CYCLIC COVERING MORPHISMS ON $\overline{M}_{0,n}$

MAKSYM FEDORCHUK

Abstract. We study cyclic covering morphisms from $\overline{M}_{0,n}$ to moduli spaces of unpointed stable curves of positive genus or compactified moduli spaces of principally polarized abelian varieties. Our main application is a construction of new semipositive vector bundles and nef divisors on $\overline{M}_{0,n}$, with a view toward the F-conjecture. In particular, we construct new extremal rays of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$. We also find an alternate description of all $\mathfrak{sl}$ level 1 conformal blocks divisors on $\overline{M}_{0,n}$.

1. Introduction

The purpose of this paper is to describe a new geometric construction of nef line bundles on $\overline{M}_{0,n}$, the moduli space of stable n-pointed rational curves. The construction is motivated by the F-conjecture, posed by Fulton, that describes $\mathrm{Nef}(\overline{M}_{0,n})$ as the polyhedral cone, called the *F-cone*, cut out by the hyperplanes dual to 1-dimensional boundary strata in $\overline{M}_{0,n}$. The F-conjecture was first addressed in [KM96] and has attracted a great deal of interest since; see [GKM02, FG03, Gib09, Lar09, Fon09]. In particular, by [GKM02] the S_n-symmetric variant of the F-conjecture implies the analog of the F-conjecture for $\overline{M}_g$ formulated by Faber in [Fab96].

Our main tool is the *cyclic covering morphism* that associates to a stable n-pointed rational curve the cyclic degree p cover branched at the marked points (p needs to divide n). The literature on the subject of cyclic covers of a projective line is vast. We mention recent papers of Chen [Che10], Bouw and Möller [BM10], Eskin, Kontsevich, and Zorich [EKZ10], McMullen [McM], Moonen, and Oort [Moo10, MO11]. In particular, [EKZ10] studies the eigenbundle decomposition of the Hodge bundle over a family of cyclic covers of $\mathbb{P}^1$ with 4 branch points. So far, families of cyclic covers were used to study the geometry of the moduli spaces of stable curves, abelian differentials, of principally polarized abelian varieties, etc. For example, cyclic covers are used to study special subvarieties in the moduli space of abelian varieties that arise from families of cyclic covers in [Moo10, MO11]. The novelty of our approach is that we use the cyclic covering construction in a way that bears on the cone of nef divisors of $\overline{M}_{0,n}$. That this simple construction elucidates the structure of the symmetric nef cone of $\overline{M}_{0,n}$ comes as a pleasant surprise.

We now present three theorems that give a flavor of our results. Our most general results are contained in the main body of this paper. We also refer the reader to Section 1.1 where the illustrative example of $\overline{M}_{0,6}$ is worked out.

Key words and phrases. moduli of curves, nef cone, cyclic cover.

Briefly, for every $p \mid n$, we define the cyclic p-covering morphism $f_{n,p} \colon \overline{M}_{0,n} \to \overline{M}_g$, where $g = (n-2)(p-1)/2$, that associates to a stable n-pointed rational curve its cyclic degree p cover branched over n marked points. After passing to a root stack of $\overline{M}_{0,n}$, the cyclic p-covering morphism lifts to $\overline{\mathcal{M}}_g$. Our first observation concerning cyclic covering morphisms gives a determinantal description of $\mathfrak{sl}_n$ level 1 conformal blocks bundles on $\overline{M}_{0,n}$ recently studied by Fakhruddin in [Fak09], and Arap, Gibney, Stankewicz, and Swinarski in [AGSS10] (see also [AGS10]); we follow notation of [AGSS10].

Theorem A. (char 0) *Let $\mathbb{E}$ be the restriction of the Hodge bundle over $\overline{\mathcal{M}}_g$ to $f_{n,p}(\overline{M}_{0,n})$. There is an eigenbundle decomposition $\mathbb{E} = \bigoplus_{j=1}^{p-1} \mathbb{E}_j$ with respect to a natural μ_p-action on $\mathbb{E}$. The vector bundle $f^*_{n,p}(\mathbb{E}_j)$ is semipositive on $\overline{M}_{0,n}$ and $\det(f^*_{n,p}(\mathbb{E}_j)) = \mathbb{D}^1_{n,jn/p}/p$, where $\mathbb{D}^1_{n,jn/p}$ is the symmetric $\mathfrak{sl}_n$ level 1 conformal blocks divisor associated to the fundamental weight $w_{jn/p}$.*

By a recent work of Arap, Gibney, Stankewicz, and Swinarski [AGSS10], the $\mathfrak{sl}_n$ level 1 conformal blocks divisor $\mathbb{D}^1_{n,j}$ generates an extremal ray of the nef cone of $\overline{M}_{0,n}/S_n$ for every fundamental weight w_j, $j = 1, \dots, n-1$. Theorem A, and its generalization Theorem 4.5, provides an algebro-geometric construction of these line bundles and gives an independent proof of nefness of $\mathbb{D}^1_{n,j}$. By a recent work of Giansiracusa [Gia10], the line bundle $\mathbb{D}^1_{n,j}$ is a pullback to $\overline{M}_{0,n}$ of a polarization on a GIT quotient of the compactified parameter space of n points lying on rational normal curves of degree $j-1$ in $\mathbb{P}^{j-1}$. We observe that Theorem A expresses the Hodge class $c_1(f^*_{n,p}(\mathbb{E}))$ as an effective linear combination of the conformal blocks divisors $\mathbb{D}^1_{n,jn/p}$, $j = 1, \dots, p-1$. The following result gives a geometric interpretation of this Hodge class. Its relation with the GIT quotients of Giansiracusa is a complete mystery.[1]

Theorem B. *The line bundle $\lambda_{n,p} := \det(f^*_{n,p}(\mathbb{E}))$ is semiample on $\overline{M}_{0,n}$ and defines a morphism to the Satake compactification of $\mathcal{A}_{(n-2)(p-1)/2}$. It contracts the boundary divisors Δ_k with $p \mid k$, and its divisor class is*

$$f^*_{n,p}(\lambda) = \frac{p}{12}\left(\left(1 - \frac{1}{p^2}\right)\psi - \sum_k \left(1 - \frac{\gcd(k,p)^2}{p^2}\right)\Delta_k\right).$$

Theorem A shows that all extremal rays of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$ generated by $\mathfrak{sl}_n$ level 1 conformal blocks bundles $\mathbb{D}^1_{n,j}$ arise from the cyclic covering construction. Theorem 4.5 in the sequel shows that *every* $\mathfrak{sl}$ level 1 conformal blocks divisor on $\overline{M}_{0,n}$ arises in a similar fashion via a generalized cyclic covering morphism.

The following theorem produces a new extremal ray of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$. It lies outside of the cone generated by the determinants of conformal blocks bundles computed so far [Swi].

Theorem C. *Suppose $3 \mid n$. The divisor $f^*_{n,3}(9\lambda - \delta_{irr})$ is an extremal ray of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$.*

---

[1] Except in the case of $p = 2$ where $\mathbb{D}^1_{n,n/2} \sim f^*_{n,2}(\lambda)$, and $p = 3$ where $\mathbb{D}^1_{n,n/3} = \mathbb{D}^1_{n,2n/3} \sim f^*_{n,3}(\lambda)$; see Proposition 4.8.

Theorem C is proved in Section 5. We believe that there are analogs of Theorem C that produce new extremal rays of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$ for other values of p: We prove one for $p=5$, producing nef divisors on $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$ that were previously unattainable with other techniques. (Almost definitely, these divisors generate extremal rays of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$, for $5 \mid n$.)

Taken together, Theorems A and C raise a fascinating possibility that the extremal rays of the symmetric nef cone of $\overline{M}_{0,n}$ can be understood by studying families of cyclic covers. This is the subject of our ongoing investigation.

1.1. **Cyclic covering morphisms on $\overline{M}_{0,6}$.** We proceed to describe an illustrative example of our approach in the case of $\widetilde{M}_{0,6} := \overline{M}_{0,6}/S_6$. The Neron-Severi space of $\widetilde{M}_{0,6}$ is generated by divisor classes Δ_2 and Δ_3. There are only two F-curves: $F_{2,2,1,1}$ and $F_{3,1,1,1}$. So the extremal rays of the F-cone are spanned by

$$D_1 = 2\Delta_2 + \Delta_3 \ \text{ and } D_2 = \Delta_2 + 3\Delta_3.$$

The divisor D_2 is easily seen to be semiample: it defines the morphism to $\mathrm{Sym}^6(\mathbb{P}^1)/\!\!/\,\mathrm{SL}(2)$. This morphism contracts every F-curve of type $(3,1,1,1)$, and more generally contracts Δ_3 to the unique point in the GIT quotient corresponding to strictly semistable orbits.

To prove that D_1 is semiample, we introduce the morphism $f_2\colon \widetilde{M}_{0,6} \to \overline{M}_2$ that sends a stable 6-pointed rational curve to its cyclic cover of degree 2 totally ramified over 6 marked points. For example, a $\mathbb{P}^1$ marked by points $[x_i : 1]$ is mapped to the genus 2 curve

$$y^2 = \prod_{i=1}^{6}(x - x_i).$$

This morphism extends to the boundary because $\overline{M}_2$ is proper and the indeterminacy of f_2 at the boundary of $\widetilde{M}_{0,6}$ is a priori at worst finite. If we take now a stable 6-pointed rational curve in the 1-dimensional boundary stratum of type $(2,2,1,1)$, then the corresponding double cover has three rational components and its stabilization is a 2-nodal rational curve. In particular, the normalization of the double cover does not vary in moduli. Consider now the Hodge class $\lambda \in \mathrm{Pic}(\overline{M}_2)$. It is well-known to define the morphism from $\overline{M}_2$ to the Satake compactification of $\mathcal{A}_2$, also known as the quotient of the *Igusa quartic* by the action of S_6. Pulling back λ via f_2 we obtain a semiample divisor on $\widetilde{M}_{0,6}$ that intersects every F-curve of type $(2,2,1,1)$ in degree 0. It follows that

$$D_1 = 2\Delta_2 + \Delta_3 \sim f_2^*\lambda,$$

where $\sim$ stands for numerical proportionality.

