero $L \in S_1$ such that $L^{\nu-\ell} \in I_{\nu-\ell}$, and $L^{\nu-\ell-1} \notin I_{\nu-\ell-1}$. In particular, $f_{\mathcal{A}}$ has no points of multiplicity $\ell+1$ or higher if and only if*

$$I_{\nu-\ell} \cap \mathcal{C}_{\nu-\ell} = (0).$$

Proof. By Proposition 3.5, the ℓ^{th} gradient point of $f_{\mathcal{A}}$ is dual to the $(\nu-\ell)^{th}$ Hilbert point of $\mathcal{A}$

$$H_{\nu-\ell}\colon S_{\nu-\ell} \twoheadrightarrow \mathcal{A}_{\nu-\ell}.$$

We conclude by Remark 3.3 that all partial derivatives of $f_{\mathcal{A}}$ of order ℓ vanish at $(a_1, \dots, a_n)$ if and only if

$$(a_1x_1 + \cdots + a_nx_n)^{\nu-\ell} \in \ker H_{\nu-\ell} = I_{\nu-\ell}.$$

It follows that $L = a_1x_1 + \cdots + a_nx_n$ satisfies $L^{\nu-\ell} \in I_{\nu-\ell}$ and $L^{\nu-\ell-1} \notin I_{\nu-\ell-1}$ if and only if $f_{\mathcal{A}}$ has multiplicity exactly $\ell+1$ at the point $(a_1, \dots, a_n)$. ☐

4. The gradient morphism ∇

In this section, we prove Theorem 2.1. Recall that we have the commutative diagram

$$\begin{tikzcd}
\mathbb{P}\big(S_{d+1}\big)^{ss} \arrow[r, "\nabla"] \arrow[d, "\pi_0"] & \mathrm{Grass}\big(n, S_d\big)^{ss} \arrow[d, "\pi_1"] \\
\mathbb{P}\big(S_{d+1}\big)^{ss} /\!/ \mathrm{SL}(n) \arrow[r, "\overline{\nabla}"] & \mathrm{Grass}\big(n, S_d\big)^{ss} /\!/ \mathrm{SL}(n).
\end{tikzcd}$$

Let $\mathfrak{D}\mathfrak{S}_{d+1}^{ss} := \mathbb{P}(\mathfrak{D}\mathfrak{S}_{d+1})^{ss}$ be the locus of semistable direct sums in $\mathbb{P}\big(S_{d+1}\big)^{ss}$. By [8, Section 3], the set $\mathfrak{D}\mathfrak{S}_{d+1}^{ss}$ is precisely the closed locus in $\mathbb{P}\big(S_{d+1}\big)^{ss}$ where ∇ has positive fiber dimension.

Suppose $f \in S_{d+1}$ is a semistable form. Then, after a linear change of variables, we have a maximally fine direct sum decomposition

$$f = \sum_{i=1}^{k} f_i(\mathbf{x}^i), \tag{4.1}$$

where $V_i = \langle \mathbf{x}^i \rangle$ are such that $V = \oplus_{i=1}^k V_i$, and where each f_i is not a direct sum in $\mathrm{Sym}\, V_i$. Set $n_i := \dim_{\mathbb{C}} V_i$. We define the canonical torus $\Theta(f) \subset \mathrm{SL}(n)$ associated to f as the connected component of the identity of the subgroup

$$\{g \in \mathrm{SL}(n) \mid V_i \text{ is an eigenspace of } g, \text{ for every } i = 1, \dots, k\} \subset \mathrm{SL}(n).$$

Clearly, $\Theta(f) \simeq (\mathbb{C}^*)^{k-1}$, and since

$$\nabla([f]) = \nabla([f_1]) \oplus \cdots \oplus \nabla([f_k]), \text{ where } \nabla([f_i]) \in \mathrm{Grass}(n_i, \mathrm{Sym}^d V_i),$$

we also have $\Theta(f) \subset \mathrm{Stab}(\nabla([f]))$, where Stab denotes the stabilizer under the $\mathrm{SL}(n)$-action.

From the definition of $\Theta(f)$, it is clear that $\Theta(f) \cdot [f] \subset \nabla^{-1}(\nabla([f]))$, and in fact [8, Corollary 3.12] gives a set-theoretic equality $\nabla^{-1}(\nabla([f])) = \Theta(f) \cdot [f]$. We will now obtain a stronger result:

Lemma 4.1. *One has $\nabla^{-1}(\nabla([f])) = \Theta(f) \cdot [f]$ scheme-theoretically, or, equivalently,*

$$\ker(d\nabla_{[f]}) = \mathbf{T}_{[f]}(\Theta(f) \cdot [f]),$$

where $\mathbf{T}_{[f]}$ denotes the tangent space at $[f]$.

Proof. Under the standard identification of $\mathbf{T}_{[f]}\mathbb{P}(S_{d+1})$ with $S_{d+1}/\langle f \rangle$, the subspace $\mathbf{T}_{[f]}(\Theta(f) \cdot [f])$ is identified with $\langle f_1, \dots, f_k \rangle / \langle f \rangle$. It now suffices to show that every $g \in S_{d+1}$ that satisfies $\nabla[g] \subset \nabla[f]$ must lie in $\langle f_1, \dots, f_k \rangle$, where $\nabla[g] := \langle \partial g/\partial x_1, \dots, \partial g/\partial x_n \rangle \subset S_d$. This is precisely the statement of [8, Corollary 3.12]. ☐

We note an immediate consequence:

Corollary 4.2. *If $f \in S_{d+1}^{ss}$ is not a direct sum, then ∇ is unramified at $[f]$.*

Further, since ∇ is equivariant with respect to the $\mathrm{SL}(n)$-action, we have the inclusion $\mathrm{Stab}([f]) \subset \mathrm{Stab}(\nabla([f]))$. As the following result shows, the difference between $\mathrm{Stab}([f])$ and $\mathrm{Stab}(\nabla([f]))$ is controlled by the torus $\Theta(f)$.

