ASSOCIATED FORM MORPHISM

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Abstract. We study the geometry of the morphism that sends a smooth hypersurface of degree \(d + 1\) in \(\mathbb{P}^{n-1}\) to its associated hypersurface of degree \(n(d - 1)\) in the dual space \((\mathbb{P}^{n-1})^\vee\).

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1. Introduction

One of the first applications of Geometric Invariant Theory is a construction of the moduli space of smooth degree \(m\) hypersurfaces in a fixed projective space \(\mathbb{P}^{n-1}\) [15]. This moduli space is an affine GIT quotient

\[ U_{m,n} := (\mathbb{P}H^0(\mathbb{P}^{n-1}, \mathcal{O}(m)) \setminus \Delta) \sslash \text{PGL}(n), \]

where \(\Delta\) is the discriminant divisor parameterizing singular hypersurfaces. The GIT construction produces a natural compactification

\[ U_{m,n} \subset V_{m,n} := (\mathbb{P}H^0(\mathbb{P}^{n-1}, \mathcal{O}(m)))^{ss} \sslash \text{PGL}(n), \]

given by a categorical quotient of the locus of GIT semistable hypersurfaces. We call \(V_{m,n}\) the GIT compactification of \(U_{m,n}\).

The subject of this paper is a certain rational map \(V_{m,n} \to V_{n(m-2),n}\), where \(n \geq 2, m \geq 3\) and where we exclude the (trivial) case \((n, m) = (2, 3)\). While this map has a purely algebraic construction, which we shall recall soon, it has several surprising geometric properties that we establish in this paper. In particular, this rational map restricts to a locally closed immersion \(\tilde{A} : U_{m,n} \to V_{n(m-2),n}\), and often

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contracts the discriminant divisor in $V_{m,n}$. Consequently, the closure of the image of $\bar{A}$ in $V_{n(m-2),n}$ is a compactification of the GIT moduli space $U_{m,n}$ that is different from the GIT compactification $V_{m,n}$.

To define $\bar{A}$, we consider the associated form morphism defined on the space of smooth homogeneous forms $f \in \mathbb{C}[x_1, \ldots, x_n]$ of fixed degree $m \geq 3$. Given such an $f$, its associated form $A(f)$ is a degree $n(m-2)$ homogeneous form in the graded dual polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$. In our recent paper [10], we proved that the associated form $A(f)$ is always polystable in the sense of GIT. Consequently, we obtain a morphism $\bar{A}$ from $U_{m,n}$ to $V_{n(m-2),n}$ sending the image of $f$ in $U_{m,n}$ to the image of $A(f)$ in $V_{n(m-2),n}$.

Our first result is that the morphism $\bar{A}$ is an isomorphism onto its image, a locally closed subvariety in the target.

**Theorem 1.1.** The morphism

$$\bar{A}: U_{m,n} \to V_{n(m-2),n}$$

is a locally closed immersion.

In the process of establishing Theorem 1.1, we generalize results of [2] to the case of an arbitrary number of variables, and, in particular, prove that the auxiliary gradient morphism sending a semistable form to the span of its partial derivatives gives rise to a closed immersion on the level of quotients (see Theorem 2.1).

Our second main result is Theorem 2.2, which describes the rational map $\bar{A}: V_{m,n} \dashrightarrow V_{n(m-2),n}$ in codimension one. Namely, we study how $\bar{A}$ extends to the generic point of the discriminant divisor in the GIT compactification (see Corollary 5.8), and prove that for $n = 2, 3$ and $m \geq 4$, as well as for $n \geq 4$, $m \gg 0$, the morphism $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety in the target (see Corollary 5.9). In the process, we prove that the image of $\bar{A}$ contains the orbit of the Fermat hypersurface in its closure and as a result obtain a new proof of the generic smoothness of associated forms (see Corollary 5.10).

1.1. **Notation and conventions.** Let $S := \text{Sym } V \simeq \mathbb{C}[x_1, \ldots, x_n]$ be a symmetric algebra of an $n$-dimensional vector space $V$, with its standard grading. Let $D := \text{Sym } V^\vee \simeq \mathbb{C}[z_1, \ldots, z_n]$ be the graded dual of $S$, with the structure of the $S$-module given by the polar pairing $S \times D \to D$, which is defined by

$$g(x_1, \ldots, x_n) \circ F(z_1, \ldots, z_n) := g(\partial/\partial z_1, \ldots, \partial/\partial z_1)F(z_1, \ldots, z_n).$$

A homogeneous polynomial $f \in S_m$ is called a direct sum if, after a linear change of variables, it can be written as the sum of two non-zero polynomials in disjoint sets of variables:

$$f = f_1(x_1, \ldots, x_a) + f_2(x_{a+1}, \ldots, x_n).$$

We will use the recognition criteria for direct sums established in [8], and so we keep the pertinent terminology of that paper. We will say that $f \in S_m$ is a $k$-partial
Fermat form for some \( k \leq n \), if, after a linear change of variables, it can be written as follows:

\[
f = x_1^m + \cdots + x_k^m + g(x_{k+1}, \ldots, x_n).
\]

Clearly, any \( n \)-partial Fermat form is linearly equivalent to the standard Fermat form. Furthermore, all \( k \)-partial Fermat forms are direct sums. We denote by \( \mathcal{DS}_m \) the locus of direct sums in \( S_m \).

2. Associated form of a balanced complete intersection

Fix \( d \geq 2 \). In what follows the trivial case \((n, d) = (2, 2)\) will be excluded. A length \( n \) regular sequence \( g_1, \ldots, g_n \) of elements of \( S_d \) will be called a balanced complete intersection of type \((d)^n\). It defines a graded Gorenstein Artin \( \mathbb{C} \)-algebra \( A(g_1, \ldots, g_n) := S/(g_1, \ldots, g_n) \), whose socle lies in degree \( n(d - 1) \). In \([2]\) an element \( A(g_1, \ldots, g_n) \in \mathcal{D}_{n(d-1)} \), called the associated form of \( g_1, \ldots, g_n \), was introduced. The form \( A(g_1, \ldots, g_n) \) is a homogeneous Macaulay inverse system, or a dual socle generator, of the algebra \( A(g_1, \ldots, g_n) \). It follows that \( [A(g_1, \ldots, g_n)] \in \mathbb{P}\mathcal{D}_{n(d-1)} \) depends only on the linear span \( \langle g_1, \ldots, g_n \rangle \), which we regard as a point in \( \text{Grass}(n, S_d) \).

Recall that \( g_1, \ldots, g_n \) is a regular sequence in \( S_d \) if and only if \( \langle g_1, \ldots, g_n \rangle \) does not lie in the resultant divisor \( \text{Res} \subset \text{Grass}(n, S_d) \). Setting \( \text{Grass}(n, S_d)_{\text{Res}} := \text{Grass}(n, S_d) \setminus \text{Res} \), we obtain a morphism

\[
A : \text{Grass}(n, S_d)_{\text{Res}} \to \mathbb{P}\mathcal{D}_{n(d-1)}.
\]

Given \( f \in S_{d+1} \), the partial derivatives \( \partial f/\partial x_1, \ldots, \partial f/\partial x_n \) form a regular sequence if and only if \( f \) is non-degenerate. For a non-degenerate \( f \in S_{d+1} \), in \([1, 3]\) the associated form of \( f \) was defined to be

\[
A(f) := A(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \in \mathcal{D}_{n(d-1)}.
\]

Summarizing, we obtain a commutative diagram

\[
\begin{tikzcd}
\mathbb{P}(S_{d+1})_{\Delta} & \mathbb{P}(\mathcal{D}_{n(d-1)}) \\
\text{Grass}(n, S_d)_{\text{Res}} \ar[ru, \mathbf{A}] &
\end{tikzcd}
\]

where \( \mathbb{P}(S_{d+1})_{\Delta} \) denotes the complement to the discriminant divisor in \( \mathbb{P}(S_{d+1}) \) and \( \nabla \) is the morphism sending a form into the linear span of its first partial derivatives.