This is the essence of our method. Two aspects of the described construction can now be varied. One is the degree of the cyclic covering, the other is the line bundle that we pullback from a moduli space of positive genus curves. To illustrate the first point, consider the morphism $f_3\colon \widetilde{M}_{0,6} \to \overline{M}_4$ that now sends a stable 6-pointed rational curve to its cyclic cover of degree 3 totally ramified over 6 marked points: a $\mathbb{P}^1$ marked by points $[x_i : 1]$ is mapped to the genus 4 curve

$$y^3 = \prod_{i=1}^{6}(x - x_i).$$

Every curve in the family of μ_3-covers over an F-curve of type $(3,1,1,1)$ has an elliptic component with j-invariant 0 attached at three points to another elliptic component with j-invariant 0. Thus, the degree of $\lambda \in \mathrm{Pic}(\overline{M}_4)$ on such a family is 0. It follows that

$$D_2 \sim f_3^*\lambda.$$

In particular, the GIT quotient of 6 unordered points on $\mathbb{P}^1$ has another geometric interpretation: It is (the normalization of) the image of the cyclic 3-covering morphism from $\widetilde{M}_{0,6}$ to the Satake compactification of principally polarized abelian varieties of dimension 4. It is also the quotient of the *Segre cubic* by the action of S_6. By [Pro10, p.257], this quotient is isomorphic to $\mathbb{P}(2,4,5,6)$.

To illustrate the second point, we return to the morphism $f_2\colon \widetilde{M}_{0,6} \to \overline{M}_2$. It is well-known that the line bundle $12\lambda - \delta_{\mathrm{irr}}$ is nef on $\overline{M}_2$ and has degree 0 on any family whose moving components are all of genus 1. Observe now that the moving component of the family of μ_2-covers over an F-curve of type $(3,1,1,1)$ is of genus 1. It follows that

$$D_2 \sim f_2^*(12\lambda - \delta_{\mathrm{irr}}).$$

Finally, we know from Theorem C that the line bundle $9\lambda - \delta_{\mathrm{irr}}$ is nef on the locus of μ_3-covers in $\overline{M}_4$ and that

$$D_1 \sim f_3^*(9\lambda - \delta_{\mathrm{irr}}).$$

1.2. **Notation and conventions.** We work over an algebraically closed field of characteristic 0, which we denote by $\mathbb{C}$. Some results, such as computations of Section 6, extend to sufficiently high positive characteristic.

We denote by $\widetilde{M}_{0,n}$ the quotient scheme $\overline{M}_{0,n}/S_n$. We have $\mathrm{Pic}(\widetilde{M}_{0,n}) = \mathbb{Z}\{\Delta_2, \Delta_3, \dots, \Delta_{\lfloor n/2 \rfloor}\}$.

We denote by $\langle a \rangle_p$ the representative in $\{0, 1, \dots, p-1\}$ of the residue of a modulo p.

If $\vec{d} = (d_1, \dots, d_n)$ and $I \subset \{1, \dots, n\}$, we set $d(I) := \sum_{i \in I} d_i$.

An F-curve of type (a,b,c,d) in $\overline{M}_{0,n}$ is a family of stable n-pointed rational curves obtained from the universal family $(\mathcal{C}; \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ over $\overline{M}_{0,4}$ by identifying sections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with sections of constant stable $(a+1)$, $(b+1)$, $(c+1)$, and $(d+1)$-pointed rational curves.[2] For $t \in \mathbb{P}^1$, we call the 4-pointed component $\mathcal{C}_t$ a *backbone*; we also denote $p_A := \sigma_1(t)$, $p_B := \sigma_2(t)$, $p_C := \sigma_3(t)$, $p_D := \sigma_4(t)$. The *class* of any F-curve in $\widetilde{M}_{0,n}$ corresponding to the decomposition $n = a+b+c+d$ is denoted by $F_{a,b,c,d}$. We also call $F_{a,b,c,d}$ an F-curve of type (a,b,c,d). In the non-symmetric case, an F-curve corresponding to the partition $\{1, \dots, n\} = I_1 \cup I_2 \cup I_3 \cup I_4$ is denoted F_{I_1,I_2,I_3,I_4}. A divisor that intersects every F-curve non-negatively is called *F-nef*. The F-nef divisors form an *F-cone* inside $N^1(\overline{M}_{0,n})$. Fulton's conjecture posits that the F-cone of $\overline{M}_{0,n}$ is equal to the nef cone.

[2]We do not require constant families to be maximally degenerate.

Given $\mathbb{Q}$-Cartier divisors D_1 and D_2 on a projective variety X, we say that $D_1 \sim D_2$ if D_1 and D_2 generate the same ray in $N^1(X)$, or, equivalently, if $D_1 \equiv_{num} cD_2$ for some $c > 0$. Finally, we refer to [KM96] and [HM98] for the intersection theory on $\widetilde{M}_{0,n}$ and $\overline{M}_{0,n}$.

2. Cycling covering morphisms

We introduce a sequence of natural morphisms, the so-called *cyclic covering morphisms* from $\overline{M}_{0,n}$ to moduli spaces of *unpointed* Deligne-Mumford stable curves of positive genus. The key example is easy to describe: Given n points $p_1, \dots, p_n$ on $\mathbb{P}^1$, for every $p \mid n$ there is a μ_p-cover of $\mathbb{P}^1$ totally ramified over points p_i. If $p_i = [x_i : 1]$, this cover is the regular model of the function field extension of $\mathbb{C}(x)$ defined by $y^p = (x - x_1) \cdots (x - x_n)$. The resulting smooth curve has genus $g = (n-2)(p-1)/2$ by the Riemann-Hurwitz formula.

Definition 2.1 ((Cyclic p-covering morphism))**.** Let $p \mid n$. We define a regular morphism

$$f_{n,p} \colon \overline{M}_{0,n} \to \overline{M}_g, \quad g = (n-2)(p-1)/2,$$

to be a unique morphism that sends $(\mathbb{P}^1; p_1, \dots, p_n) \in M_{0,n}$ to its degree p cyclic cover totally ramified over $\sum_{i=1}^n p_i$.

There are two related ways to see that $f_{n,p}$ indeed extends to a morphism $\overline{M}_{0,n} \to \overline{M}_g$: One way is to use the theory of admissible covers of Harris and Mumford [HM82]. By [HM98, p.186], there exists a coarse moduli scheme H parameterizing pseudo-admissible μ_p-covers (i.e., branched covers where each branching profile at a smooth point is a p-cycle in S_p). By construction, H maps finitely to $\overline{M}_{0,n}$ and admits a morphism to $\overline{M}_g$. Taking the closure of the section $M_{0,n} \to H$ given by the cyclic covering trick, and using the normality of $\overline{M}_{0,n}$, we obtain the necessary extension. Note that one either has to work with coarse moduli schemes or to take an appropriate root stack of $\overline{M}_{0,n}$ to lift the morphism $f_{n,p}$ to $\overline{\mathcal{M}}_g$.

We now sketch a related direct construction.

2.1. The universal μ_p-cover over $\overline{M}_{0,n}$.

As above, consider $p \mid n$. Let $\mathcal{C} \to \overline{M}_{0,n}$ be the universal family of stable n-pointed curves and let $\{\sigma_i\}_{i=1}^n$ be the universal sections. Observe that $\mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n \sigma_n)$ is not divisible by p in $\mathrm{Pic}(\mathcal{C})$: One obstruction to divisibility comes from any curve in Δ_k with $p \nmid k$; another obstruction is due to the fact that $\psi_i \in \mathrm{Pic}(\overline{M}_{0,n})$ is not divisible by p. For any given 1-parameter family $B \subset \overline{M}_{0,n}$, this difficulty can be overcome by making a finite base change of an appropriate order and blowing up the total family at the offending nodes. Instead of specifying the base change for every B, we indicate a global construction.

Definition 2.2. Let $p \mid n$. We say that $(\mathcal{X}; p_1, \dots, p_n)$ is a *p-divisible n-pointed orbicurve* if

(1) $\mathcal{X}$ is an orbicurve whose coarse moduli space X, marked by $p_1, \dots, p_n$, is a stable n-pointed rational curve.

(2) Étale locally at a node of X of type Δ_k the orbicurve $\mathcal{X}$ is isomorphic to
$$[\operatorname{Spec}\mathbb{C}[x,y]/(xy)/\mu_r]\,,$$
where $r = p/\gcd(k,p)$ and μ_r acts by $(x,y) \mapsto (\alpha^{r-k}, \alpha^k y)$, with α a primitive r^{th} root of unity.

(3) Étale locally over $p_1 \in X$, the orbicurve $\mathcal{X}$ is isomorphic to $[\operatorname{Spec}\mathbb{C}[x]/\mu_p]$, where μ_p acts by $x \mapsto \beta x$, with β a primitive p^{th} root of unity. We also require the existence of a section $\sigma_1\colon \operatorname{Spec}\mathbb{C} \to [\operatorname{Spec}\mathbb{C}[x]/\mu_p]$.

The notion of a p-divisible orbicurve is a mild generalization of the notion of even rational orbicurves considered in [Fed10]. With Definition 2.2, for every family $(\mathcal{X} \to B; \sigma_1, \dots, \sigma_n)$ of p-divisible orbicurves, there is a unique line bundle $\mathcal{L}$ satisfying $\mathcal{L}^p \simeq \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n \sigma_i)$ and $\sigma_1^*\mathcal{L} \simeq \mathcal{O}_B$. (In other words, $\mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n \sigma_i)$ has a unique, up to pullbacks from the base, p^{th} root.) Consider now the Deligne-Mumford stack $\mathcal{M}$ of p-divisible n-pointed orbicurves with its universal family $\mathcal{Y} \to \mathcal{M}$. We can construct a μ_p-cover $\mathcal{Z} \to \mathcal{Y}$ simply by applying the cyclic covering construction (see [Laz04, Proposition 4.1.6]) to the divisor $\sum_{i=1}^n \sigma_i$. We omit the details of the construction and refer to [Fed10] for the special case of $p = 2$.