Corollary 4.3. *The subgroup $\mathrm{Stab}(\nabla([f]))$ is generated by $\Theta(f)$ and $\mathrm{Stab}([f])$.*

Proof. Suppose $\sigma \in \mathrm{Stab}(\nabla([f]))$. Then $\nabla(\sigma \cdot [f]) = \nabla([f])$ implies by Lemma 4.1 that $\sigma \cdot [f] = \tau \cdot [f]$ for some $\tau \in \Theta(f)$. Consequently, $\tau^{-1} \circ \sigma \in \mathrm{Stab}([f])$ as desired. ☐

Next, we obtain the following generalization of [2, Proposition 6.3], whose proof we follow almost verbatim.

Proposition 4.4. *The morphism ∇ is a closed immersion along the open locus $\mathcal{U} := \mathbb{P}(S_{d+1})^{ss} \setminus \mathfrak{D}\mathfrak{S}_{d+1}^{ss}$ of all elements that are not direct sums.*

Proof. Since for every $[f] \in \mathcal{U}$ we have that ∇ is unramified at $[f]$ and $\nabla^{-1}(\nabla([f])) = [f]$, it suffices to show that ∇ is a finite morphism when restricted to $\mathcal{U}$. Since, by [9], the induced morphism on the GIT quotients is finite, by [13, p. 89, Lemme] it suffices to verify that ∇ is quasi-finite and that ∇ sends closed orbits to closed orbits. The former has already been established, and the latter is proved below in Proposition 4.5. ☐

Proposition 4.5. *Suppose $f \in S_{d+1}^{ss}$ is polystable and not a direct sum. Then the image $\nabla([f]) \in \mathrm{Grass}(n, S_d)^{ss}$ is polystable.*

The above result is a generalization of [9, Theorem 1.1], whose method of proof we follow; we also keep the notation of *loc.cit.*, especially as it relates to monomial orderings. We begin with a preliminary observation.

Lemma 4.6. *Suppose $f \in S_{d+1}$ is such that there exists a non-trivial one-parameter subgroup λ of $\mathrm{SL}(n)$ acting diagonally on $x_1, \dots, x_n$ with weights $\lambda_1, \dots, \lambda_n$ and satisfying*

$$w_\lambda(\mathrm{in}_\lambda(\partial f/\partial x_i)) = d\lambda_i.$$

Then f is a direct sum.

Proof. We can assume that

$$\lambda_1 \leq \cdots \leq \lambda_a < \lambda_{a+1} = \cdots = \lambda_n$$

for some $1 \leq a < n$. Then the fact that

$$w_\lambda(\mathrm{in}_\lambda(\partial f/\partial x_i)) = d\lambda_i = d\lambda_n,$$

for all $i = a+1, \dots, n$, implies

$$\partial f/\partial x_{a+1}, \dots, \partial f/\partial x_n \in \mathbb{C}[x_{a+1}, \dots, x_n].$$

Consequently, $f = g_1(x_1, \dots, x_a) + g_2(x_{a+1}, \dots, x_n)$ is a direct sum. ☐

Proof of Proposition 4.5. Since f is polystable, by [9, Theorem 1.1] it follows that $\nabla([f])$ is semistable. Suppose $\nabla([f])$ is not polystable. Then there exists a one-parameter subgroup λ acting on the coordinates $x_1, \dots, x_n$ with the weights $\lambda_1, \dots, \lambda_n$ such that the limit of $\nabla([f])$ under λ exists and does not lie in the orbit of $\nabla([f])$. In particular, the limit of $[f]$ under λ does not exist.

Then by [9, Lemma 3.5], there is an upper triangular unipotent coordinate change

$$\begin{aligned}
x_1 &\mapsto x_1 + c_{12}x_2 + \cdots + c_{1n}x_n,\\
x_2 &\mapsto \phantom{x_1 + c_{12}} x_2 + \cdots + c_{2n}x_n,\\
&\vdots\\
x_n &\mapsto \phantom{x_1 + c_{12}x_2 + \cdots + c_{1n}} x_n
\end{aligned}$$

such that for the transformed form

$$h(x_1, \dots, x_n) := f(x_1 + c_{12}x_2 + \cdots + c_{1n}x_n, x_2 + \cdots + c_{2n}x_n, \dots, x_n)$$

the initial monomials

$$\mathrm{in}_\lambda(\partial h/\partial x_1), \dots, \mathrm{in}_\lambda(\partial h/\partial x_n)$$

are distinct. Now, setting

$$\mu_i := w_\lambda(\mathrm{in}_\lambda(\partial h/\partial x_i)),$$

by [9, Lemma 3.2] we have

$$\mu_1 + \cdots + \mu_n = 0.$$

It follows that with the respect to the one-parameter subgroup λ' acting on x_i with the weight $d\lambda_i - \mu_i$, all monomials of h have non-negative weights (cf. [9, the proof of Lemma 3.6]). Write $h = h_0 + h_1$, where all monomials of h_0 have zero λ'-weights and all monomials of h_1 have positive λ'-weights. Then $h_0 \in \overline{\mathrm{SL}(n) \cdot h} = \mathrm{SL}(n) \cdot h$, by the polystability assumption on f. Furthermore, h_0 is stabilized by λ'.