The above diagram is equivariant with respect to the standard \( \text{SL}(n) \)-actions on \( S \) and \( \mathcal{D} \). By \([2]\), the morphism \( \mathbf{A} \) is a locally closed immersion, and it was proved in \([10]\) that \( \mathbf{A} \) sends polystable orbits to polystable orbits. Passing to the GIT
quotients, we thus obtain a commutative diagram

\[
\begin{array}{ccccc}
\mathbb{P}(S_{d+1}) & \xrightarrow{\bar{\nabla}} & \mathbb{P}(D_{n(d-1)}) & \xrightarrow{\bar{A}} & SL(n) \\
\Delta & \nearrow & & \searrow & \\
\text{Grass}(n, S_d) & \xrightarrow{\nabla} & \text{Res} & \xrightarrow{A} & SL(n),
\end{array}
\]

(2.1)

where \(\bar{\nabla} := \nabla//SL(n)\) is a finite injective morphism (see [9]) and \(\bar{A} := A//SL(n)\) is a locally closed immersion. The main focus of this paper is the geometry of diagram (2.1).

Noting that by [9] the map \(\nabla\) extends to a morphism from \(\mathbb{P}(S_{d+1})^s\) to \(\text{Grass}(n, S_d)^s\) and thus induces a map \(\nabla\) of the corresponding GIT quotients, we will now state our two main results as follows:

**Theorem 2.1.** The morphism \(\bar{\nabla}: \mathbb{P}(S_{d+1})^s//SL(n) \to \text{Grass}(n, S_d)^s//SL(n)\) is a closed immersion.

**Theorem 2.2.** The rational map

\[
\bar{A}: \mathbb{P}(S_{d+1})^s//SL(n) \dashrightarrow \mathbb{P}(D_{n(d-1)})^s//SL(n)
\]

extends to the generic point of the discriminant divisor \(\Delta//SL(n)\) in the GIT compactification and contracts the discriminant divisor to a lower-dimensional variety for all sufficiently large \(d\) as described in Corollaries 5.8 and 5.9.

### 3. Preliminaries on dualities

In this section we collect results on Macaulay inverse systems of graded Gorenstein Artin \(\mathbb{C}\)-algebras. We also recall the duality between the Hilbert points of such algebras and the gradient points of their inverse systems.

Recall that we regard \(S = \mathbb{C}[x_1, \ldots, x_n]\) as a ring of polynomial differential operators on the graded dual ring \(D := \mathbb{C}[z_1, \ldots, z_n]\) via polar pairing (1.1). For every positive \(m\), the restricted pairing

\[
S_m \times D_m \to \mathbb{C}
\]

is perfect and so defines an isomorphism

\[
D_m \cong S_m^\vee,
\]

where, as usual, \(V^\vee\) stands for the dual of a vector space \(V\).

Given \(W \subset D\), we define

\[
W^\perp := \{ f \in S \mid f \circ g = 0, \text{ for all } g \in W \} \subset S.
\]

Similarly given \(U \subset S\), we define

\[
U^\perp := \{ g \in D \mid f \circ g = 0, \text{ for all } f \in U \} \subset D.
\]
Claim 3.1. Isomorphism (3.1) sends an element \( \omega \in S_m^\nu \) to the element
\[
\mathcal{D}_\omega := \sum_{i_1 + \cdots + i_n = m} \frac{\omega(x_1^{i_1} \cdots x_n^{i_n})}{i_1! \cdots i_n!} z_1^{i_1} \cdots z_n^{i_n} \in D_m.
\]

Conversely, an element \( g \in D_m \) is mapped by isomorphism (3.1) to the projection
\[
S_m \to S_m/(g^\perp)_m \simeq \mathbb{C},
\]
where the isomorphism with \( \mathbb{C} \) is chosen so that \( 1 \in \mathbb{C} \) pairs to \( 1 \) with \( g \).

Proof. One observes that \( f \circ \mathcal{D}_\omega = \omega(f) \) for every \( f \in S_m \), and the first part of the claim follows. The second part is immediate from definitions. \( \square \)

Corollary 3.2. Given \( \omega \in S_m^\nu \), for every \( (a_1, \ldots, a_n) \in \mathbb{C}^n \) we have
\[
(3.2) \quad \mathcal{D}_\omega(a_1, \ldots, a_n) = \omega((a_1 x_1 + \cdots + a_n x_n)^m/m!).
\]

Proof. \[
\omega((a_1 x_1 + \cdots + a_n x_n)^m/m!) = \left( \frac{a_1 x_1 + \cdots + a_n x_n}{m!} \right) \circ \mathcal{D}_\omega = \left( \frac{a_1 \partial/\partial z_1 + \cdots + a_n \partial/\partial z_n}{m!} \right) \mathcal{D}_\omega = \mathcal{D}_\omega(a_1, \ldots, a_n),
\]
where the last equality is easily checked, say on monomials. \( \square \)

Remark 3.3. It follows from Corollary 3.2 that all forms in a subset \( W \subset D_m \) vanish at a given point \( (a_1, \ldots, a_n) \in \mathbb{C}^n \) if and only if \( (a_1 x_1 + \cdots + a_n x_n)^m \in W^\perp \).

Notice that the maps
\[
[(\mathcal{D}_\omega) \subset D_m] \mapsto [(\mathcal{D}_\omega)^\perp_m \subset S_m] = [\ker(\omega) \subset S_m]
\]
define isomorphisms
\[
\text{Grass}(1, D_m) \simeq \text{Grass}(\dim_{\mathbb{C}} S_m - 1, S_m).
\]
More generally, for any \( 1 \leq m \leq (m+n-1)! - 1 \) the correspondence
\[
[W \subset D_m] \mapsto [(W^\perp)_m \subset S_m]
\]
yields an isomorphism
\[
(3.3) \quad \text{Grass}(k, D_m) \simeq \text{Grass}(\dim_{\mathbb{C}} S_m - k, S_m).
\]

Let \( I \subset S \) be a Gorenstein ideal and \( \nu \) the socle degree of the algebra \( \mathcal{A} = S/I \). Recall that a (homogeneous) Macaulay inverse system of \( \mathcal{A} \) is an element \( f_\mathcal{A} \in D_\nu \) such that
\[
f_\mathcal{A}^\perp = I
\]
(see [11, Lemma 2.12] or [6, Exercise 21.7]). As \( (f_\mathcal{A}^\perp)_\nu = I_\nu \), we see that all Macaulay inverse systems are mutually proportional and \( (f_\mathcal{A})_\nu = ((I_\nu)^\perp)_\nu \). Clearly, the line \( \langle f_\mathcal{A} \rangle \in \text{Grass}(1, D_\nu) \) maps to the \( \nu \)th Hilbert point \( H_\nu \in \text{Grass}(\dim_{\mathbb{C}} S_\nu - 1, S_\nu) \) of \( \mathcal{A} \) under isomorphism (3.3) with \( k = 1 \).
Remark 3.4. Papers \cite{3, 4}, for any \( \omega \in S_{\nu}^\vee \) with \( \ker \omega = I_{\nu} \), introduced the associated form of \( \mathcal{A} \) as the element of \( \mathcal{D}_\nu \) given by the right-hand side of formula (3.2) with \( m = \nu \) (up to the factor \( \nu! \)). By Corollary 3.2, under isomorphism (3.3) with \( k = 1 \) the span of every associated form in \( \mathcal{D}_\nu \) also maps to the \( \nu \)-th Hilbert point \( H_{\nu} \in \text{Grass}(\dim \mathbb{C} S_{\nu} - 1, S_{\nu}) \) of \( \mathcal{A} \). In particular, for the algebra \( \mathcal{A} \) any associated form is simply one of its Macaulay inverse systems, and equation (3.2) with \( m = \nu \) and \( \ker \omega = I_{\nu} \) is an explicit formula for a Macaulay inverse system of \( \mathcal{A} \) (see \cite{12} for more details).