Since $\mathcal{M} \to \overline{M}_{0,n}$ is bijective on geometric points, we will not distinguish $\operatorname{Pic}(\mathcal{M}) \otimes \mathbb{Q}$ and $\operatorname{Pic}(\overline{M}_{0,n}) \otimes \mathbb{Q}$. We will informally call $\mathcal{Z} \to \mathcal{M}$, the *universal stable μ_p-cover* over $\overline{M}_{0,n}$, and the fibers of $\mathcal{Z} \to \mathcal{M}$ will be called *stable μ_p-covers.* The universal stable μ_p-cover induces a morphism $\mathcal{M} \to \overline{\mathcal{M}}_g$ that descends to give the desired morphism $f_{n,p}\colon \overline{M}_{0,n} \to \overline{M}_g$.

2.2. **Divisor classes on $\overline{M}_{0,n}$ via cyclic covering morphisms.** By pulling back nef divisors on $\overline{\mathcal{M}}_g$, we obtain nef divisors on $\overline{M}_{0,n}$. The most interesting, from our point of view, divisors are obtained by pulling back: the Hodge class λ, which itself is well-known to be semiample (it defines a morphism from $\overline{M}_g$ to the Satake compactification of $\mathcal{A}_g$), the determinants $\lambda(j)$ of the eigenbundles of the Hodge bundle (these are discussed in the sequel), and linear combinations of $\lambda(j)$ and the boundary divisor δ_{irr}.

Definition 2.3 ((The Hodge class))**.** Given $p \mid n$, we define $\lambda_{n,p} := f_{n,p}^*\lambda \in \operatorname{Pic}(\overline{M}_{0,n}) \otimes \mathbb{Q}$.

As we have already observed, the divisor class $\lambda_{n,p}$ is semiample and has a simple geometric interpretation: It defines a morphism from $\overline{M}_{0,n}$ to the Satake compactification of $\mathcal{A}_g$ by sending a rational n-pointed curve to the abelian part of the generalized Jacobian of its μ_p-cover. Such a geometric interpretation of $\lambda_{n,p}$ lents itself to a description of $\lambda_{n,p}^{\perp} \subset N_1(\overline{M}_{0,n})$. Namely, we have the following observation.

Proposition 2.4. *A curve $B \subset \overline{M}_{0,n}$ satisfies $\lambda_{n,p} \cdot B = 0$ if and only if every moving component of the family of μ_p-covers over B is rational.*

Proof. Clear from the definition of the Satake compactification. □

Next, we recall a construction of covering families for the boundary divisors $\Delta_k \subset \overline{M}_{0,n}$.

Construction 2.5. For every $k \in \{3, \dots, \lfloor n/2 \rfloor\}$, we consider the family of n-pointed stable rational curves obtained by attaching a constant family of $(n-k+1)$-pointed $\mathbb{P}^1$ along one of the sections to the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, where $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ has k horizontal sections. Let $T_k \subset \overline{M}_{0,n}$ be the stabilization of this family. We then have

$$\Delta_i \cdot T_k = \begin{cases} 2-k & \text{if } i = k, \\ k & \text{if } i = k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Corollary 2.6.
(1) *If p divides one of a, b, c, d, then $\lambda_{n,p}$ has degree 0 on an F-curve $F_{a,b,c,d}$.*
(2) *If p divides k, then $\lambda_{n,p}$ has degree 0 on the curve T_k of Construction 2.5.*

Proof. The only potential moving component of the μ_p-cover over $F_{a,b,c,d}$ is a cyclic cover of the backbone $\mathbb{P}^1$ branched exactly over points p_A, p_B, p_C, p_D (see Section 1.2 for notation). Say p divides a, then the ramification profile over p_A is trivial. It follows that the cover of the backbone does not vary in moduli.

The family over T_k is obtained by gluing two constant families along a section. Since the ramification profile over the gluing section is trivial, the resulting cover does not vary in moduli. □

Definition 2.7. Let $p \mid n$. Set $g = (n-2)(p-1)/2$. We define $\tau_{n,p} \colon \overline{M}_{0,n} \to \mathcal{A}_g^S$ to be the composition of the cyclic covering morphism $f_{n,p} \colon \overline{M}_{0,n} \to \overline{M}_g$ and the extended Torelli morphism $\tau \colon \overline{M}_g \to \mathcal{A}_g^S$ from $\overline{M}_g$ to the Satake compactification of $\mathcal{A}_g$.

We are now ready to prove the main result of this section (Theorem B from the introduction).

Theorem 2.8. *The line bundle $\lambda_{n,p}$ is semiample on $\overline{M}_{0,n}$ and defines the morphism*

$$\tau_{n,p} \colon \overline{M}_{0,n} \to \mathcal{A}^S_{(n-2)(p-1)/2}$$

to the Satake compactification of $\mathcal{A}_{(n-2)(p-1)/2}$. It contracts the boundary divisors Δ_k with $p \mid k$, and its class is

$$\lambda_{n,p} = \frac{p}{12}\left(\left(1 - \frac{1}{p^2}\right)\psi - \sum_k \left(1 - \frac{\gcd(k,p)^2}{p^2}\right)\Delta_k\right).$$

Proof. Set $g = (n-2)(p-1)/2$ and consider the composition $\tau_{n,p} \colon \overline{M}_{0,n} \to \overline{M}_g \to \mathcal{A}_g^S$. On $\overline{M}_g$, we have that $\lambda = \tau^*(\mathcal{O}_{\mathcal{A}_g^S}(1))$. It follows that $\lambda_{n,p} = f_{n,p}^*(\lambda)$ is semiample. The class of $\lambda_{n,p}$ is computed in Proposition 3.1 below. Finally, for every k divisible by p, we know from Corollary 2.6 that $\lambda_{n,p}$ has degree 0 on the curve class T_k described in Construction 2.5. Since deformations of T_k cover Δ_k, we conclude that the morphism $\tau_{n,p}$ contracts Δ_k. □

Remark 2.9. When p is prime, we obtain an aesthetically pleasing formula

$$\lambda_{n,p} = \frac{p^2-1}{12p}\left(\psi - \Delta + \sum_{p \mid k} \Delta_k\right).$$

It is an amusing exercise to verify that $\lambda_{n,p}$ is F-nef.

2.3. **A remark on the birational geometry of $\widetilde{M}_{0,n}$.** The effective cone of $\widetilde{M}_{0,n}=\overline{M}_{0,n}/S_n$ has been studied already in [KM96]. Keel and McKernan prove that $\overline{\mathrm{Eff}}(\widetilde{M}_{0,n})$ is simplicial and is generated by the boundary divisors Δ_i. Further, every movable divisor is big. Intuitively, this is explained by the absence of natural rational fibrations of $\widetilde{M}_{0,n}$ analogous to the forgetful morphisms on $\overline{M}_{0,n}$. In particular, (in characteristic 0) there are no regular contractions of $\widetilde{M}_{0,n}$ affecting the interior.

Subsequently, Hu and Keel asked whether $\widetilde{M}_{0,n}$ is a Mori dream space [HK00]. If it is, then because every movable divisor is in the interior of $\overline{\mathrm{Eff}}(\widetilde{M}_{0,n})$, Mori chambers cannot share a boundary along a simplicial face of $\overline{\mathrm{Eff}}(\widetilde{M}_{0,n})$. It follows that if $\widetilde{M}_{0,n}$ is a Mori dream space, then the unique Mori chamber adjacent to the face spanned by $\{\Delta_j \mid j\neq i\}$ defines a birational contraction $f_i\colon \widetilde{M}_{0,n} \dashrightarrow X_i$ such that its exceptional locus is $\mathrm{Exc}(f_i)=\bigcup_{j\neq i}\Delta_j$, the set of all boundary divisors but Δ_i. That a regular contraction $f_2\colon \widetilde{M}_{0,n}\to \mathrm{Sym}^n(\mathbb{P}^1)/\!\!/\mathrm{SL}_2$ fits the bill for $i=2$ is already observed in [HK00]. For a general n, we are unaware of such f_i for $i\neq 2$. We note that Rulla computes candidates for the divisors R_i that could be expected to define morphisms f_i; see [Rul06, Corollary 6.4] and the subsequent remark. However, the question of whether these divisors are in fact moving and have finitely generated section rings is far from settled.

For some small n, the morphism of Definition 2.7 gives us a *regular* contraction of $\overline{M}_{0,n}$ contracting all boundary divisors but one. For example, the morphism $\tau_{8,2}\colon \widetilde{M}_{0,8}\to \mathcal{A}_3^S$ contracts boundary divisors Δ_2 and Δ_4.

2.4. **Cyclic covering morphisms II: Weighted case.** Motivated by the preceding discussion, we generalize the definition of a cyclic covering morphism. To this end, we take a sequence of non-negative integers $\vec{d}=(d_1,\dots,d_n)$, called *weights* and an integer number $p\geq 2$ such that $p\mid \sum_{i=1}^n d_i$.

Definition 2.10. We define a morphism $f_{\vec{d},p}\colon M_{0,n}\to \overline{M}_g$, by sending a $\mathbb{P}^1$ marked by n points $p_i=[x_i:1]$ to its degree p cyclic covering totally ramified over $d_1p_1+\cdots+d_np_n$, i.e. the regular model of the function field extension of $\mathbb{C}(x)$ given by $y^p=(x-x_1)^{d_1}\cdots(x-x_n)^{d_n}$. We have that $g=\frac{1}{2}\left[2-2p+\sum_{i=1}^n(p-\gcd(d_i,p)\right]$ by the Riemann-Hurwitz formula. As before, the morphism extends to $f_{\vec{d},p}\colon \overline{M}_{0,n}\to \overline{M}_g$.

We now consider a symmetric variant of the weighted cyclic covering morphism. We define

$$f^{S_n}_{\vec{d},p}=\prod_{\sigma\in S_n} f_{\sigma(\vec{d}),p}\colon \overline{M}_{0,n}\to \prod_{\sigma\in S_n}\overline{M}_g,$$

Evidently, $f^{S_n}_{\vec{d},p}$ descends to the morphism $f^{S_n}_{\vec{d},p}\colon \widetilde{M}_{0,n}\to \prod_{\sigma\in S_n}\overline{M}_g$.