If λ' is a trivial one-parameter subgroup, then $\mu_i = d\lambda_i$ for all $i = 1, \dots, n$, and by Lemma 4.6 the form h is a direct sum, which is a contradiction.

Suppose now that λ' is a non-trivial one-parameter subgroup. Clearly, we have

$$w_\lambda(\mathrm{in}_\lambda(\partial h_0/\partial x_i) \geq w_\lambda(\mathrm{in}_\lambda(\partial h/\partial x_i),$$

since the state of h_0 is a subset of the state of h. If one of the inequalities above is strict, then $\nabla([h_0])$ is destabilized by λ, contradicting the semistability of $\nabla([h_0])$ established in [9, Theorem 1.1]. Thus

$$w_\lambda(\mathrm{in}_\lambda(\partial h_0/\partial x_i)) = w_\lambda(\mathrm{in}_\lambda(\partial h/\partial x_i)) = \mu_i.$$

Moreover, since h_0 is λ'-invariant, we have that $\partial h_0/\partial x_i$ is homogeneous of degree $-w_{\lambda'}(x_i) = \mu_i - d\lambda_i$ with respect to λ'. Let μ be the one-parameter subgroup acting on $x_1, \dots, x_n$ with the weights $\mu_1, \dots, \mu_n$. It follows that

$$w_\mu(\mathrm{in}_\mu(\partial h_0/\partial x_i)) = dw_\lambda(\mathrm{in}_\lambda(\partial h_0/\partial x_i) + w_{\lambda'}(\mathrm{in}_{\lambda'}(\partial h_0/\partial x_i) = d\mu_i - \mu_i + d\lambda_i.$$

Then the one-parameter subgroup $\lambda + \mu$ acting on $x_1, \dots, x_n$ with the weights $\lambda_1 + \mu_1, \dots, \lambda_n + \mu_n$ satisfies

$$\begin{aligned} w_{\lambda+\mu}(\mathrm{in}_{\lambda+\mu}(\partial h_0/\partial x_i)) = w_\lambda(\mathrm{in}_\lambda(\partial h_0/\partial x_i)) + w_\mu(\mathrm{in}_\mu(\partial h_0/\partial x_i)) = \\ d\mu_i - \mu_i + d\lambda_i + \mu_i = d(\mu_i + \lambda_i). \end{aligned}$$

Applying Lemma 4.6, we conclude that either h_0 is a direct sum, or

$$\lambda_i + \mu_i = 0 \quad \text{for all } i = 1, \dots, n.$$

In the latter case, it follows that λ is proportional to $\lambda' = d\lambda - \mu$. Since the limit of h under λ' exists and is equal to h_0, the limit under λ of h must exist and be equal to h_0 as well. Observing that the inverse of an upper-triangular matrix with 1's on the diagonal has the same form, we see that the limit of

$$f(x_1, \dots, x_n) = h(x_1 + c'_{12}x_2 + \cdots + c'_{1n}x_n, x_2 + \cdots + c'_{2n}x_n, \dots, x_n)$$

under λ also exists. This contradiction concludes the proof. □

Corollary 4.7. *The morphism $\nabla\colon \mathbb{P}(S_{d+1})^{ss} \to \mathrm{Grass}(n, S_d)^{ss}$ preserves polystability.*

Proof. Suppose $f = f_1 + \cdots + f_k$ is the maximally fine direct sum decomposition of a polystable form f, where $f_i \in \mathrm{Sym}^{d+1} V_i$, and where $V = \oplus_{i=1}^k V_i$. Then each f_i is polystable and not a direct sum in $\mathrm{Sym}^{d+1} V_i$. Hence $\nabla([f_i])$ is polystable with respect to the $\mathrm{SL}(V_i)$-action.

Since $\Theta(f) \subset \mathrm{Stab}(\nabla([f]))$ is a reductive subgroup, to prove that $\nabla([f])$ is polystable, it suffices to verify that $\nabla([f])$ is polystable with respect to the centralizer $C_{\mathrm{SL}(n)}(\Theta(f))$ of $\mathrm{Stab}(\Theta(f))$ in $\mathrm{SL}(n)$, see [14, Corollaire 1 and Remarque 1]. We have

$$C_{\mathrm{SL}(n)}(\Theta(f)) = (\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_k)) \cap \mathrm{SL}(n).$$

Arguing as on [9, p. 456], we see that every one-parameter subgroup λ of $C_{\mathrm{SL}(n)}(\Theta(f))$ can be renormalized to a one-parameter subgroup of $\mathrm{SL}(V_1)\times\cdots\times\mathrm{SL}(V_k)$ without changing its action on $\nabla([f])$. Since $\nabla([f_i])$ is polystable with respect to $\mathrm{SL}(V_i)$, it follows that

$$\nabla([f]) = \nabla([f_1]) \oplus \cdots \oplus \nabla([f_k])$$

is polystable with respect to the action of λ thus proving the claim. $\square$

Proof of Theorem 2.1. Suppose that f is polystable, consider its maximally fine direct sum decomposition and the canonical torus $\Theta(f)$ in $\mathrm{Stab}(\nabla([f]))$ as constructed above. In what follows, we will write X to denote $\mathbb{P}(S_{d+1})^{ss}$ and Y to denote $\mathrm{Grass}(n, S_d)^{ss}$. Set $p := \pi_0([f]) \in X/\!/\,\mathrm{SL}(n)$.