3.1. Gradient points. Given a polynomial \( F \in \mathcal{D}_m \), we define the \( p \)-th gradient point of \( F \) to be the linear span of all \( p \)-th partial derivatives of \( F \) in \( \mathcal{D}_{m-p} \). We denote the \( p \)-th gradient point by \( \nabla^p(F) \). Note that \( \nabla^p(F) = \{ g \circ F \mid g \in S_p \} \) is simply the \((m-p)\)-th graded piece of the principal \( S \)-module \( SF \). The 1st gradient point \( \nabla F := \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle \) will be called simply the gradient point of \( F \).

Proposition 3.5 (Duality between gradient and Hilbert points). The \( p \)-th gradient point of a Macaulay inverse system \( f_A \in \mathcal{D}_\nu \) maps to the \((\nu-p)\)-th Hilbert point \( H_{\nu-p} \) of \( \mathcal{A} \) under isomorphism (3.3).

Proof. Let \( G \) be the \( p \)-th gradient point of \( f_A \), that is

\[
G := \left\{ \frac{\partial^p}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}} f_A \mid i_1 + \cdots + i_n = p \right\}.
\]

We need to verify that \( I_{\nu-p} = (G^\perp)_{\nu-p} \). We have

\[
(G^\perp)_{\nu-p} = \left\{ f \in S_{\nu-p} \mid f \circ \frac{\partial^p}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}} f_A = 0 \text{ for all } i_1 + \cdots + i_n = p \right\}
\]

\[
= \left\{ f \in S_{\nu-p} \mid f x_1^{i_1} \cdots x_n^{i_n} \circ f_A = 0 \text{ for all degree } p \text{ monomials} \right\}
\]

\[
= \left\{ f \in S_{\nu-p} \mid x_1^{i_1} \cdots x_n^{i_n} f \in f_A^\perp \text{ for all degree } p \text{ monomials} \right\}
\]

\[
= I_{\nu-p},
\]

where the last equality comes from the fact that \( I \) is Gorenstein. \( \square \)

As a corollary of the above duality result, we recall in Proposition 3.6 below a generalization of \cite[Lemma 4.4]{1}. Although this statement is well-known (it appears, for example, in \cite[Proposition 4.1, p. 174]{5}), we provide a short proof for completeness. We first recall that a non-zero homogeneous form \( f \) in \( n \) variables has multiplicity \( \ell + 1 \) at a point \( p \in \mathbb{P}^{n-1} \) if and only if all partial derivatives of \( f \) of order \( \ell \) (hence of all orders \( \leq \ell \)) vanish at \( p \), and some partial derivative of \( f \) of order \( \ell + 1 \) does
not vanish at \( p \). We define the Veronese cone \( C_m \) to be the variety of all degree \( m \) powers of linear forms in \( S_m \):

\[
C_m := \{ L^m \mid L \in S_1 \} \subset S_m.
\]

**Proposition 3.6.** Let \( I \subset S \) be a Gorenstein ideal and \( \nu \) the socle degree of the algebra \( A = S/I \). Then a Macaulay inverse system \( f_A \) of \( A \) has a point of multiplicity \( \ell + 1 \) if and only if there exists a non-zero \( L \in S_1 \) such that \( L^{\nu - \ell} \in I_{\nu - \ell} \), and \( L^{\nu - \ell - 1} \not\in I_{\nu - \ell - 1} \). In particular, \( f_A \) has no points of multiplicity \( \ell + 1 \) or higher if and only if

\[
I_{\nu - \ell} \cap C_{\nu - \ell} = (0).
\]

**Proof.** By Proposition 3.5, the \( \ell \)th gradient point of \( f_A \) is dual to the \( (\nu - \ell) \)th Hilbert point of \( A 
\]

\[
H_{\nu - \ell} \colon S_{\nu - \ell} \twoheadrightarrow A_{\nu - \ell}.
\]

We conclude by Remark 3.3 that all partial derivatives of \( f_A \) of order \( \ell \) vanish at \( (a_1, \ldots, a_n) \) if and only if

\[
(a_1 x_1 + \cdots + a_n x_n)^{\nu - \ell} \in \ker H_{\nu - \ell} = I_{\nu - \ell}.
\]

It follows that \( L = a_1 x_1 + \cdots + a_n x_n \) satisfies \( L^\nu \in I_{\nu - \ell} \) and \( L^{\nu - \ell - 1} \not\in I_{\nu - \ell - 1} \) if and only if \( f_A \) has multiplicity exactly \( \ell + 1 \) at the point \( (a_1, \ldots, a_n) \). \( \square \)

4. The gradient morphism \( \nabla \)

In this section, we prove Theorem 2.1. Recall that we have the commutative diagram

\[
\begin{array}{ccc}
P(S_{d+1})^{ss} & \xrightarrow{\nabla} & \text{Grass}(n, S_d)^{ss} \\
\pi_0 & & \pi_1 \\
P(S_{d+1})^{ss} \times \text{SL}(n) & \xrightarrow{\nabla} & \text{Grass}(n, S_d)^{ss} \times \text{SL}(n).
\end{array}
\]

Let \( \mathfrak{D} \mathfrak{S}_{d+1}^{ss} := P(\mathfrak{D} \mathfrak{S}_{d+1})^{ss} \) be the locus of semistable direct sums in \( P(S_{d+1})^{ss} \). By [8, Section 3], the set \( \mathfrak{D} \mathfrak{S}_{d+1}^{ss} \) is precisely the closed locus in \( P(S_{d+1})^{ss} \) where \( \nabla \) has positive fiber dimension.

Suppose \( f \in S_{d+1} \) is a semistable form. Then, after a linear change of variables, we have a maximally fine direct sum decomposition

\[
f = \sum_{i=1}^{k} f_i(x^i),
\]

where \( V_i = \langle x^i \rangle \) are such that \( V = \bigoplus_{i=1}^{k} V_i \), and where each \( f_i \) is not a direct sum in \( \text{Sym} V_i \). Set \( n_i := \text{dim}_\mathbb{C} V_i \). We define the canonical torus \( \Theta(f) \subset \text{SL}(n) \) associated to \( f \) as the connected component of the identity of the subgroup

\[
\{ g \in \text{SL}(n) \mid V_i \text{ is an eigenspace of } g, \text{ for every } i = 1, \ldots, k \} \subset \text{SL}(n).
\]
Clearly, $\Theta(f) \simeq (\mathbb{C}^*)^{k-1}$, and since
$$\nabla([f]) = \nabla([f_1]) \oplus \cdots \oplus \nabla([f_k]), \text{ where } \nabla([f_i]) \in \text{Grass}(n, \text{Sym}^d V_i),$$
we also have $\Theta(f) \subset \text{Stab}(\nabla([f]))$, where $\text{Stab}$ denotes the stabilizer under the $\text{SL}(n)$-action.

From the definition of $\Theta(f)$, it is clear that $\Theta(f) \cdot [f] \subset \nabla^{-1}(\nabla([f]))$, and in fact [8, Corollary 3.12] gives a set-theoretic equality $\nabla^{-1}(\nabla([f])) = \Theta(f) \cdot [f]$. We will now obtain a stronger result:

**Lemma 4.1.** One has $\nabla^{-1}(\nabla([f])) = \Theta(f) \cdot [f]$ scheme-theoretically, or, equivalently,
$$\ker(d\nabla([f])) = T_{[f]}(\Theta(f) \cdot [f]),$$
where $T_{[f]}$ denotes the tangent space at $[f]$.