Definition 2.11 ((Weighted Hodge class))**.** We define

$$\lambda_{\vec{d},p} := (f_{\vec{d},p}^{S_n})^* \bigotimes_{\sigma \in S_n} \mathrm{pr}_\sigma^* \lambda.$$

Proposition 2.12. *If for every partition of $\{1, \dots, n\}$ into subsets of sizes a, b, c, d, the weight of one of the partitions is divisible by p, then the divisor $\lambda_{\vec{d},p}$ has degree zero on all F-curves of type (a, b, c, d).*

Proof. The proof is the same as in Corollary 2.6. □

Example 2.13 (Divisors on $\overline{\mathcal{M}}_{0,7}$)**.** We consider the weight vector $\vec{d} = (1, 1, 1, 1, 1, 1, 0)$. By Proposition 2.12, the divisor $\lambda_{\vec{d},2}$ has degree 0 on $F_{2,2,2,1}$. This implies that $\lambda_{\vec{d},2} \sim \Delta_2 + \Delta_3$, which is easily seen to be an extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,7})$. Again by Proposition 2.12, the divisor $\lambda_{\vec{d},3}$ has degree 0 on $F_{4,1,1,1}$. This implies that $\lambda_{\vec{d},3} \sim \Delta_2 + 3\Delta_3$, which is another extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,7})$. Thus $\lambda_{\vec{d},2}$ and $\lambda_{\vec{d},3}$ account for all extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,7})$.

3. Divisor classes associated to families of stable cyclic covers

In this section, we prove our main technical intersection-theoretic results.

Proposition 3.1 (Pullback formulae for cyclic covering morphisms)**.** *Let $\vec{d} = (d_1, \dots, d_n)$. For any $p \mid \sum_{i=1}^n d_i$, consider the cyclic covering morphism $f_{\vec{d},p} \colon \overline{M}_{0,n} \to \overline{M}_g$ of Definition 2.10. For every boundary divisor $\Delta_{I,J}$, set $d(I) = \sum_{i \in I} d_i$. Then the divisor classes $\lambda, \delta_{irr} = \delta_0$, and $\delta_{red} = \delta_1 + \dots + \delta_{\lfloor g/2 \rfloor}$ pullback as follows:*

$$\begin{aligned} f_{\vec{d},p}^*(\lambda) &= \frac{1}{12p} \left[\sum_{i=1}^n \left(p^2 - \gcd(d_i, p)^2\right) \psi_i - \sum_{I,J} \left(p^2 - \gcd(d(I), p)^2\right) \Delta_{I,J} \right] \\ f_{\vec{d},p}^*(\delta_{irr}) &= \frac{1}{p} \sum_{I,J\colon \gcd(d(I),p)>1} \gcd(d(I), p)^2 \Delta_{I,J} \\ f_{\vec{d},p}^*(\delta_{red}) &= \frac{1}{p} \sum_{I,J\colon \gcd(d(I),p)=1} \Delta_{I,J}. \end{aligned} \tag{3.1}$$

In particular, in the unweighted case we have:

$$\begin{aligned} f_{n,p}^*(\lambda) &= \frac{p}{12} \left(\left(1 - \frac{1}{p^2}\right) \psi - \sum_k \left(1 - \frac{\gcd(k,p)^2}{p^2}\right) \Delta_k \right), \\ f_{n,p}^*(\delta_{irr}) &= \frac{1}{p} \sum_{k\colon \gcd(k,p)>1} \gcd(k,p)^2 \Delta_k, \\ f_{n,p}^*(\delta_{red}) &= \frac{1}{p} \sum_{k\colon \gcd(k,p)=1} \Delta_k. \end{aligned} \tag{3.2}$$

Proof. Due to the inductive nature of the boundary of $\overline{M}_{0,n}$, it suffices to prove Formulae (3.1) for generically smooth families of a special form. Namely, we consider a family $\mathcal{X}' \to B$ of stable n-pointed rational curves with $B \simeq \mathbb{P}^1$ and with

$\mathcal{X}'$ constructed as follows: Let $\mathcal{X} \to B$ be a $\mathbb{P}^1$-bundle with a section Σ_1 of negative self-intersection $\Sigma_1^2 = -r$ and $(n-1)$ sections $\{\Sigma_i\}_{i=2}^n$ disjoint from Σ_1 and in general position. Then we take $\mathcal{X}'$ to be simply the stabilization of $\mathcal{X} \to B$, i.e. the ordinary blow-up of $\mathcal{X}$ at the points where sections $\{\Sigma_i\}_{i=2}^n$ intersect. We also let $S = \{t_1, \dots, t_m\}$ be the points in B over which sections $\{\Sigma_i\}_{i=2}^n$ intersect. (Evidently, $m = \binom{n-1}{2}r$ but we will not need this in the sequel.)

The family $\mathcal{X}' \to B$ induces a map $B \to \overline{M}_{0,n}$. We now follow what happens to B under the cyclic p-covering morphism $f_{n,p}\colon \overline{M}_{0,n} \to \overline{M}_g$.

Observe that $D := \sum_{i=1}^n d_i \Sigma_i$ will be divisible by p in $\mathrm{Pic}(\mathcal{X})$ as long as $p \mid r$. The family of stable μ_p-covers over B is constructed out of $(\mathcal{X} \to B; \{\Sigma_i\}_{i=1}^n)$ in the following steps:

(Step 1) Apply the cyclic covering trick to construct a μ_p-cover ramified over D. Obtain the family $\mathcal{Y} \to B$ of generically smooth μ_p-covers over B, with singular fibers over S.

(Step 2) Consider a finite base change $B' \to B$ of degree p totally ramified over points in S. Denote $\mathcal{Y}' := \mathcal{Y} \times_B B'$.

(Step 3) Perform weighted blow-ups on $\mathcal{Y}'$ to arrive at the stable family $\mathcal{Z} \to B'$.

We proceed to elaborate on the last step. The problem is local and so we work locally around a point where two sections Σ_i and Σ_j meet. The local equation of $\mathcal{Y}'$ at this point is $y^p = (x - at^p)^{d_i}(x - bt^p)^{d_j}$, where t is the uniformizer on B'.

Set $q = \gcd(p, d_i + d_j)$. A weighted blow-up with weights $w(x, y, t) = (p/q, (d_i + d_j)/q, 1)$ followed by the normalization replaces the singularity $y^p = (x - at^p)^{d_i}(x - bt^p)^{d_j}$ by a smooth curve G_{ij} which now meets the rest of the fiber in q points. The self-intersection of G_{ij} in $\mathcal{Z}$ is (-1). Observe that at each of the points where G_{ij} meets the rest of the fiber, $\mathcal{Z}$ has an A_{q-1} singularity. It follows that this singular fiber contributes q^2 to $\delta_{\mathrm{irr}} \cdot B'$ if $q > 1$, and contributes 1 to $\delta_{\mathrm{red}} \cdot B'$ if $q = 1$. Summarizing:

$$\delta_{\mathrm{irr}} \cdot B' = \sum_{\gcd(d_i+d_j,p)>1} \gcd(d_i + d_j, p)^2 \Delta_{ij}, \quad \delta_{\mathrm{red}} \cdot B' = \sum_{\gcd(d_i+d_j,p)=1} \Delta_{ij}. \tag{3.3}$$

We proceed to compute the remaining numerical invariants of the family $\mathcal{Z} \to B'$. First, we set $r_i = p/\gcd(d_i, p)$ and consider the branch divisor

$$\mathrm{Br} = \sum_{i=1}^n \left(\frac{r_i - 1}{r_i}\right) \Sigma_i.$$

If $\pi\colon \mathcal{Y} \to \mathcal{X}$ is the cyclic cover constructed in (Step 1), then $\omega_{\mathcal{Y}/B} = \pi^*(\omega_{\mathcal{X}/B} + \mathrm{Br})$. Since $\mathcal{Y}'/B'$ is obtained from $\mathcal{Y}/B$ by a finite base change of degree p, we conclude that

$$\omega^2_{\mathcal{Y}'/B'} = p\omega^2_{\mathcal{Y}/B} = p^2(\omega_{\mathcal{X}/B} + \mathrm{Br})^2. \tag{3.4}$$

Next, if we let $\xi\colon \mathcal{Z} \to \mathcal{Y}'$ be the composition of the weighted blow-ups in (Step 3), then the exceptional divisors of ξ are precisely curves G_{ij} described above. We have $\omega_{\mathcal{Z}/B'} = \xi^*(\omega_{\mathcal{Y}'/B'}) + \sum_{i<j} a_{ij} G_{ij}$ and our immediate goal is to compute a_{ij}. Observe that each exceptional divisor G_{ij} is a degree p cover of $\mathbb{P}^1$ with 3

branch points and the branching profile $(\tau^{-d_i-d_j}, \tau^{d_i}, \tau^{d_j})$ where τ is a p-cycle in S_p. Therefore, by the Riemann-Hurwitz formula, we have

$$2g(G_{ij}) + 2p - 2 = (r_i - 1)p/r_i + (r_i - 1)p/r_j + (p/q - 1)q$$

and so $2g(G_{ij}) - 2 = p - q - p/r_i - p/r_j$ (as before, $q = \gcd(d_i + d_j, p)$). To determine a_{ij} we apply adjunction: because the singularities of $\mathcal{Z}$ are Du Val, we have $\omega_{\mathcal{Z}/B'} \cdot G_{ij} = \deg \omega_{G_{ij}} + q = p - p/r_i - p/r_j$. Recalling that $G_{ij}^2 = -1$, we obtain $a_{ij} = p - p/r_i - p/r_j$. It follows that

$$\omega^2_{\mathcal{Z}/B'} = \omega^2_{\mathcal{Y}'/B'} - p^2 \sum_{i<j} (1 - 1/r_i - 1/r_j)^2 \Delta_{ij}. \tag{3.5}$$