We will prove that $\overline{\nabla}$ is unramified at p. Let $N_{[f]}$ be the normal space to the $\mathrm{SL}(n)$-orbit of $[f]$ in X at the point $[f]$, and $N_{\nabla([f])}$ the normal space to the $\mathrm{SL}(n)$-orbit of $\nabla([f])$ in Y at the point $\nabla([f])$. We have a natural map

$$\iota\colon N_{[f]} \to N_{\nabla([f])}$$

induced by the differential of ∇. The map ι is injective by Lemma 4.1.

Since both $[f]$ and $\nabla([f])$ have closed orbits in X and Y, respectively (see Corollary 4.7), to verify that $\overline{\nabla}$ is unramified at p, it suffices, by Luna's étale slice theorem, to prove that the morphism

$$s(f)\colon N_{[f]}/\!/\,\mathrm{Stab}([f]) \to N_{\nabla([f])}/\!/\,\mathrm{Stab}(\nabla([f]) \tag{4.2}$$

is unramified.

As ∇ is not necessarily stabilizer-preserving at $[f]$ (i.e., $\mathrm{Stab}([f])$ may not be equal to $\mathrm{Stab}(\nabla([f]))$), we cannot directly appeal to the injectivity of ι. Instead, consider the $\Theta(f)$-orbit, say F, of $[f]$ in X. Let $\mathcal{N}_{F/X}$ be the $\Theta(f)$-invariant normal bundle of F in X. Since by Lemma 4.1 we have $\nabla^{-1}(\nabla([f])) = F$, there is a natural $\Theta(f)$-equivariant map $J : \mathcal{N}_{F/X} \to N_{\nabla([f])}$. We now make a key observation that for the induced map $\tilde{J} : \mathcal{N}_{F/X}/\!/\Theta(f) \to N_{\nabla([f])}$ one has

$$\tilde{J}(\mathcal{N}_{F/X}/\!/\Theta(f)) = \iota\left(N_{[f]}\right).$$

Since $\overline{\nabla}$ is finite by [9, Proposition 2.1], the morphism $s(f)$ from Equation (4.2) is quasi-finite. Applying Lemma 4.8 (proved below), with $\mathrm{Spec}\, A = N_{[f]}$, $\mathrm{Spec}\, B = N_{\nabla([f])}$, $T = \Theta(f)$, $H = \mathrm{Stab}([f])$, $G = \mathrm{Stab}(\nabla([f]))$, as well as Corollary 4.3, we obtain that $s(f)$ is in fact a closed immersion, and so is unramified. Note that here the group G is reductive by Matsushima's criterion. This proves that $\overline{\nabla}$ is unramified at p.

We now note that $\overline{\nabla}$ is injective. Indeed, this follows as in the proof of [9, Part (2) of Proposition 2.1] from Corollary 4.7 and the finiteness of $\overline{\nabla}$. We then conclude that $\overline{\nabla}$ is a closed immersion. $\square$

Lemma 4.8 (GIT lemma)**.** *Suppose G is a reductive group. Suppose $T \subset G$ is a connected reductive subgroup, and $H \subset G$ is a reductive subgroup such that G*

is generated by T and H. Suppose we have a G-equivariant closed immersion of normal affine schemes admitting an action of G

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B.$$

such that $\operatorname{Spec} A^H \to \operatorname{Spec} B^G$ *is quasi-finite. Then* $\operatorname{Spec} A^G \simeq \operatorname{Spec} A^H$ *and, consequently,* $\operatorname{Spec} A^H \to \operatorname{Spec} B^G$ *is a closed immersion.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} A^H \arrow[rr, hook] \arrow[d] \arrow[drr] & & \operatorname{Spec} B^H \arrow[d] \\
(\operatorname{Spec} A^H)/\!\!/T \simeq \operatorname{Spec} A^G \arrow[rr, hook] & & (\operatorname{Spec} B^H)/\!\!/T \simeq \operatorname{Spec} B^G.
\end{tikzcd}$$

Since the diagonal arrow is quasi-finite by assumption, and the bottom arrow is a closed immersion, we conclude that the GIT quotient $\operatorname{Spec} A^H \to (\operatorname{Spec} A^H)/\!\!/T$ is quasi-finite as well. Since this is a good quotient by a connected group, the morphism $\operatorname{Spec} A^H \to (\operatorname{Spec} A^H)/\!\!/T \simeq \operatorname{Spec} A^G$ must be an isomorphism. $\square$

Corollary 4.9 (Theorem 1.1)**.** *The morphism*

$$\bar{A}\colon \mathbb{P}(S_{d+1})_\Delta /\!\!/ \operatorname{SL}(n) \to \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!\!/ \operatorname{SL}(n)$$

is a locally closed immersion.

5. The morphism $\mathbf{A}_{\mathrm{Gr}}$

In this section, we prove Theorem 2.2. In fact, we study in detail the rational map $\bar{A}\colon (\mathbb{P}S_{d+1})^{ss} /\!\!/ \operatorname{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!\!/ \operatorname{SL}(n)$ in codimension one.