**Proof.** Under the standard identification of $T_{[f]}\mathbb{P}(S_{d+1})$ with $S_{d+1}/\langle f \rangle$, the subspace $T_{[f]}\Theta(f) \cdot [f]$ is identified with $\langle f_1, \ldots, f_k \rangle/\langle f \rangle$. It now suffices to show that every $g \in S_{d+1}$ that satisfies $\nabla[g] \subset \nabla[f]$ must lie in $\langle f_1, \ldots, f_k \rangle$, where $\nabla[g] := \langle \partial g/\partial x_1, \ldots, \partial g/\partial x_n \rangle \subset S_d$. This is precisely the statement of [8, Corollary 3.12]. \qed

We note an immediate consequence:

**Corollary 4.2.** If $f \in S_{d+1}^{ss}$ is not a direct sum, then $\nabla$ is unramified at $[f]$.

Further, since $\nabla$ is equivariant with respect to the $\text{SL}(n)$-action, we have the inclusion $\text{Stab}([f]) \subset \text{Stab}(\nabla([f]))$. As the following result shows, the difference between $\text{Stab}([f])$ and $\text{Stab}(\nabla([f]))$ is controlled by the torus $\Theta(f)$.

**Corollary 4.3.** The subgroup $\text{Stab}(\nabla([f]))$ is generated by $\Theta(f)$ and $\text{Stab}([f])$.

**Proof.** Suppose $\sigma \in \text{Stab}(\nabla([f]))$. Then $\nabla(\sigma \cdot [f]) = \nabla([f])$ implies by Lemma 4.1 that $\sigma \cdot [f] = \tau \cdot [f]$ for some $\tau \in \Theta(f)$. Consequently, $\tau^{-1} \circ \sigma \in \text{Stab}([f])$ as desired. \qed

Next, we obtain the following generalization of [2, Proposition 6.3], whose proof we follow almost verbatim.

**Proposition 4.4.** The morphism $\nabla$ is a closed immersion along the open locus $U := \mathbb{P}(S_{d+1})^{ss} \setminus \text{OS}_{d+1}^{ss}$ of all elements that are not direct sums.

**Proof.** Since for every $[f] \in U$ we have that $\nabla$ is unramified at $[f]$ and $\nabla^{-1}(\nabla([f])) = [f]$, it suffices to show that $\nabla$ is a finite morphism when restricted to $U$. Since, by [9], the induced morphism on the GIT quotients is finite, by [13, p. 89, Lemme] it suffices to verify that $\nabla$ is quasi-finite and that $\nabla$ sends closed orbits to closed orbits. The former has already been established, and the latter is proved below in Proposition 4.5. \qed

**Proposition 4.5.** Suppose $f \in S_{d+1}^{ss}$ is polystable and not a direct sum. Then the image $\nabla([f]) \subset \text{Grass}(n, S_d)^{ss}$ is polystable.
The above result is a generalization of [9, Theorem 1.1], whose method of proof we follow; we also keep the notation of loc.cit., especially as it relates to monomial orderings. We begin with a preliminary observation.

**Lemma 4.6.** Suppose \( f \in S_{d+1} \) is such that there exists a non-trivial one-parameter subgroup \( \lambda \) of \( \text{SL}(n) \) acting diagonally on \( x_1, \ldots, x_n \) with weights \( \lambda_1, \ldots, \lambda_n \) and satisfying
\[
 w_\lambda(\text{in}_\lambda(\partial f/\partial x_i)) = d\lambda_i.
\]
Then \( f \) is a direct sum.

**Proof.** We can assume that
\[
 \lambda_1 \leq \cdots \leq \lambda_a < \lambda_{a+1} = \cdots = \lambda_n
\]
for some \( 1 \leq a < n \). Then the fact that
\[
 w_\lambda(\text{in}_\lambda(\partial f/\partial x_i)) = d\lambda_i = d\lambda_n,
\]
for all \( i = a+1, \ldots, n \), implies
\[
 \partial f/\partial x_{a+1}, \ldots, \partial f/\partial x_n \in \mathbb{C}[x_{a+1}, \ldots, x_n].
\]
Consequently, \( f = g_1(x_1, \ldots, x_a) + g_2(x_{a+1}, \ldots, x_n) \) is a direct sum. \( \square \)

**Proof of Proposition 4.5.** Since \( f \) is polystable, by [9, Theorem 1.1] it follows that \( \nabla([f]) \) is semistable. Suppose \( \nabla([f]) \) is not polystable. Then there exists a one-parameter subgroup \( \lambda \) acting on the coordinates \( x_1, \ldots, x_n \) with the weights \( \lambda_1, \ldots, \lambda_n \) such that the limit of \( \nabla([f]) \) under \( \lambda \) exists and does not lie in the orbit of \( \nabla([f]) \). In particular, the limit of \( [f] \) under \( \lambda \) does not exist.

Then by [9, Lemma 3.5], there is an upper triangular unipotent coordinate change
\[
 x_1 \mapsto x_1 + c_{12} x_2 + \cdots + c_{1n} x_n,
 x_2 \mapsto x_2 + \cdots + c_{2n} x_n,
 \vdots
 x_n \mapsto x_n
\]
such that for the transformed form
\[
 h(x_1, \ldots, x_n) := f(x_1 + c_{12} x_2 + \cdots + c_{1n} x_n, x_2 + \cdots + c_{2n} x_n, \ldots, x_n)
\]
the initial monomials
\[
 \text{in}_\lambda(\partial h/\partial x_1), \ldots, \text{in}_\lambda(\partial h/\partial x_n)
\]
are distinct. Now, setting
\[
 \mu_i := w_\lambda(\text{in}_\lambda(\partial h/\partial x_i)),
\]
by [9, Lemma 3.2] we have
\[
 \mu_1 + \cdots + \mu_n = 0.
\]
It follows that with the respect to the one-parameter subgroup \( \lambda' \) acting on \( x_i \) with the weight \( d\lambda_i - \mu_i \), all monomials of \( h \) have non-negative weights (cf. [9, the proof of Lemma 3.6]). Write \( h = h_0 + h_1 \), where all monomials of \( h_0 \) have zero \( \lambda' \)-weights.
and all monomials of \( h_1 \) have positive \( \lambda' \)-weights. Then \( h_0 \in \overline{\SL(n) \cdot h} = \SL(n) \cdot h \), by the polystability assumption on \( f \). Furthermore, \( h_0 \) is stabilized by \( \lambda' \).

If \( \lambda' \) is a trivial one-parameter subgroup, then \( \mu_i = d \lambda_i \) for all \( i = 1, \ldots, n \), and by Lemma 4.6 the form \( h \) is a direct sum, which is a contradiction.