Recall that the family $\mathcal{X}' \to B$ of stable n-pointed rational curves associated to $\mathcal{X} \to B$ is obtained by blowing up points where sections $\{\Sigma_i\}_{i=1}^n$ intersect. It follows that

$$\mathrm{Br}^2 = -\sum_{i=1}^n \left(\frac{r_i - 1}{r_i}\right)^2 \psi_i + \sum_{i<j} \left(2 - \frac{1}{r_i} - \frac{1}{r_j}\right)^2 \Delta_{ij},$$

$$\omega_{\mathcal{X}/B} \cdot \mathrm{Br} = \sum_{i=1}^n \frac{r_i - 1}{r_i} \psi_i - \sum_{i<j} \left(2 - \frac{1}{r_i} - \frac{1}{r_j}\right) \Delta_{ij}.$$

Using $\omega^2_{\mathcal{Y}'/B'} = p^2(\omega^2_{\mathcal{X}/B} + 2\omega_{\mathcal{X}/B}\mathrm{Br} + \mathrm{Br}^2)$ from Equation (3.4), we compute

$$\omega^2_{\mathcal{Y}'/B'} = p^2 \left[\sum_{i=1}^n \frac{r_i^2 - 1}{r_i^2} \psi_i\right] - p^2 \left[\sum_{i<j} \left(2 - \frac{1}{r_i} - \frac{1}{r_j}\right)\left(\frac{1}{r_i} + \frac{1}{r_j}\right) \Delta_{ij}\right] \tag{3.6}$$

We finally compute the degree of λ on the family $\mathcal{Z} \to B'$. Combining Equations (3.3), (3.5), (3.6), and Mumford's formula $12\lambda = \kappa + \delta$, we obtain:

$$\begin{aligned}\lambda = \frac{1}{12}\Bigg[& p^2 \sum_{i=1}^n \frac{r_i^2 - 1}{r_i^2} \psi_i - p^2 \sum_{i<j} \left(2 - \frac{1}{r_i} - \frac{1}{r_j}\right)\left(\frac{1}{r_i} + \frac{1}{r_j}\right) \Delta_{ij} \\ & - p^2 \sum_{i<j} (1 - 1/r_i - 1/r_j)^2 \Delta_{ij} + \sum_{i<j} \gcd(p, d_i + d_j)^2 \Delta_{ij}\Bigg] \\ = \frac{p^2}{12} & \left[\sum_{i=1}^n \left(1 - \frac{\gcd(d_i, p)^2}{p^2}\right) \psi_i - \sum_{i<j} \left(1 - \frac{\gcd(p, d_i + d_j)^2}{p^2}\right) \Delta_{ij}\right]\end{aligned}$$

□

4. Eigenbundles of Hodge bundles over $\overline{M}_{0,n}$

We continue the study of cyclic covering morphisms $f_{n,p} \colon \overline{M}_{0,n} \to \overline{M}_g$, defined for every $p \mid n$ and $g = (n-2)(p-1)/2$. In Section 3, we have studied the Hodge class $\lambda_{n,p}$, which by Definition 2.3 is the determinant of $\mathbb{E}$ – the pullback of the Hodge bundle from $\overline{\mathcal{M}}_g$ via $f_{n,p}$. We now turn our attention to $\mathbb{E}$ itself.

By construction, $\mathbb{E}$ is the Hodge bundle associated to the family of stable μ_p-covers over $\overline{M}_{0,n}$ constructed in Section 2.1. The presence of the μ_p-action urges us to consider the eigenbundle decomposition of $\mathbb{E}$. We identify characters of μ_p

with integers $\{0,1,\dots,p-1\}$. Let α be a generator of μ_p. For every character j, and every stable cyclic μ_p-cover C, we define

$$\mathrm{H}^0(C,\omega_C)_j := \{\omega\in \mathrm{H}^0(C,\omega_C) \mid \alpha\cdot\omega=\alpha^j\omega\}.$$

We refer to the 1-forms in $\mathrm{H}^0(C,\omega_C)_j$ as *forms of weight* j. Since $\mathrm{H}^0(C,\omega_C)_0=(0)$, the eigenbundle decomposition of $\mathbb{E}$ with respect to the μ_p-action is

$$\mathbb{E}=\bigoplus_{j=1}^{p-1}\mathbb{E}_j.$$

4.1. **Determinants of eigenbundles $\mathbb{E}_j$.** Since the Hodge bundle $\mathbb{E}$ is semipositive [Kol90], the eigenbundles $\mathbb{E}_j$ are semipositive as well. Hence, their determinants are nef divisors on $\overline{M}_{0,n}$. At the first sight, the task of computing the determinant of $\mathbb{E}_j$ appears daunting. However, there is one situation where this can be done explicitly. Namely, we restrict ourselves to the family of stable μ_p-covers over an F-curve of type (a,b,c,d). Then the moving component of the stable μ_p-cover in question is much studied family of cyclic covers defined by the equation

$$C_\lambda:\quad y^p=x^a(x-1)^b(x-\lambda)^c,\quad \lambda\in\mathbb{P}^1. \tag{4.1}$$

Still how does one compute the degree of $\mathbb{E}_j$ on this family? Before we answer this question completely, we describe a situation when the computation is made without any effort.

Observe that a rational 1-form of weight j on C_λ is necessarily of the form $y^j dx/f(x)$. Assuming for simplicity that p is coprime to a, b, c, and $a+b+c$, we see that $(dx)=(p-1)[0]+(p-1)[1]+(p-1)[\lambda]-(p+1)[\infty]$ and $(y)=a[0]+b[1]+c[\lambda]-(a+b+c)[\infty]$. Evidently, $y^j dx/f(x)$ can be regular if and only if there is an effective integer linear combination of vectors

$$(p-1,p-1,p-1,-(p+1)),(ja,jb,jc,-j(a+b+c)),(p,0,0,-p),(0,p,0,-p),(0,0,p,-p).$$

To see whether an effective linear combination exists, we make the first three entries (corresponding to orders of vanishing at $0,1,\lambda$) as small as possible. This is clearly achieved by the vector

$$(\langle aj-1\rangle_p,\langle bj-1\rangle_p,\langle cj-1\rangle_p,2p-4-\langle aj-1\rangle_p-\langle bj-1\rangle_p-\langle cj-1\rangle_p)$$

It follows that if $\langle aj-1\rangle_p+\langle bj-1\rangle_p+\langle cj-1\rangle_p\geq 2p-3$, then there are no forms of weight j on the generic C_λ, i.e. $\mathrm{H}^0(C_\lambda,\omega_{C_\lambda})_j=(0)$ for the generic λ. Thus $\det\mathbb{E}_j\cdot F_{a,b,c,d}=0$. We summarize the discussion so far in the following proposition.

Proposition 4.1. *If $\langle aj\rangle_p+\langle bj\rangle_p+\langle cj\rangle_p+\langle dj\rangle_p=3p$, then*

$$\det\mathbb{E}_j\cdot F_{a,b,c,d}=0.$$

Proof. By Lemma 6.5, we have $\mathrm{H}^0(C,\omega_C)_j=(0)$ for the generic curve in Family (4.1). □

We now proceed to generalize this observation. We note that the results of the following proposition are not new and can be found in [BM10] and [EKZ10]. For completeness, we include proofs in Section 6. We use the notation of Construction 6.6.

Proposition 4.2. *Let $\mathbb{E}$ be the Hodge bundle of the universal cyclic μ_p-cover of type (a,b,c,d) over $\overline{M}_{0,4}$. Then:*

(1) *The eigenbundle $\mathbb{E}_j$ has rank 2 and $c_1(\mathbb{E}_j)=0$ for all $j\in\{0,1,\dots,p-1\}$ such that*
$$\langle aj\rangle_p+\langle bj\rangle_p+\langle cj\rangle_p+\langle dj\rangle_p=p.$$
(2) *The eigenbundle $\mathbb{E}_j$ has rank 1 for all $j\in\{0,1,\dots,p-1\}$ such that*
$$\langle aj\rangle_p+\langle bj\rangle_p+\langle cj\rangle_p+\langle dj\rangle_p=2p.$$
Moreover,
$$\deg \mathbb{E}_j=\frac{1}{p}min\{\langle aj\rangle_p,\langle bj\rangle_p,\langle cj\rangle_p,\langle dj\rangle_p,\langle -aj\rangle_p,\langle -bj\rangle_p,\langle -cj\rangle_p,\langle -dj\rangle_p\}.$$

Proof. This is Proposition 6.7 from Section 6. □

Using Propositions 4.1 and 4.2, we now prove Theorem A from the introduction. We begin with a preliminary lemma.

Lemma 4.3. *Suppose $p\mid n$. Regard an F-curve of type (a,b,c,d) as a map $\iota\colon \overline{M}_{0,4}\to\overline{M}_{0,n}$. Let $\mathbb{E}$ be the pullback to $\overline{M}_{0,n}$ of the Hodge bundle under the cyclic covering morphism $f_{n,p}\colon \overline{M}_{0,n}\to\overline{M}_{(n-2)(p-1)/2}$ and let $\mathbb{F}$ be the pullback to $\overline{M}_{0,4}$ of the Hodge bundle under the weighted cyclic covering morphism $f_{(a,b,c,d),p}\colon \overline{M}_{0,4}\to\overline{M}_h$. Then for every character j of μ_p, we have*
$$c_1(\iota^*\mathbb{E}_j)=c_1(\mathbb{F}_j).$$

Proof. In the situation of the lemma, $\mathbb{E}$ is an extension of $\mathbb{F}$ by a trivial vector bundle on $\overline{M}_{0,4}$. The statement follows. □

Theorem 4.4. *Let $\mathbb{E}$ be the pullback to $\overline{M}_{0,n}$ of the Hodge bundle over $\overline{\mathcal{M}}_g$ via the cyclic p-covering morphism $f_{n,p}$. Let $\mathbb{E}_j$ be the eigenbundle of $\mathbb{E}$ associated to the character j of μ_p. Then the eigenbundle $\mathbb{E}_j$ is semipositive. The determinant line bundle $\lambda_{n,p}(j):=\det\mathbb{E}_j$ is nef and*
$$\lambda_{n,p}(j)=\frac{1}{p}\mathbb{D}^1_{n,jn/p},$$
where $\mathbb{D}^1_{n,jn/p}$ is the symmetric $\mathfrak{sl}_n$ level 1 conformal blocks divisor associated to the fundamental weight jn/p.