As in Section 2, fix $d \geq 2$. As always, we assume that $n \geq 2$ and disregard the trivial case $(n, d) = (2, 2)$. Given $U \in \operatorname{Grass}(n, S_d)$, we take I_U to be the ideal in S generated by the elements in U. Consider the following locus in $\operatorname{Grass}(n, S_d)$:

$$W_{n,d} = \{U \in \operatorname{Grass}(n, S_d) \mid \dim_{\mathbb{C}}(S/I_U)_{n(d-1)-1} = n\}.$$

Since $\dim_{\mathbb{C}}(S/I_U)_{n(d-1)-1}$ is an upper semi-continuous function on $\operatorname{Grass}(n, S_d)$ and for every $U \in \operatorname{Grass}(n, S_d)$ one has $\dim_{\mathbb{C}}(S/I_U)_{n(d-1)-1} \geq n$, we conclude that $W_{n,d}$ is an open subset of $\operatorname{Grass}(n, S_d)$. Moreover, since for $U \in \operatorname{Grass}(n, S_d)_{\mathrm{Res}}$ the ideal I_U is Gorenstein of socle degree $n(d-1)$, we have $\operatorname{Grass}(n, S_d)_{\mathrm{Res}} \subset W_{n,d}$.

Applying polar pairing, we obtain a morphism

$$\mathbf{A}_{\mathrm{Gr}}\colon W_{n,d} \to \operatorname{Grass}(n, \mathcal{D}_{n(d-1)-1}),$$
$$\mathbf{A}_{\mathrm{Gr}}(U) = \left[(I_U)^{\perp}_{n(d-1)-1} \subset \mathcal{D}_{n(d-1)-1}\right].$$

From the duality between Hilbert and gradient points it follows that

$$\nabla(\mathbf{A}(U)) = \mathbf{A}_{\mathrm{Gr}}(U) \text{ for every } U \in \operatorname{Grass}(n, S_d)_{\mathrm{Res}}.$$

We conclude that we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{P}(S_{d+1})^{ss}/\!\!/\,\mathrm{SL}(n) & & \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}/\!\!/\,\mathrm{SL}(n) \\
\mathbb{P}(S_{d+1})^{ss} \arrow[u, "\pi_0"] \arrow[d, "\nabla"'] & \mathbb{P}(S_{d+1})_\Delta \arrow[l, hook'] \arrow[r, "A"] \arrow[d, "\nabla"] & \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} \arrow[u, "\pi_2"'] \arrow[d, "\nabla"] \\
\mathrm{Grass}(n, S_d)^{ss} \arrow[d, "\pi_1"'] & \mathrm{Grass}(n, S_d)_{\mathrm{Res}} \arrow[l, hook'] \arrow[r] \arrow[ur, "\mathbf{A}"] \arrow[d, hook] & \mathrm{Grass}(n, \mathcal{D}_{n(d-1)-1})^{ss} \arrow[d, "\pi_3"] \\
\mathrm{Grass}(n, S_d)^{ss}/\!\!/\,\mathrm{SL}(n) & W_{n,d} \arrow[ur, "\mathbf{A}_{\mathrm{Gr}}"] & \mathrm{Grass}(n, \mathcal{D}_{n(d-1)-1})^{ss}/\!\!/\,\mathrm{SL}(n).
\end{tikzcd}$$

Proposition 5.1. *Suppose $U \in \mathrm{Grass}(n, S_d)$ is such that*

$$\mathbb{V}(I_U) = \{p_1, \dots, p_k\}$$

is scheme-theoretically a set of k distinct points in general linear position in $\mathbb{P}^{n-1}$. Then $U \in W_{n,d}$.

Remark 5.2. A set $\{p_1, \dots, p_k\}$ points in $\mathbb{P}^{n-1}$ is in general linear position if and only if $k \le n$, and, up to the $\mathrm{PGL}(n)$-action,

$$p_i = \{x_1 = \cdots = \widehat{x_i} = \cdots = x_n = 0\}, \quad i = 1, \dots, k,$$

in the homogeneous coordinates $[x_1 : \cdots : x_n]$ on $\mathbb{P}^{n-1}$.

Proof of Proposition 5.1. Since $\mathrm{depth}(I_U) = n - 1$, we can choose degree d generators $g_1, \dots, g_n$ of I_U such that $g_1, \dots, g_{n-1}$ form a regular sequence. Then $\Gamma := \mathbb{V}(g_1, \dots, g_{n-1})$ is a finite-dimensional subscheme of $\mathbb{P}^{n-1}$. By Bézout's theorem, Γ is a set of d^{n-1} points, counted with multiplicities.

Set $R := S/(g_1, \dots, g_{n-1})$. Consider the Koszul complex $K_\bullet := K_\bullet(g_1, \dots, g_n)$. We have

$$\mathrm{H}_0(K_\bullet) = S/(g_1, \dots, g_n) = S/I_U.$$

Since $g_1, \dots, g_{n-1}$ is a regular sequence, we also have

$$\mathrm{H}_i(K_\bullet) = 0 \quad \text{for all } i > 0$$

and

$$\mathrm{H}_1(K_\bullet) = \big(((g_1, \dots, g_{n-1}) :_S (g_1, \dots, g_n))/(g_1, \dots, g_{n-1})\big)(-d) \simeq \mathrm{Ann}_R(g_n)(-d).$$

To establish the identity

$$\mathrm{codim}\big((I_U)_{n(d-1)-1}, S_{n(d-1)-1}\big) = n$$

it suffices to prove

$$\mathrm{H}_1(K_\bullet)_{n(d-1)-1} = 0.$$

Indeed, in this case the graded degree $n(d-1)-1$ part of the Koszul complex will be an exact complex of vector spaces and so the dimension of $(S/I_U)_{n(d-1)-1}$ will coincide with that in the situation when $g_1, \dots, g_n$ is a regular sequence, that is, with n.