Suppose now that \( \lambda' \) is a non-trivial one-parameter subgroup. Clearly, we have

\[
    w_\lambda (\in\lambda (\partial h_0 / \partial x_i)) \geq w_\lambda (\in\lambda (\partial h / \partial x_i)),
\]

since the state of \( h_0 \) is a subset of the state of \( h \). If one of the inequalities above is strict, then \( \nabla(\in(h_0)) \) is destabilized by \( \lambda \), contradicting the semistability of \( \nabla(\in(h_0)) \) established in [9, Theorem 1.1]. Thus

\[
    w_\lambda (\in\lambda (\partial h_0 / \partial x_i)) = w_\lambda (\in\lambda (\partial h / \partial x_i)) = \mu_i.
\]

Moreover, since \( h_0 \) is \( \lambda' \)-invariant, we have that \( \partial h_0 / \partial x_i \) is homogeneous of degree \( -w_\lambda (x_i) = \mu_i - d \lambda_i \) with respect to \( \lambda' \). Let \( \mu \) be the one-parameter subgroup acting on \( x_1, \ldots, x_n \) with the weights \( \mu_1, \ldots, \mu_n \). It follows that

\[
    w_\mu (\in\mu (\partial h_0 / \partial x_i)) = d w_\lambda (\in\lambda (\partial h_0 / \partial x_i)) + w_\lambda (\in\lambda (\partial h_0 / \partial x_i)) = d \mu_i - \mu_i + d \lambda_i.
\]

Then the one-parameter subgroup \( \lambda + \mu \) acting on \( x_1, \ldots, x_n \) with the weights \( \lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n \) satisfies

\[
    w_{\lambda + \mu} (\in_{\lambda + \mu} (\partial h_0 / \partial x_i)) = w_\lambda (\in\lambda (\partial h_0 / \partial x_i)) + w_\mu (\in\mu (\partial h_0 / \partial x_i)) = d \mu_i - \mu_i + d \lambda_i + \mu_i = d (\mu_i + \lambda_i).
\]

Applying Lemma 4.6, we conclude that either \( h_0 \) is a direct sum, or

\[
    \lambda_i + \mu_i = 0 \quad \text{for all} \ i = 1, \ldots, n.
\]

In the latter case, it follows that \( \lambda \) is proportional to \( \lambda' = d \lambda - \mu \). Since the limit of \( h \) under \( \lambda' \) exists and is equal to \( h_0 \), the limit under \( \lambda \) of \( h \) must exist and be equal to \( h_0 \) as well. Observing that the inverse of an upper-triangular matrix with 1’s on the diagonal has the same form, we see that the limit of

\[
    f(x_1, \ldots, x_n) = h(x_1 + c'_{12} x_2 + \cdots + c'_{1n} x_n, x_2 + \cdots + c'_{2n} x_n, \ldots, x_n)
\]

under \( \lambda \) also exists. This contradiction concludes the proof.

**Corollary 4.7.** The morphism \( \nabla : \PP(S_{d+1})^{ss} \to \Grass(n, S_d)^{ss} \) preserves polystability.

**Proof.** Suppose \( f = f_1 + \cdots + f_k \) is the maximally fine direct sum decomposition of a polystable form \( f \), where \( f_i \in \Sym^{d+1} V_i \), and where \( V = \bigoplus_{i=1}^k V_i \). Then each \( f_i \) is polystable and not a direct sum in \( \Sym^{d+1} V_i \). Hence \( \nabla([f_i]) \) is polystable with respect to the \( \SL(V_i) \)-action.

Since \( \Theta(f) \subset \mathrm{Stab}(\nabla([f])) \) is a reductive subgroup, to prove that \( \nabla([f]) \) is polystable, it suffices to verify that \( \nabla([f]) \) is polystable with respect to the centralizer \( C_{\SL(n)}(\Theta(f)) \) of \( \mathrm{Stab}(\Theta(f)) \) in \( \SL(n) \), see [14, Corollaire 1 and Remarque 1]. We have

\[
    C_{\SL(n)}(\Theta(f)) = (\GL(V_1) \times \cdots \times \GL(V_k)) \cap \SL(n).
\]
Arguing as on [9, p. 456], we see that every one-parameter subgroup \( \lambda \) of \( C_{\text{SL}(n)}(\Theta(f)) \) can be renormalized to a one-parameter subgroup of \( \text{SL}(V_1) \times \cdots \times \text{SL}(V_k) \) without changing its action on \( \nabla([f]) \). Since \( \nabla([f]) \) is polystable with respect to \( \text{SL}(V_i) \), it follows that

\[
\nabla([f]) = \nabla([f_1]) \oplus \cdots \oplus \nabla([f_k])
\]

is polystable with respect to the action of \( \lambda \) thus proving the claim. \( \square \)

**Proof of Theorem 2.1.** Suppose that \( f \) is polystable, consider its maximally fine direct sum decomposition and the canonical torus \( \Theta(f) \) in \( \text{Stab}(\nabla([f])) \) as constructed above. In what follows, we will write \( X \to \text{Spec} \Theta(f) \) to verify that \( \text{Stab}([f]) \) is unramified. Applying Lemma 4.8 (proved below), with \( \text{Spec} A = N_{[f]} \) and \( \text{Spec} B = \nabla([f]) \), we see that every one-parameter subgroup can be renormalized to a one-parameter subgroup of \( \text{SL}(V_i) \). Since both \( \Theta(f) \)-invariant normal bundle of \( f \) in \( X \) at the point \( [f] \), and \( N_{\nabla([f])} \) the normal space to the \( \text{SL}(n) \)-orbit of \( \nabla([f]) \) in \( Y \) at the point \( \nabla([f]) \). We have a natural map

\[
\iota: N_{[f]} \to N_{\nabla([f])}
\]

induced by the differential of \( \nabla \). The map \( \iota \) is injective by Lemma 4.1.

Since both \( [f] \) and \( \nabla([f]) \) have closed orbits in \( X \) and \( Y \), respectively (see Corollary 4.7), to verify that \( \nabla \) is unramified at \( p \), it suffices, by Luna’s étale slice theorem, to prove that the morphism

\[
s(f): N_{[f]}//\text{Stab}([f]) \to N_{\nabla([f])}//\text{Stab}([f])
\]

is unramified.

As \( \nabla \) is not necessarily stabilizer-preserving at \( [f] \) (i.e., \( \text{Stab}([f]) \) may not be equal to \( \text{Stab}(\nabla([f])) \)), we cannot directly appeal to the injectivity of \( \iota \). Instead, consider the \( \Theta(f) \)-orbit, say \( F \), of \( [f] \) in \( X \). Let \( N_{F/X} \) be the \( \Theta(f) \)-invariant normal bundle of \( F \) in \( X \). Since by Lemma 4.1 we have \( \nabla^{-1}(\nabla([f])) = F \), there is a natural \( \Theta(f) \)-equivariant map \( J: N_{F/X} \to N_{\nabla([f])} \). We now make a key observation that for the induced map \( J: N_{F/X}//\Theta(f) \to N_{\nabla([f])} \) one has

\[
J(N_{F/X}//\Theta(f)) = \iota(N_{[f]}).
\]

Since \( \nabla \) is finite by [9, Proposition 2.1], the morphism \( s(f) \) from Equation (4.2) is quasi-finite. Applying Lemma 4.8 (proved below), with \( \text{Spec} A = N_{[f]} \), \( \text{Spec} B = N_{\nabla([f])} \), \( T = \Theta(f) \), \( H = \text{Stab}([f]) \), \( G = \text{Stab}(\nabla([f])) \), as well as Corollary 4.3, we obtain that \( s(f) \) is in fact a closed immersion, and so is unramified. Note that here the group \( G \) is reductive by Matsushima’s criterion. This proves that \( \nabla \) is unramified at \( p \).

We now note that \( \nabla \) is injective. Indeed, this follows as in the proof of [9, Part (2) of Proposition 2.1] from Corollary 4.7 and the finiteness of \( \nabla \). We then conclude that \( \nabla \) is a closed immersion. \( \square \)

**Lemma 4.8 (GIT lemma).** Suppose \( G \) is a reductive group. Suppose \( T \subset G \) is a connected reductive subgroup, and \( H \subset G \) is a reductive subgroup such that \( G \).
is generated by \( T \) and \( H \). Suppose we have a \( G \)-equivariant closed immersion of normal affine schemes admitting an action of \( G \)

\[
\text{Spec } A \hookrightarrow \text{Spec } B.
\]
such that \( \text{Spec } A^H \to \text{Spec } B^G \) is quasi-finite. Then \( \text{Spec } A^G \simeq \text{Spec } A^H \) and, consequently, \( \text{Spec } A^H \to \text{Spec } B^G \) is a closed immersion.