Proof. By Kollár's semipositivity results [Kol90, Theorem 4.3 and Remark 4.4], the Hodge bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_g$ is semipositive in characteristic 0. Thus every eigenbundle $\mathbb{E}_j$ is semipositive. We conclude that $\det(\mathbb{E}_j)$ is nef. Finally, by Propositions 4.1–4.2, Lemma 4.3, and [Fak09, Proposition 5.2] the degrees of $p\lambda_{n,p}(j)$ and $\mathbb{D}^1_{n,jn/p}$ are equal on every F-curve. □

The following result shows that by considering weighted cyclic covering morphisms every $\mathfrak{sl}_p$ level 1 conformal blocks line bundle on $\overline{M}_{0,n}$ arises as the determinant of the eigenbundle corresponding to the character $j=1$, after a suitable choice of a weighted covering morphism.

Theorem 4.5. *For a weight vector $\vec{d}=(d_1,\dots,d_n)$, let p be an integer dividing $\sum_{i=1}^n d_i$. Denote by $\mathbb{E}$ the pullback to $\overline{M}_{0,n}$ of the Hodge bundle over $\overline{\mathcal{M}}_g$ via the weighted cyclic p-covering morphism $f_{\vec{d},p}$. Let $\mathbb{E}_1$ be the eigenbundle of $\mathbb{E}$ associated to the character $j=1$ of μ_p. Then $\mathbb{E}_1$ is semipositive and its determinant $\lambda_{\vec{d},p}(1):=\det\mathbb{E}_1$ is nef. Moreover,*

$$\lambda_{\vec{d},p}(1)=\frac{1}{p}\mathbb{D}(\mathfrak{sl}_p,1,(w_{d_1},\dots,w_{d_n})),$$

where $\mathbb{D}(\mathfrak{sl}_p,1,(w_{d_1},\dots,w_{d_n}))$ is the $\mathfrak{sl}_p$ level 1 conformal blocks divisor associated to the sequence $(w_{d_1},\dots,w_{d_n})$ of fundamental weights.

Proof. The proof is the same as that of Theorem 4.4. □

As a corollary of Theorem 2.8 and Theorem 4.4, we obtain the following result.

Proposition 4.6. *Let $\mathbb{E}$ be the pullback of the Hodge bundle over $\overline{\mathcal{M}}_g$ to $\overline{M}_{0,n}$ via the cyclic p-covering morphism $f_{n,p}$. Then for every $j=1,\dots,p-1$, the morphism associated to the semiample line bundle $\lambda_{n,p}(j)\sim\mathbb{D}^1_{n,jn/p}$ contracts boundary divisors Δ_k with $p\mid k$.*

Proof. Each line bundle $\lambda_{n,p}(j)$ is semiample because by Theorem 4.4 it is a multiple of a conformal blocks line bundle $\mathbb{D}^1_{n,jn/p}$, which is generated by global sections [Fak09, Lemma 2.5]. From the eigenbundle decomposition $\mathbb{E}=\bigoplus_{j=1}^{p-1}\mathbb{E}_j$, we deduce that $f_{n,p}^*(\lambda)$ is an effective combination of $\lambda_{n,p}(j)$ for $j=1,\dots,p-1$. Since the morphism associated to $f_{n,p}^*(\lambda)$ contracts all Δ_k with $p\mid k$ by Theorem 2.8, the same holds for each $\lambda_{n,p}(j)$. □

Example 4.7 (The ψ-class)**.** Take $\vec{d}=(n-2,2,1,1,\dots,1)$ and $p=n-1$. Consider the weighted cyclic covering morphism $f_{\vec{d},p}\colon\overline{M}_{0,n}\to\overline{M}_g$ and let $\mathbb{E}$ be the pullback of the Hodge bundle from $\overline{M}_g$. Let $\lambda_{\vec{d},p}(1)$ be the determinant of the eigenbundle $\mathbb{E}_1$ corresponding to the character $j=1$ of μ_p. Then

$$\psi\sim\sum_{\sigma\in S_n}\lambda_{\sigma(\vec{d}),p}(1).$$

We finish this section by giving a new formula for the classes of line bundles $\lambda_{\vec{d},p}(j)$ and, therefore, for all $\mathfrak{sl}_p$ level 1 conformal blocks divisors. As will be clear from the proof of the proposition, the novelty here is in finding an expression for the divisor classes in question that behaves nicely under restriction to the boundary.

Proposition 4.8. *For a weight vector $\vec{d}=(d_1,\dots,d_n)$, let p be an integer dividing $\sum_{i=1}^n d_i$. Let $\mathbb{E}$ be the pullback to $\overline{M}_{0,n}$ of the Hodge bundle over $\overline{\mathcal{M}}_g$ via the weighted cyclic p-covering morphism $f_{\vec{d},p}$ and let $\mathbb{E}_j$ be the eigenbundle of $\mathbb{E}$*

associated to the character j of μ_p. Then[3]

$$\lambda_{\vec{d},p}(j) := \det \mathbb{E}_j = \frac{1}{2p^2}\left[\sum_{i=1}^{n}\langle jd_i\rangle_p\langle p-jd_i\rangle_p\psi_i - \sum_{I,J}\langle jd(I)\rangle_p\langle jd(J)\rangle_p\Delta_{I,J}\right].$$

Remark 4.9. To get a formula for $\mathbb{D}(\mathfrak{sl}_p, 1, (d_1,\dots,d_n))$, take $j=1$ and multiply by p.

Proof. Set $\mathcal{D}(d_1,\dots,d_n) := \sum_{i=1}^{n}\langle jd_i\rangle_p\langle p-jd_i\rangle_p\psi_i - \sum_{I,J}\langle jd(I)\rangle_p\langle jd(J)\rangle_p\Delta_{I,J}$. We note immediately that both $\mathbb{E}_j$ and $\mathcal{D}(d_1,\dots,d_n)$ behave functorially under restriction to the boundary. Namely, if $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_{n-k})$ and $B\subset\Delta_{I,J}\subset\overline{M}_{0,n}$ is a family of generically reducible curves obtained by gluing families $B_1\subset\overline{M}_{0,k+1}$ and $B_2\subset\overline{M}_{0,n-k+1}$, then

$$\mathcal{D}(d_1,\dots,d_n)\cdot B = D(d_{i_1},\dots,d_{i_k},\sum_{r\in J}d_r)\cdot B_1 + \mathcal{D}(\sum_{r\in I}d_r, d_{j_1},\dots,d_{j_{n-k}})\cdot B_2.$$

The same holds for $\lambda_{\vec{d},p}(j)$. Therefore, it suffices to show that the degrees of two divisor classes are the same on all F-curves.

Consider the F-curve $F := F_{I_1,I_2,I_3,I_4}$. The moving family over any F-curve has exactly 4 sections of self-intersection (-1) and exactly 3 nodal fibers. Denote

$$a := \langle jd(I_1)\rangle_p,\ b := \langle jd(I_2)\rangle_p,\ c := \langle jd(I_3)\rangle_p,\ d := \langle jd(I_4)\rangle_p,$$

and suppose without loss of generality that $a\le b\le c\le d$. Since both $\mathcal{D}$ and $\lambda_{\vec{d},p}(j)$ are invariant under substitution $j\mapsto p-j$, it suffices to treat the following cases:

Case 1: $a+b+c+d=2p$. In this case the degree of $\mathcal{D}$ on F is

$$\begin{aligned}&a(p-a)+b(p-b)+c(p-c)+d(p-d)\\&\qquad -(a+b)(p-a-b)-(a+c)(p-a-c)-(a+d)(p-a-d)=2pa,\end{aligned}$$

if $a+d\le p$, and is

$$\begin{aligned}&a(p-a)+b(p-b)+c(p-c)+d(p-d)\\&\qquad -(a+b)(p-a-b)-(a+c)(p-a-c)-(b+c)(p-b-c)=2p(p-d),\end{aligned}$$

if $a+d\ge p$.

Case 2: $a+b+c+d=p$. In this case the degree of $\mathcal{D}$ on F is

$$\begin{aligned}&a(p-a)+b(p-b)+c(p-c)+d(p-d)\\&\qquad -(a+b)(p-a-b)-(a+c)(p-a-c)-(a+d)(p-a-d)=0.\end{aligned}$$

By comparing these formulae with the formulae of Proposition 6.7 we conclude the proof.

∎

[3] As before, we denote by $\langle a\rangle_p$ the representative in $\{0,1,\dots,p-1\}$ of the residue of a modulo p.

5. New extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,n})$

In this section, we prove Theorem C from the introduction and thus construct new extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,n})$. Our approach is directly via the intersection theory for one-parameter families of curves. The key ingredient of our construction is the following result due to Stankova:

Proposition 5.1 ([SF00, Theorem 7.3]). *If $\mathcal{X} \to C$ is a family of generically smooth trigonal curves of genus g, then $(\delta_{irr} \cdot C)/(\lambda \cdot C) \leq 36(g+1)/(5g+1)$.*

Proof. In fact, Stankova proves a stronger inequality

$$(\delta \cdot C) \leq \frac{36(g+1)}{(5g+1)}(\lambda \cdot C).$$

The statement now follows from $(\delta \cdot C) = (\delta_{\mathrm{irr}} \cdot C) + (\delta_{\mathrm{red}} \cdot C) \geq (\delta_{\mathrm{irr}} \cdot C)$ for any generically smooth family of stable curves. □

Theorem 5.2. *Suppose $3 \mid n$. Consider the cyclic covering morphism $f_{n,3}\colon \overline{M}_{0,n} \to \overline{M}_{n-2}$. The line bundle*

$$f_{n,3}^*(9\lambda - \delta_{irr}) = 2\psi - 2\Delta - \sum_{3|k} \Delta_k$$

generates an extremal ray of $\widetilde{M}_{0,n}$.