As we have already observed, we have

$$\mathrm{H}_1(K_\bullet)_{n(d-1)-1} = \mathrm{Ann}_R(g_n)_{n(d-1)-1}(-d) = \mathrm{Ann}_R(g_n)_{n(d-1)-1-d}.$$

Hence it suffices to prove that $\mathrm{Ann}_R(g_n)_{n(d-1)-1-d} = 0$. Write $\Gamma = \Gamma' \cup \Gamma''$, where $\Gamma' \neq \varnothing$ and $\Gamma'' := \{p_1, \dots, p_k\}$. Since g_n vanishes on all of Γ'' but does not vanish at any point of Γ', every element of $\mathrm{Ann}_R(g_n)_{n(d-1)-1-d}$ comes from a degree $n(d-1)-1-d$ form that vanishes on all of Γ'. We apply the Cayley-Bacharach Theorem [7, Theorem CB6], which implies the following statement:

Claim 5.3. *Set $s := d(n-1)-(n-1)-1 = n(d-1)-d$. If $r \leq s$ is a non-negative integer, then the dimension of the family of projective hypersurfaces of degree r containing Γ' modulo those containing all of Γ is equal to the failure of Γ'' to impose independent conditions on projective hypersurfaces of complementary degree $s-r$.*

In our situation $r = s-1$, and Γ'' imposes independent conditions on hyperplanes by the general linear position assumption. Hence we conclude by Claim 5.3 that every form of degree $n(d-1)-1-d$ that vanishes on all of Γ' also vanishes on all of Γ'' and therefore, as the ideal $(g_1, \dots, g_{n-1})$ is saturated, maps to 0 in R. We thus see that $\mathrm{Ann}_R(g_n)_{n(d-1)-1-d} = 0$. This finishes the proof. $\square$

Motivated by the result above, we consider the following partial stratification of the resultant divisor $\mathfrak{Res} \subset \mathrm{Grass}(n, S_d)$. For $1 \leq k \leq n$, define Z_k to be the locally closed subset of $\mathrm{Grass}(n, S_d)$ consisting of all subspaces U such that $\mathbb{V}(I_U)$ is scheme-theoretically a set of k distinct points in general linear position in $\mathbb{P}^{n-1}$. Clearly, Z_1 is dense in $\mathfrak{Res}$, and

$$\overline{Z}_k \supset Z_{k+1} \cup \cdots \cup Z_n.$$

We will also set $\Sigma_k := \nabla^{-1}(Z_k) \subset \mathbb{P}(S_{d+1})$. By the Jacobian criterion, Σ_k is the locus of hypersurfaces with only k ordinary double points in general linear position and no other singularities.

Lemma 5.4. *For every $1 \leq k \leq n$, one has that Z_k is a non-empty and irreducible subset of $\mathrm{Grass}(n, S_d)$, and Σ_k is a non-empty and irreducible subset of $\mathbb{P}(S_{d+1})^{ss}$.*

Proof. It follows from the Hilbert-Mumford numerical criterion that any hypersurface in $\mathbb{P}^{n-1}$ of degree $d+1$ with at worst ordinary double point singularities is semistable.

Having k singularities at k fixed points $p_1, \dots, p_k$ (resp., having k fixed base points $p_1, \dots, p_k$) in general linear position is a linear condition on the elements of $\mathbb{P}(S_{d+1})$ (resp., the elements of the Stiefel variety over $\mathrm{Grass}(n, S_d)$) and so defines an irreducible closed subvariety $\Sigma(p_1, \dots, p_k)$ in $\mathbb{P}(S_{d+1})$ (resp., $Z(p_1, \dots, p_k)$ in $\mathrm{Grass}(n, S_d)$). The property of having exactly ordinary double points at $p_1, \dots, p_k$ (resp., having the base locus being equal to $\{p_1, \dots, p_k\}$ scheme-theoretically) is an open condition in $\Sigma(p_1, \dots, p_k)$ in $\mathbb{P}(S_{d+1})$ (resp., $Z(p_1, \dots, p_k)$ in $\mathrm{Grass}(n, S_d)$)

and so defines an irreducible subvariety $\Sigma^0(p_1, \dots, p_k)$ (resp., $Z^0(p_1, \dots, p_k)$). We conclude the proof of irreducibility by noting that $\Sigma_k = \mathrm{PGL}(n) \cdot \Sigma^0(p_1, \dots, p_k)$ (resp., $Z_k = \mathrm{PGL}(n) \cdot Z^0(p_1, \dots, p_k)$).