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Spec } A^H & \longrightarrow & \text{Spec } B^H \\
\downarrow & & \downarrow \\
(\text{Spec } A^H)\!//\!T & \simeq & (\text{Spec } A^G)\!//\!T
\end{array}
\]

Since the diagonal arrow is quasi-finite by assumption, and the bottom arrow is a closed immersion, we conclude that the GIT quotient \( \text{Spec } A^H \to (\text{Spec } A^H)\!//\!T \) is quasi-finite as well. Since this is a good quotient by a connected group, the morphism \( \text{Spec } A^H \to (\text{Spec } A^H)\!//\!T \simeq \text{Spec } A^G \) must be an isomorphism. \( \square \)

**Corollary 4.9** (Theorem 1.1). The morphism

\[
\bar{A} : \mathbb{P}(S_{d+1})_{\Delta} // \text{SL}(n) \to \mathbb{P}(D_{n(d-1)})^{ss} // \text{SL}(n)
\]
is a locally closed immersion.

5. **The Morphism \( \mathbf{A}_{\text{Gr}} \)**

In this section, we prove Theorem 2.2. In fact, we study in detail the rational map \( \bar{A} : (\mathbb{P}S_{d+1})^{ss} // \text{SL}(n) \to (\mathbb{P}D_{n(d-1)})^{ss} // \text{SL}(n) \) in codimension one.

As in Section 2, fix \( d \geq 2 \). As always, we assume that \( n \geq 2 \) and disregard the trivial case \((n, n) = (2, 2)\). Given \( U \in \text{Grass}(n, S_d) \), we take \( I_U \) to be the ideal in \( S \) generated by the elements in \( U \). Consider the following locus in \( \text{Grass}(n, S_d) \):

\[
W_{n,d} = \{ U \in \text{Grass}(n, S_d) \mid \dim_c(S/I_U)_{n(d-1)-1} = n \}.
\]

Since \( \dim_c(S/I_U)_{n(d-1)-1} \) is an upper semi-continuous function on \( \text{Grass}(n, S_d) \) and for every \( U \in \text{Grass}(n, S_d) \) one has \( \dim_c(S/I_U)_{n(d-1)-1} \geq n \), we conclude that \( W_{n,d} \) is an open subset of \( \text{Grass}(n, S_d) \). Moreover, since for \( U \in \text{Grass}(n, S_d) \) the ideal \( I_U \) is Gorenstein of socle degree \( n(d-1) \), we have \( \text{Grass}(n, S_d)_{\text{Res}} \subset W_{n,d} \).

Applying polar pairing, we obtain a morphism

\[
\mathbf{A}_{\text{Gr}} : W_{n,d} \to \text{Grass}(n, D_{n(d-1)-1}),
\]

\[
\mathbf{A}_{\text{Gr}}(U) = \left( (I_U)_{n(d-1)-1} \subset D_{n(d-1)-1} \right).
\]

From the duality between Hilbert and gradient points it follows that

\[
\nabla(\mathbf{A}(U)) = \mathbf{A}_{\text{Gr}}(U) \text{ for every } U \in \text{Grass}(n, S_d)_{\text{Res}}.
\]
We conclude that we have the commutative diagram:

\[
\begin{array}{ccc}
\mathbb{P}(S_{d+1})^{ss} & \xrightarrow{\pi_0} & \mathbb{P}(S_{d+1})^{ss} \\
\mathbb{P}(S_{d+1})^{ss} & \xleftarrow{\nabla} & \mathbb{P}(S_{d+1})^{ss} \\
\gamma & \xrightarrow{\nabla} & \gamma \\
\text{Grass}(n, S_d)^{ss} & \xleftarrow{\pi_1} & \text{Grass}(n, S_d)^{ss} \\
\text{Grass}(n, S_d)^{ss} & \xrightarrow{A} & \text{Grass}(n, D_{n(d-1)-1})^{ss} \\
W_{n,d} & \xrightarrow{A_{Gr}} & \text{Grass}(n, D_{n(d-1)-1})^{ss} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{P}(S_{d+1})^{ss} & \xrightarrow{\pi_0} & \mathbb{P}(D_{n(d-1)})^{ss} \\
\mathbb{P}(D_{n(d-1)})^{ss} & \xleftarrow{\nabla} & \mathbb{P}(D_{n(d-1)})^{ss} \\
\gamma & \xrightarrow{\nabla} & \gamma \\
\text{Grass}(n, S_d)^{ss} & \xleftarrow{\pi_1} & \text{Grass}(n, D_{n(d-1)-1})^{ss} \\
\text{Grass}(n, D_{n(d-1)-1})^{ss} & \xrightarrow{\pi_3} & \text{Grass}(n, D_{n(d-1)-1})^{ss} \\
\end{array}
\]

**Proposition 5.1.** Suppose \( U \in \text{Grass}(n, S_d) \) is such that

\[
\mathbb{V}(I_U) = \{p_1, \ldots, p_k\}
\]

is scheme-theoretically a set of \( k \) distinct points in general linear position in \( \mathbb{P}^{n-1} \). Then \( U \in W_{n,d} \).

**Remark 5.2.** A set \( \{p_1, \ldots, p_k\} \) points in \( \mathbb{P}^{n-1} \) is in general linear position if and only if \( k \leq n \), and, up to the \( \text{PGL}(n) \)-action,

\[
p_i = \{x_1 = \cdots = \hat{x}_i = \cdots = x_n = 0\}, \quad i = 1, \ldots, k,
\]

in the homogeneous coordinates \([x_1 : \cdots : x_n] \) on \( \mathbb{P}^{n-1} \).

**Proof of Proposition 5.1.** Since \( \text{depth}(I_U) = n - 1 \), we can choose degree \( d \) generators \( g_1, \ldots, g_n \) of \( I_U \) such that \( g_1, \ldots, g_{n-1} \) form a regular sequence. Then \( \Gamma := \mathbb{V}(g_1, \ldots, g_{n-1}) \) is a finite-dimensional subscheme of \( \mathbb{P}^{n-1} \). By Bézout’s theorem, \( \Gamma \) is a set of \( d^{n-1} \) points, counted with multiplicities.

Set \( R := S/(g_1, \ldots, g_{n-1}) \). Consider the Koszul complex \( K_\bullet := K_\bullet(g_1, \ldots, g_n) \). We have

\[
H_0(K_\bullet) = S/(g_1, \ldots, g_n) = S/I_U.
\]

Since \( g_1, \ldots, g_{n-1} \) is a regular sequence, we also have

\[
H_i(K_\bullet) = 0 \quad \text{for all } i > 0
\]

and

\[
H_1(K_\bullet) = \left( (g_1, \ldots, g_{n-1}) : s(g_1, \ldots, g_n)/(g_1, \ldots, g_{n-1}) \right)(-d) \simeq \text{Ann}_R(g_n)(-d).
\]

To establish the identity

\[
\text{codim}((I_U)_{n(d-1)-1}, S_{n(d-1)-1}) = n
\]

it suffices to prove

\[
H_1(K_\bullet)_{n(d-1)-1} = 0.
\]

Indeed, in this case the graded degree \( n(d-1) - 1 \) part of the Koszul complex will be an exact complex of vector spaces and so the dimension of \((S/I_U)_{n(d-1)-1}\) will
coincide with that in the situation when \(g_1, \ldots, g_n\) is a regular sequence, that is, with \(n\).