Proof. We compute the divisor class using Proposition 3.1 to obtain

$$f_{n,3}^*(9\lambda - \delta_{\mathrm{irr}}) = 2(\psi - \Delta + \sum_{3|k} \Delta_k) - 3\sum_{3|k} \Delta_k = 2\psi - 2\Delta - \sum_{3|k} \Delta_k.$$

Proof of nefness: Does writing divisor $2\psi - 2\Delta - \sum_{3|k} \Delta_k$ as a pullback of $9\lambda - \delta_{\mathrm{irr}}$ from $\overline{M}_{n-2}$ help to establish its nefness on $\widetilde{M}_{0,n}$? By now even a casual reader will guess that the answer is yes. Suppose that a family $\mathcal{X} \to B$ of stable μ_3-covers is generically reducible. Then we can write $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_i \to B$ are themselves families of stable μ_3-covers, and where the union is formed by identifying sections. We have then the inequality

$$(9\lambda - \delta_{\mathrm{irr}})_{\mathcal{X}/B} \geq (9\lambda - \delta_{\mathrm{irr}})_{\mathcal{X}_1/B} + (9\lambda - \delta_{\mathrm{irr}})_{\mathcal{X}_2/B}.$$

Therefore, it suffices to show that $9\lambda - \delta_{\mathrm{irr}}$ is non-negative on every family of generically smooth cyclic triple covers. Clearly, the genus of a μ_3-cover is at least 2. If the genus is 3 or more, then we are done by the inequality $\dfrac{36(g+1)}{5g+1}\lambda - \delta_{\mathrm{irr}} \geq 0$ of Proposition 5.1. It remains to treat the genus 2 case. The generic μ_3-cover of genus 2 is a degree 3 cover of $\mathbb{P}^1$ branched over 4 points with ramification profile $(\tau^2, \tau^2, \tau, \tau)$ where τ is a 3-cycle in S_3. A family of genus 2 stable μ_3-covers is obtained by varying the cross-ratio of 4 points on $\mathbb{P}^1$. But the divisor $f_{n,3}^*(9\lambda - \delta_{\mathrm{irr}})$ is zero on such a family by an explicit computation, or by observing that $f_{n,3}^*(9\lambda - \delta_{\mathrm{irr}}) = 2\psi - 2\Delta - \sum_{3|k} \Delta_k$ is zero on any F-curve congruent to $(2, 2, 1, 1)$ modulo 3 (and positive on any other F-curve)! □

Proof of extremality: Since the Picard number of $\widetilde{M}_{0,n}$ is $\lfloor n/2 \rfloor - 1$, a nef divisor generates an extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,n})$ if it intersects trivially $\lfloor n/2 \rfloor - 2$ linearly independent effective curves in $N_1(\widetilde{M}_{0,n})$. We follow the approach of [AGSS10] and look for $\lfloor n/2 \rfloor - 2$ linearly independent F-curves which $2\psi - 2\Delta - \sum_{3|k} \Delta_k$ intersects trivially. By above, we have to consider F-curves congruent to $(2,2,1,1)$ modulo 3.

We treat the case of $n \equiv 0 \mod 12$ in full detail and indicate the necessary modifications for the remaining cases.

Case of $n = 12t$: It is easy to verify that for $n = 12$ the F-curves $F_{5,5,1,1}, F_{4,4,2,2}, F_{1,2,2,7}, F_{1,1,2,8}$ are linearly independent. Suppose now $t \geq 2$. We let

$$\mathcal{N}_i = \{F_{6i+1,1,2,n-4-6i}, F_{6i+1,2,2,n-5-6i}, F_{6i+2,1,1,n-4-6i}, F_{6i+5,1,1,n-7-6i}, F_{6i+4,2,2,n-8-6i}, F_{6i+5,1,2,n-8-6i}\}$$

for $i = 0, \ldots, t-2$, and let $\mathcal{N}_{t-1} = \{F_{6t-5,1,2,6t+2}, F_{6t-4,1,1,6t+2}, F_{6t-5,2,2,6t+1}, F_{6t-1,1,1,6t-1}\}$. Then for $1 \leq i \leq t-2$, the intersection pairing of $\mathcal{N}_i$ with the divisors $\{\Delta_{6i+k}\}_{k=3}^{8}$ is the following:

	Δ_{6i+3}	Δ_{6i+4}	Δ_{6i+5}	Δ_{6i+6}	Δ_{6i+7}	Δ_{6i+8}
$F_{6i+1,1,2,n-4-6i}$	1	−1	0	0	0	0
$F_{6i+2,1,1,n-4-6i}$	2	−1	0	0	0	0
$F_{6i+1,2,2,n-5-6i}$	2	0	−1	0	0	0
$F_{6i+4,2,2,n-8-6i}$	0	−1	0	2	0	−1
$F_{6i+5,1,1,n-7-6i}$	0	0	−1	2	−1	0
$F_{6i+5,1,2,n-8-6i}$	0	0	−1	1	1	−1

When $i = 0$, the matrix is slightly modified:

	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
$F_{1,1,2,n-4}$	2	−1	0	0	0	0
$F_{1,2,2,n-5}$	2	1	−1	0	0	0
$F_{4,1,2,n-7}$	1	−1	1	1	−1	0
$F_{5,1,1,n-7}$	0	0	−1	2	−1	0
$F_{4,2,2,n-8}$	0	−1	0	2	0	−1
$F_{5,1,2,n-8}$	1	0	−1	1	1	−1

Finally, the intersection pairing of $\mathcal{N}_{t-1}$ with Δ_{6t-3}, Δ_{6t-2}, Δ_{6t-1}, Δ_{6t} is

	Δ_{6t-3}	Δ_{6t-2}	Δ_{6t-1}	Δ_{6t}
$F_{6t-5,1,2,6t+2}$	1	−1	0	0
$F_{6t-4,1,1,6t+2}$	2	−1	0	0
$F_{6t-5,2,2,6t+1}$	2	0	−1	0
$F_{6t-1,1,1,6t-1}$	0	0	−2	2

Remaining cases: If $n = 12t + 3$, we take

$$\mathcal{N}_{t-1} = \{F_{6t-5,1,2,6t+5}, F_{6t-4,1,1,6t+5}, F_{6t-1,1,1,6t+2}, F_{6t-4,1,2,6t+4}, F_{6t-1,1,2,6t+1}\}.$$

Then the intersection pairing of $\mathcal{N}_{t-1}$ with Δ_{6t-3}, Δ_{6t-2}, Δ_{6t-1}, Δ_{6t}, Δ_{6t+1} is nondegenerate. If $n = 12t + 6$, we take

$$\mathcal{N}_{t-1} = \{F_{6t-5,1,2,6t+8}, F_{6t-4,1,1,6t+8}, F_{6t-5,2,2,6t+7}, F_{6t-1,1,1,6t+5}, F_{6t-2,1,2,6t+5}, F_{6t-2,2,2,6t+4}, F_{6t+1,2,2,6t+1}\}.$$

Then the intersection pairing of $\mathcal{N}_{t-1}$ with Δ_{6t-3}, Δ_{6t-2}, Δ_{6t-1}, Δ_{6t}, Δ_{6t+1}, Δ_{6t+2}, Δ_{6t+3} is nondegenerate. Finally, if $n = 12t + 9$, we take

$$\mathcal{N}_{t-1} = \{F_{6t-5,1,2,6t+11}, F_{6t-4,1,1,6t+11}, F_{6t-5,2,2,6t+10}, F_{6t-1,1,1,6t+8}, F_{6t-2,1,2,6t+8}, F_{6t-2,2,2,6t+7}, F_{6t+1,1,2,6t+5}, F_{6t+1,2,2,6t+4}\}.$$

Then the intersection pairing of $\mathcal{N}_{t-1}$ with Δ_{6t-3}, Δ_{6t-2}, Δ_{6t-1}, Δ_{6t}, Δ_{6t+1}, Δ_{6t+2}, Δ_{6t+3}, Δ_{6t+4} is nondegenerate.

□

Remark 5.3. We remark that $2\psi - 2\Delta - \sum_{3|k} \Delta_k$ is clearly F-nef: It intersects every F-curve non-negatively and has degree 0 precisely on F-curves congruent to $(2, 2, 1, 1)$ modulo 3.

The divisor $D := 2\psi - 2\Delta - \sum_{3|k} \Delta_k$ of Theorem 5.2 can be rewritten as

$$2\big(K_{\overline{M}_{0,n}} + \sum_{3\nmid k} \Delta_k + \frac{1}{2}\sum_{3|k} \Delta_k\big).$$

Because $D/2$ is F-nef and of the form $K_{\overline{M}_{0,n}} + G$ where $\Delta - G \geq 0$, a theorem of Farkas and Gibney [FG03, Theorem 4] implies that D is nef. Nef and big log canonical divisors are expected to be semiample. Whether this is the case is still an open question.

In a similar vein, if we allow ourselves to use the results of higher-dimensional birational geometry, such as the Contraction Theorem, as in [KM96, FG03], we obtain the following.

Proposition 5.4. *Suppose $p \mid n$. Then the divisor*

$$\psi - \Delta - \frac{1}{2}\sum_{p|k} \Delta_k = K_{\overline{M}_{0,n}} + \sum_{p\nmid k} \Delta_k + \frac{1}{2}\sum_{p|k} \Delta_k$$

is nef.

Proof. Indeed, the divisor in question is easily seen to be F-nef. A theorem of Farkas and Gibney [FG03, Theorem 4] finishes the proof. □

We note that the divisor of Proposition 5.4 clearly lies on the boundary of $\mathrm{Nef}(\widetilde{M}_{0,n})$. However, it does not generate an extremal ray for $p \geq 4$. We also note that if $p \geq 3$ is prime, then

$$\psi - \Delta - \frac{1}{2}\sum_{p|k} \Delta_k \sim f_{n,p}^* \left(\frac{8p^2}{p^2-1}\lambda - \delta_{\mathrm{irr}}\right).$$

5.1. **Extensions and examples.** We believe that Theorem 5.2 admits a generalization that produces a manifold of new extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,n})$ coinciding with extremal rays of the F-cone. We prove one such generalization in the case $p = 5$ below. Proving that an extremal ray of the F-cone is actually generated by a nef divisor serves two purposes. On the one hand, it brings us closer to the proof of the F-conjecture. On the other hand, it delineates the region of the F-cone where one should look (or not look) for counterexamples.