Since $\Sigma_k = \nabla^{-1}(Z_k)$, it suffices to check the non-emptiness of Σ_k. If $F \in \Sigma_n$ has ordinary double points at $p_1, \dots, p_n$, then by the deformation theory of hypersurfaces, there exists a deformation of F with ordinary double points at $p_1, \dots, p_k$ and no other singularities. Indeed, if $G \in S_{d+1}$ is a general form vanishing at $p_1, \dots, p_k$ and non-vanishing at $p_{k+1}, \dots, p_n$, then $F + tG \in \Sigma^0(p_1, \dots, p_k)$ will have ordinary double points at $p_1, \dots, p_k$ and no other singularities for $0 < t \ll 1$.

It remains to prove that Σ_n is non-empty. Indeed, the following is an element of Σ_n:

$$(d-1)(x_1+\cdots+x_n)^{d+1}-(d+1)(x_1+\cdots+x_n)^{d-1}(x_1^2+\cdots+x_n^2)+2(x_1^{d+1}+\cdots+x_n^{d+1}).$$

In fact, a generic linear combination of all degree $(d+1)$ monomials with the exception of x_i^{d+1}, for $i = 1, \dots, n$, and $x_i^d x_j$, for $i, j = 1, \dots, n$, $i < j$, is a form with precisely n ordinary double point singularities in general linear position. $\square$

By Proposition 5.1, we know that $\mathbf{A}_{\mathrm{Gr}}$ is defined at all points of $Z_1 \cup \cdots \cup Z_n$. In fact, we can explicitly compute $\mathbf{A}_{\mathrm{Gr}}(U)$ for all $U \in Z_n$, as well as the orbit closure of $\mathbf{A}_{\mathrm{Gr}}(U)$ for all $U \in Z_{n-1}$. We need a preliminary fact.

Proposition 5.5. *Suppose $U \in \mathrm{Grass}(n, S_d)$ and $p \in \mathbb{V}(I_U) \subset \mathbb{P}V^\vee$. Let $L \in V^\vee$ be a non-zero linear form corresponding to p. Then $L^{n(d-1)-1} \in (I_U)^\perp_{n(d-1)-1}$.*

Proof. Since $p \in \mathbb{V}(I_U)$, all elements of $(I_U)_{n(d-1)-1}$ vanish at p, and it follows that $F \circ L^{n(d-1)-1} = 0$ for all $F \in (I_U)_{n(d-1)-1}$ (cf. Remark 3.3). $\square$

Corollary 5.6. *Suppose $U \in Z_k$ is such that*

$$\mathbb{V}(I_U) = \{p_1 := [1 : 0 : \cdots : 0], p_2 := [0 : 1 : \cdots : 0], \dots, p_k := [0 : \cdots : 1 : \cdots : 0]\}.$$

Then

$$\mathbf{A}_{\mathrm{Gr}}(U) = \langle z_1^{n(d-1)-1}, \dots, z_k^{n(d-1)-1}, g_{k+1}(z_1, \dots, z_n), \dots, g_n(z_1, \dots, z_n)\rangle,$$

for some $g_{k+1}, \dots, g_n \in \mathcal{D}_{n(d-1)-1}$. In particular, for $U \in Z_n$ one has

$$\mathbf{A}_{\mathrm{Gr}}(U) = \langle z_1^{n(d-1)-1}, \dots, z_n^{n(d-1)}\rangle = \nabla\big(\big[z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}\big]\big).$$

Moreover, for a generic $U \in Z_k$, we have $\mathbf{A}_{\mathrm{Gr}}(U) \in \mathrm{Grass}(n, \mathcal{D}_{n(d-1)})_{\mathrm{Res}}$.

Proof. Since the point $p_i = \mathbb{V}(x_1, \dots, \widehat{x_i}, \dots, x_n) \in \mathbb{P}V^\vee$ corresponds to the linear form $z_i \in V^\vee$, Proposition 5.5 implies that $z_i^{n(d-1)-1} \in \mathbf{A}_{\mathrm{Gr}}(U)$ for every $i = 1, \dots, k$.

As $Z_n \subset \overline{Z}_k$ and $\mathbf{A}_{\mathrm{Gr}}(U) \in \mathrm{Grass}(n, \mathcal{D}_{n(d-1)})_{\mathrm{Res}}$ for every $U \in Z_n$, it follows that $\mathbf{A}_{\mathrm{Gr}}(U)$ is also spanned by a regular sequence for a generic $U \in Z_k$. The claim follows. $\square$

Consider the rational maps

$$\begin{tikzcd}
\mathbb{P}(S_{d+1})^{ss} /\!\!/ \mathrm{SL}(n) \arrow[rr, dashed, "\bar{A}"] \arrow[dr, "\overline{\nabla}"'] & & \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!\!/ \mathrm{SL}(n) \\
& \mathrm{Grass}(n, S_d)^{ss} /\!\!/ \mathrm{SL}(n) \arrow[ur, dashed, "\bar{\mathbf{A}}"'] &
\end{tikzcd}$$

of projective GIT quotients.