As we have already observed, we have
\[
H_1(K)_{n(d-1)-1} = \text{Ann}_R(g_n)_{n(d-1)-1}(-d) = \text{Ann}_R(g_n)_{n(d-1)-1-d}.
\]

Hence it suffices to prove that \(\text{Ann}_R(g_n)_{n(d-1)-1-d} = 0\). Write \(\Gamma = \Gamma' \cup \Gamma''\), where \(\Gamma' \neq \emptyset\) and \(\Gamma'' := \{p_1, \ldots, p_k\}\). Since \(g_n\) vanishes on all of \(\Gamma''\) but does not vanish at any point of \(\Gamma'\), every element of \(\text{Ann}_R(g_n)_{n(d-1)-1-d}\) comes from a degree \(n(d-1) - 1 - d\) form that vanishes on all of \(\Gamma'\). We apply the Cayley-Bacharach Theorem [7, Theorem CB6], which implies the following statement:

**Claim 5.3.** Set \(s := d(n-1) - (n-1) - 1 = n(d-1) - d\). If \(r \leq s\) is a non-negative integer, then the dimension of the family of projective hypersurfaces of degree \(r\) containing \(\Gamma'\) modulo those containing all of \(\Gamma\) is equal to the failure of \(\Gamma''\) to impose independent conditions on projective hypersurfaces of complementary degree \(s-r\).

In our situation \(r = s-1\), and \(\Gamma''\) imposes independent conditions on hyperplanes by the general linear position assumption. Hence we conclude by Claim 5.3 that every form of degree \(n(d-1) - 1 - d\) that vanishes on all of \(\Gamma'\) also vanishes on all of \(\Gamma''\) and therefore, as the ideal \((g_1, \ldots, g_{n-1})\) is saturated, maps to 0 in \(R\). We thus see that \(\text{Ann}_R(g_n)_{n(d-1)-1-d} = 0\). This finishes the proof.

Motivated by the result above, we consider the following partial stratification of the resultant divisor \(\mathcal{R}\) in \(\text{Grass}(n, S_d)\). For \(1 \leq k \leq n\), define \(Z_k\) to be the locally closed subset of \(\text{Grass}(n, S_d)\) consisting of all subspaces \(U\) such that \(\mathcal{R}(U)\) is scheme-theoretically a set of \(k\) distinct points in general linear position in \(\mathbb{P}^{n-1}\).

Clearly, \(Z_1\) is dense in \(\mathcal{R}\), and
\[
Z_k \supset Z_{k+1} \cup \cdots \cup Z_n.
\]

We will also set \(\Sigma_k := \nabla^{-1}(Z_k) \subset \mathbb{P}(S_{d+1})\). By the Jacobian criterion, \(\Sigma_k\) is the locus of hypersurfaces with only \(k\) ordinary double points in general linear position and no other singularities.

**Lemma 5.4.** For every \(1 \leq k \leq n\), one has that \(Z_k\) is a non-empty and irreducible subset of \(\text{Grass}(n, S_d)\), and \(\Sigma_k\) is a non-empty and irreducible subset of \(\mathbb{P}(S_{d+1})^{ss}\).

**Proof.** It follows from the Hilbert-Mumford numerical criterion that any hypersurface in \(\mathbb{P}^{n-1}\) of degree \(d+1\) with at worst ordinary double point singularities is semistable.

Having \(k\) singularities at \(k\) fixed points \(p_1, \ldots, p_k\) (resp., having \(k\) fixed base points \(p_1, \ldots, p_k\)) in general linear position is a linear condition on the elements of \(\mathbb{P}(S_{d+1})\) (resp., the elements of the Stiefel variety over \(\text{Grass}(n, S_d)\)) and so defines an irreducible closed subvariety \(\Sigma(p_1, \ldots, p_k)\) in \(\mathbb{P}(S_{d+1})\) (resp., \(Z(p_1, \ldots, p_k)\) in \(\text{Grass}(n, S_d)\)). The property of having exactly ordinary double points at \(p_1, \ldots, p_k\) (resp., having the base locus being equal to \(\{p_1, \ldots, p_k\}\) scheme-theoretically) is an open condition in \(\Sigma(p_1, \ldots, p_k)\) in \(\mathbb{P}(S_{d+1})\) (resp., \(Z(p_1, \ldots, p_k)\) in \(\text{Grass}(n, S_d)\))
and so defines an irreducible subvariety $\Sigma^0(p_1, \ldots, p_k)$ (resp., $Z^0(p_1, \ldots, p_k)$). We conclude the proof of irreducibility by noting that $\Sigma_k = \text{PGL}(n) \cdot \Sigma^0(p_1, \ldots, p_k)$ (resp., $Z_k = \text{PGL}(n) \cdot Z^0(p_1, \ldots, p_k)$).

Since $\Sigma_k = \nabla^{-1}(Z_k)$, it suffices to check the non-emptiness of $\Sigma_k$. If $F \in \Sigma_n$ has ordinary double points at $p_1, \ldots, p_n$, then by the deformation theory of hypersurfaces, there exists a deformation of $F$ with ordinary double points at $p_1, \ldots, p_k$ and no other singularities. Indeed, if $G \in S_{d+1}$ is a general form vanishing at $p_1, \ldots, p_k$ and non-vanishing at $p_{k+1}, \ldots, p_n$, then $F + tG \in \Sigma^0(p_1, \ldots, p_k)$ will have ordinary double points at $p_1, \ldots, p_k$ and no other singularities for $0 < t \ll 1$.

It remains to prove that $\Sigma_n$ is non-empty. Indeed, the following is an element of $\Sigma_n$:

$$(d-1)(x_1 + \cdots + x_n)^{d+1} - (d+1)(x_1 + \cdots + x_n)^{d-1}(x_1^2 + \cdots + x_n^2) + 2(x_1^{d+1} + \cdots + x_n^{d+1}).$$

In fact, a generic linear combination of all degree $(d+1)$ monomials with the exception of $x_i^{d+1}$, for $i = 1, \ldots, n$, and $x_i^2 x_j$, for $i, j = 1, \ldots, n, i < j$, is a form with precisely $n$ ordinary double point singularities in general linear position. □

By Proposition 5.1, we know that $A_{\text{Gr}}$ is defined at all points of $Z_1 \cup \cdots \cup Z_n$. In fact, we can explicitly compute $A_{\text{Gr}}(U)$ for all $U \in Z_n$, as well as the orbit closure of $A_{\text{Gr}}(U)$ for all $U \in Z_{n-1}$. We need a preliminary fact.

**Proposition 5.5.** Suppose $U \in \text{Grass}(n, S_d)$ and $p \in \nabla(I_U) \subset \mathbb{P}V^\vee$. Let $L \in V^\vee$ be a non-zero linear form corresponding to $p$. Then $L^{n(d-1)} \in (I_U)^{\perp}_{n(d-1)-1}$.

**Proof.** Since $p \in \nabla(I_U)$, all elements of $(I_U)_{n(d-1)-1}$ vanish at $p$, and it follows that $F \circ L^{n(d-1)-1} = 0$ for all $F \in (I_U)_{n(d-1)-1}$ (cf. Remark 3.3). □

**Corollary 5.6.** Suppose $U \in Z_k$ is such that

$$\nabla(I_U) = \{p_1 := [1 : 0 : \cdots : 0], p_2 := [0 : 1 : \cdots : 0], \ldots, p_k := [0 : \cdots : 1 : \cdots : 0]\}.$$

Then

$$A_{\text{Gr}}(U) = \langle z_1^{n(d-1)-1}, \ldots, z_k^{n(d-1)-1}, g_{k+1}(z_1, \ldots, z_n), \ldots, g_n(z_1, \ldots, z_n) \rangle,$$

for some $g_{k+1}, \ldots, g_n \in D_{n(d-1)-1}$. In particular, for $U \in Z_n$ one has

$$A_{\text{Gr}}(U) = \langle z_1^{n(d-1)-1}, \ldots, z_n^{n(d-1)} \rangle = \nabla([z_1^{n(d-1)-1} + \cdots + z_n^{n(d-1)}]).$$

Moreover, for a generic $U \in Z_k$, we have $A_{\text{Gr}}(U) \in \text{Grass}(n, D_{n(d-1)})_{\text{Res}}$.