Theorem 5.7. *There is a dense open subset Y_k of Z_k such that*

$$\bar{\mathbf{A}}\colon \mathrm{Grass}(n, S_d)^{ss} /\!\!/ \mathrm{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!\!/ \mathrm{SL}(n)$$

is defined on $\pi_1(Y_k)$, $k = 1, \dots, n$. Moreover, for $U \in Y_k$ the value $\bar{\mathbf{A}}(\pi_1(U))$ is the image under π_2 of a polystable k-partial Fermat form. In particular, for every $U \in Z_n$ and for a generic $U \in Z_{n-1}$

$$\bar{\mathbf{A}}(\pi_1(U)) = \pi_2\left(z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}\right).$$

Proof. Recall that Z_k is non-empty by Lemma 5.4. Suppose $U \in Z_k$ is generic, then by Corollary 5.6 in suitable coordinates we have

$$\mathbf{A}_{\mathrm{Gr}}(U) = \langle z_1^{n(d-1)-1}, \dots, z_k^{n(d-1)-1}, g_{k+1}(z_1, \dots, z_n), \dots, g_n(z_1, \dots, z_n)\rangle,$$

and $\mathbf{A}_{\mathrm{Gr}}(U) \notin \mathfrak{Res}$. It follows (as in the proof of [10, Proposition 2.7]) that the closure of the $\mathrm{SL}(n)$-orbit of $\mathbf{A}_{\mathrm{Gr}}(U)$ contains

$$\langle z_1^{n(d-1)-1}, \dots, z_k^{n(d-1)-1}, \bar{g}_{k+1}(z_{k+1}, \dots, z_n), \dots, \bar{g}_n(z_{k+1}, \dots, z_n)\rangle, \tag{5.1}$$

where $\bar{g}_i := g_i(0, \dots, 0, z_{k+1}, \dots, z_n)$ for $i = k+1, \dots, n$. Then the claim follows for for $k = n-1$ and $k = n$ as in these cases we necessarily have $\bar{g}_n = z_n^{n(d-1)-1}$.

For k arbitrary, since $\overline{\nabla}$ is a closed immersion by Theorem 2.1, we conclude that $\bar{\mathbf{A}}$ is defined at $\pi_1(U)$. Let $F \in \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}$ be a polystable element with $\pi_2(F) = \bar{\mathbf{A}}(\pi_1(U))$. Then we must have $\nabla(F) \in \overline{\mathrm{SL}(n) \cdot \mathbf{A}_{\mathrm{Gr}}(U)}$, and so $\nabla(F)$ is linearly equivalent to an element of the form (5.1). It follows at once that

$$\bar{\mathbf{A}}(\pi_1(U)) = \pi_2\left(z_1^{n(d-1)} + \cdots + z_k^{n(d-1)-1} + G(z_{k+1}, \dots, z_n)\right)$$

is the image under π_2 of a polystable k-partial Fermat form. □

We will now establish Theorem 2.2 as detailed in the next two corollaries.

Corollary 5.8. *The rational map*

$$\bar{A}\colon (\mathbb{P}S_{d+1})^{ss} /\!\!/ \mathrm{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss} /\!\!/ \mathrm{SL}(n)$$

is defined at a generic point of $\pi_0(\Sigma_{n-1})$ and at every point of $\pi_0(\Sigma_n)$. For a generic $f \in \Sigma_{n-1}$ and for every $f \in \Sigma_n$, we have

$$\bar{A}(\pi_0(f)) = \pi_2(z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}).$$

Corollary 5.9. *When $n = 2$, the rational map $\bar{A}$ contracts the discriminant divisor to a point (corresponding to the orbit of the Fermat form in $\mathcal{D}_{2d-4}$) for all $d \geq 3$. When $n = 3$, the rational map $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety if $d \geq 3$. More generally, for every $n \geq 4$ there exists d_0 such that for all $d \geq d_0$ the map $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety.*

Proof. Notice that Σ_1 is dense in the discriminant divisor Δ. Hence, for $n = 2$ the statement follows from Corollary 5.8.

When $n = 3$, Theorem 5.7 implies that $\bar{A}(\pi_0(\Sigma_1))$ lies in the locus of a 1-partial Fermat form in $\mathcal{D}_{3(d-1)}$. The linear equivalence classes of 1-partial ternary Fermat forms are in bijection with the linear equivalence classes of binary degree $3(d-1)$ forms. The dimension of this locus is $3d - 6$, which for $d \geq 3$ is strictly less than the dimension $\binom{d+3}{2} - 10$ of the discriminant divisor.

If $n \geq 4$, by Theorem 5.7 the set $\bar{A}(\pi_0(\Sigma_1))$ lies in the locus of a 1-partial Fermat form in $\mathcal{D}_{n(d-1)}$. The linear equivalence classes of 1-partial Fermat forms in n variables are in bijection with the linear equivalence classes of degree $n(d-1)$ forms in $n-1$ variables. The dimension of this locus is $\binom{n(d-1)+(n-2)}{n-2}$, which for sufficiently large d is strictly less than the dimension of the discriminant divisor $\binom{(d+1)+(n-1)}{n-1} - (n^2 + 1)$. $\square$

We conclude the paper with an alternative proof of the main fact of [1] (see Proposition 4.3 therein).

Corollary 5.10 (Generic smoothness of associated forms)**.** *The closure of* $\operatorname{Im} A$ *in* $\mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}$ *contains the orbit*

$$\mathrm{SL}(n) \cdot \left\{ z_1^{n(d-1)} + \cdots + z_n^{n(d-1)} \right\}$$

of the Fermat hypersurface. Consequently, $A(f)$ is a smooth form for a generic smooth $f \in S_{d+1}$.

Proof. By Corollary 5.8, we have

$$\pi_2(z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}) \in \operatorname{Im}(\bar{A}).$$

Since the orbit of the Fermat hypersurface is closed in $\mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}$, it lies in the closure of $\operatorname{Im} A$. $\square$

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