**Proof.** Since the point $p_i = \nabla(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \in \mathbb{P}V^\vee$ corresponds to the linear form $z_i \in V^\vee$, Proposition 5.5 implies that $z_i^{n(d-1)-1} \in A_{\text{Gr}}(U)$ for every $i = 1, \ldots, k$.

As $Z_n \subset Z_k$ and $A_{\text{Gr}}(U) \in \text{Grass}(n, D_{n(d-1)})_{\text{Res}}$ for every $U \in Z_n$, it follows that $A_{\text{Gr}}(U)$ is also spanned by a regular sequence for a generic $U \in Z_k$. The claim follows. □
Consider the rational maps
\[
\mathbb{P}(S_{d+1})^{ss}/\text{SL}(n) \xrightarrow{\nabla} \mathbb{A} \xrightarrow{\pi} \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}/\text{SL}(n)
\]
of projective GIT quotients.

**Theorem 5.7.** There is a dense open subset $Y_k$ of $Z_k$ such that
\[
\mathbb{A}: \text{Grass}(n, S_d)^{ss}/\text{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}/\text{SL}(n)
\]
is defined on $\pi_1(Y_k)$, $k = 1, \ldots, n$. Moreover, for $U \in Y_k$ the value $\mathbb{A}(\pi_1(U))$ is the image under $\pi_2$ of a polystable $k$-partial Fermat form. In particular, for every $U \in Z_n$ and for a generic $U \in Z_{n-1}$
\[
\mathbb{A}(\pi_1(U)) = \pi_2 \left( z_1^{n(d-1)} + \cdots + z_n^{n(d-1)} \right).
\]

**Proof.** Recall that $Z_k$ is non-empty by Lemma 5.4. Suppose $U \in Z_k$ is generic, then by Corollary 5.6 in suitable coordinates we have
\[
\mathbb{A}_{\text{Gr}}(U) = \langle z_1^{n(d-1)-1}, \ldots, z_k^{n(d-1)-1}, g_{k+1}(z_1, \ldots, z_n), \ldots, g_n(z_1, \ldots, z_n) \rangle,
\]
and $\mathbb{A}_{\text{Gr}}(U) \notin \mathfrak{Res}$. It follows (as in the proof of [10, Proposition 2.7]) that the closure of the $\text{SL}(n)$-orbit of $\mathbb{A}_{\text{Gr}}(U)$ contains
\[
\langle z_1^{n(d-1)-1}, \ldots, z_k^{n(d-1)-1}, \tilde{g}_{k+1}(z_{k+1}, \ldots, z_n), \ldots, \tilde{g}_n(z_{k+1}, \ldots, z_n) \rangle,
\]
where $\tilde{g}_i := g_i(0, \ldots, 0, z_{k+1}, \ldots, z_n)$ for $i = k+1, \ldots, n$. Then the claim follows for
\[
\text{for } k = n-1 \text{ and } k = n \text{ as in these cases we necessarily have } \tilde{g}_n = z_n^{n(d-1)-1}.
\]

For $k$ arbitrary, since $\nabla$ is a closed immersion by Theorem 2.1, we conclude that $\mathbb{A}$ is defined at $\pi_1(U)$. Let $F \in \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}$ be a polystable element with $\pi_2(F) = \mathbb{A}(\pi_1(U))$. Then we must have $\nabla(F) \in \text{SL}(n) \cdot \mathbb{A}_{\text{Gr}}(U)$, and so $\nabla(F)$ is linearly equivalent to an element of the form (5.1). It follows at once that
\[
\mathbb{A}(\pi_1(U)) = \pi_2 \left( z_1^{n(d-1)} + \cdots + z_k^{n(d-1)-1} + G(z_{k+1}, \ldots, z_n) \right)
\]
is the image under $\pi_2$ of a polystable $k$-partial Fermat form. 

We will now establish Theorem 2.2 as detailed in the next two corollaries.

**Corollary 5.8.** The rational map
\[
\tilde{A}: \mathbb{P}(S_{d+1})^{ss}/\text{SL}(n) \dashrightarrow \mathbb{P}(\mathcal{D}_{n(d-1)})^{ss}/\text{SL}(n)
\]
is defined at a generic point of $\pi_0(\Sigma_{n-1})$ and at every point of $\pi_0(\Sigma_{n})$. For a generic $f \in \Sigma_{n-1}$ and for every $f \in \Sigma_n$, we have
\[
\tilde{A}(\pi_0(f)) = \pi_2(z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}).
\]
Corollary 5.9. When $n = 2$, the rational map $\bar{A}$ contracts the discriminant divisor to a point (corresponding to the orbit of the Fermat form in $D_{2d-4}$) for all $d \geq 3$. When $n = 3$, the rational map $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety if $d \geq 3$. More generally, for every $n \geq 4$ there exists $d_0$ such that for all $d \geq d_0$ the map $\bar{A}$ contracts the discriminant divisor to a lower-dimensional subvariety.

Proof. Notice that $\Sigma_1$ is dense in the discriminant divisor $\Delta$. Hence, for $n = 2$ the statement follows from Corollary 5.8.

When $n = 3$, Theorem 5.7 implies that $\bar{A}(\pi_0(\Sigma_1))$ lies in the locus of a 1-partial Fermat form in $D_{3(d-1)}$. The linear equivalence classes of 1-partial ternary Fermat forms are in bijection with the linear equivalence classes of binary degree $3(d-1)$ forms. The dimension of this locus is $3d - 6$, which for $d \geq 3$ is strictly less than the dimension $(\frac{d-5}{2}) - 10$ of the discriminant divisor.

If $n \geq 4$, by Theorem 5.7 the set $\bar{A}(\pi_0(\Sigma_1))$ lies in the locus of a 1-partial Fermat form in $D_{n(d-1)}$. The linear equivalence classes of 1-partial Fermat forms in $n$ variables are in bijection with the linear equivalence classes of degree $n(d-1)$ forms in $n-1$ variables. The dimension of this locus is $(\frac{n(d-1)+(n-2)}{n-2})$, which for sufficiently large $d$ is strictly less than the dimension of the discriminant divisor $(\frac{(d+1)+(n-1)}{n-1}) - (n^2 + 1)$.

We conclude the paper with an alternative proof of the main fact of [1] (see Proposition 4.3 therein).

Corollary 5.10 (Generic smoothness of associated forms). The closure of $\text{Im} \ A$ in $\mathbb{P}(D_{n(d-1)})^{ss}$ contains the orbit

$$\text{SL}(n) \cdot \left\{ z_1^{n(d-1)} + \cdots + z_n^{n(d-1)} \right\}$$

of the Fermat hypersurface. Consequently, $A(f)$ is a smooth form for a generic smooth $f \in S_{d+1}$.

Proof. By Corollary 5.8, we have

$$\pi_2(z_1^{n(d-1)} + \cdots + z_n^{n(d-1)}) \in \text{Im}(\bar{A}).$$

Since the orbit of the Fermat hypersurface is closed in $\mathbb{P}(D_{n(d-1)})^{ss}$, it lies in the closure of $\text{Im} \ A$. \hfill \qed

References


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