Recall that the Hodge class λ gives a measure of variation in moduli for one-parameter families of stable curves: As long as the (normalization of the) members of the family vary nontrivially, the degree of λ is positive. A related observation holds for the divisor class δ_{irr}: if the degree of δ_{irr} is positive on a one-parameter family of stable curves, then the variation in moduli in the family is nontrivial and so the degree of λ is also positive. (This property clearly fails for all other boundary divisor classes, as the divisor Δ_{a-1} and the curve T_a illustrate; see Construction 2.5).

For every closed subvariety $Z \subset \overline{M}_g$, there exists a positive constant c such that $c\lambda - \delta_{\mathrm{irr}}$ lies on the boundary of $\mathrm{Nef}(Z)$. For $Z = \overline{M}_g$, it is well-known that $12\lambda - \delta_{\mathrm{irr}}$ is nef. In fact, it generates an extremal ray of $\mathrm{Nef}(\overline{M}_g)$; see [Fab96, GKM02]. For $Z = \overline{\{\text{cyclic trigonal curves}\}}$, Theorem 5.2 shows that $9\lambda - \delta_{\mathrm{irr}}$ is an extremal ray of $\mathrm{Nef}(Z)$.

In a similar vein, there is evidence that for every prime $p \geq 3$ and $j \in \{1, \dots, p-1\}$, the divisor $2p^2\lambda_{n,p}(j) - \delta_{\mathrm{irr}}$ generates an extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,n})$. For $p = 5$, this observation is formalized in the following proposition.

Proposition 5.5. *Suppose* $5 \mid n$. *The divisor classes*

$$f_{n,5}^*(50\lambda_{n,5}(1) - \delta_{irr}) = 4\psi - 4 \sum_{k \equiv 1,4 \bmod 5} \Delta_k - 6 \sum_{k \equiv 2,3 \bmod 5} \Delta_k - 5 \sum_{5|k} \Delta_k,$$

$$f_{n,5}^*(50\lambda_{n,5}(2) - \delta_{irr}) = 6\psi - 6 \sum_{k \equiv 1,4 \bmod 5} \Delta_k - 4 \sum_{k \equiv 2,3 \bmod 5} \Delta_k - 5 \sum_{5|k} \Delta_k$$

are nef on $\overline{M}_{0,n}$.

Remark 5.6. It is almost certain that both of these divisors in fact generate an extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,n})$ (for $5 \mid n$). This is verified for $n = 10$ below and can be verified by hand for $n = 15$. We omit a verification in the general case, which, in the absence of any further insight, would be a tedious exercise in linear algebra.

Remark 5.7. It is not hard to see that the divisor

$$f_{n,5}^*(50\lambda_{n,5}(2) - \delta_{\mathrm{irr}}) = 6\psi - 6 \sum_{k \equiv 1,4 \bmod 5} \Delta_k - 4 \sum_{k \equiv 2,3 \bmod 5} \Delta_k - 5 \sum_{5|k} \Delta_k$$

is not log canonical already for $n \geq 25$. Therefore, the techniques of [KM96, FG03] cannot be used to establish its nefness for $n \geq 25$. This divisor also does not appear to be a conformal blocks divisor.

Proof. First, we verify that $f_{n,5}^*(50\lambda_{n,5}(j) - \delta_{\mathrm{irr}})$ has non-negative degree on every F-curve by considering all possible F-curves modulo 5: An F-curve of type with (a, b, c, d) with $5 \mid abcd$ is easily seen to intersect the divisors in question positively. An F-curve of type $(1, 4, 1, 4)$ modulo 5 intersects $f_{n,5}^*(50\lambda_{n,5}(1) - \delta_{\mathrm{irr}})$ in degree 0, and $f_{n,p}^*(50\lambda_{n,5}(2) - \delta_{\mathrm{irr}})$ in degree 10. An F-curve of type $(2, 3, 2, 3)$ modulo 5 intersects $f_{n,5}^*(50\lambda_{n,5}(1) - \delta_{\mathrm{irr}})$ in degree 10, and $f_{n,p}^*(50\lambda_{n,5}(2) - \delta_{\mathrm{irr}})$ in degree 0. All remaining F-curves intersect divisors in question nontrivially.

From now on, the proof parallels that of Theorem 5.2 above. We begin by observing that classes $50\lambda_{n,5}(j) - \delta_{\mathrm{irr}}$ are superadditive under the operation of normalization along generic nodes. Therefore, it suffices to treat the case of a generically smooth family. We consider a family $(\mathcal{X} \to B; \{\sigma_i\}_{i=1}^m)$ with sections $\{\sigma_i\}_{i=1}^m$ endowed with weights $d_i \in \{1,2,3,4\}$. (This indicates that a weighted cyclic covering morphism is lurking in the background.) For $k \in \{1,2,3,4\}$, let $D_k := \sum_{I,J} \Delta_{I,J}$, where the sum is taken over partitions $I \cup J = \{1,\dots,m\}$ such that $\sum_{i\in I} d_i = k \mod 5$. We also set $\Psi_k = \sum_{i\colon d_i=k} \psi_i$. Then in view of Proposition 4.8, the two divisors $f_{n,5}^*(50\lambda_{n,5}(j) - \delta_{\mathrm{irr}})$, for $j=1$ and $j=2$, become incarnations of the same divisor on the moduli stack of pointed curves with weights. Namely, we have the divisor

$$Q := 4(\Psi_1+\Psi_4) + 6(\Psi_2+\Psi_3) - 4(D_1+D_4) - 6(D_2+D_3) - 5D_5.$$

For a generically smooth stable m-pointed family of rational curves with $m \geq 5$, we always have the inequality $4\psi - 6\Delta \geq 0$, which follows immediately from the identity $(m-1)\psi = \sum_k k(m-k)\Delta_k$ on $\overline{M}_{0,m}$. It follows that Q is non-negative on such families. The generically smooth stable 4-pointed families of rational curves are precisely the F-curves, and Q is non-negative on such families by the inspection above. The proposition follows.

□

We proceed to show that the divisors obtained from the cyclic covering morphisms span all extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,n})$ for $n=9$ and $n=10$. All computations with convex polytopes were done using `lrs` [Avi].

5.1.1. *Nef cone of* $\widetilde{M}_{0,9}$. Let $\vec{d} = (1,1,1,1,1,1,1,1,2)$. The extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,9})$ are as follows:

$$\begin{aligned}
D_1 &= \Delta_2 + 3\Delta_3 + 6\Delta_4 \sim (f_{\vec{d},2}^{S_9})^*(10\lambda - \delta_{\mathrm{irr}} - 2\delta_{\mathrm{red}}),\\
D_2 &= 3\Delta_2 + 3\Delta_3 + 4\Delta_4 \sim (f_{\vec{d},2}^{S_9})^*(\lambda),\\
D_3 &= \Delta_2 + 3\Delta_3 + 2\Delta_4 \sim (f_{9,3})^*(\lambda),\\
D_4 &= \Delta_2 + \Delta_3 + 2\Delta_4 \sim (f_{9,3})^*(9\lambda - \delta_{\mathrm{irr}}).
\end{aligned}$$

We note that the divisor class $10\lambda - \delta_{\mathrm{irr}} - 2\delta_{\mathrm{red}}$ generates an extremal ray of $\mathrm{Nef}(\overline{M}_g)$ for every $g \geq 2$ by [GKM02].

5.1.2. *Nef cone of* $\widetilde{M}_{0,10}$. The F-curves on $\widetilde{M}_{0,10}$ and their coordinates in the standard basis $\Delta_2, \Delta_3, \Delta_4, \Delta_5$ are as follows:

$C_1 = F_{7,1,1,1} = (3,-1,0,0)$	$C_2 = F_{6,2,1,1} = (0,2,-1,0)$	$C_3 = F_{5,3,1,1} = (1,-1,2,-1)$
$C_4 = F_{5,2,2,1} = (-2,2,1,-1)$	$C_5 = F_{4,4,1,1} = (1,0,-2,2)$	$C_6 = F_{4,3,2,1} = (-1,0,0,1)$
$C_7 = F_{4,2,2,2} = (-3,0,2,0)$	$C_8 = F_{3,3,3,1} = (0,-3,3,0)$	$C_9 = F_{3,3,2,2} = (-2,-2,1,2)$

Using `lrs`, we compute the extremal rays of the F-cone. Using cyclic covering morphisms, we prove that every extremal ray is generated by nef divisors. The results are listed in Table 1. We refer to [AGSS10, AGS10] for background on $\mathfrak{sl}_n$ and $\mathfrak{sl}_2$ conformal blocks divisors on $\overline{M}_{0,n}$.

Extremal ray of $\mathrm{Nef}(\widetilde{M}_{0,10})$	Orthogonal F-curves	Cyclic covering interpretation	Conformal blocks interpretation
$4\Delta_2+6\Delta_3+6\Delta_4+7\Delta_5$	C_7, C_8, C_9	$50\lambda_{10,5}(2)-\delta_{\mathrm{irr}}$ (Prop. 5.5)	N/A
$2\Delta_2+6\Delta_3+6\Delta_4+5\Delta_5$	C_1, C_5, C_8	$(f^{S_9}_{(0,1,\dots,1),3})^*\lambda \sim \lambda_{10,10}(3)$	$\mathbb{D}(\mathfrak{sl}_{10}, 1, w_3^{10})$
$4\Delta_2+3\Delta_3+6\Delta_4+4\Delta_5$	$C_2, C_4, C_5, C_6, C_7, C_9$	$f^*_{10,2}\lambda=\lambda_{10,2}(1)\sim\lambda_{10,10}(5)$	$\mathbb{D}(\mathfrak{sl}_{10}, 1, w_5^{10})$ $\sim$ $\mathbb{D}(\mathfrak{sl}_2, k, k^{10})$
$2\Delta_2+6\Delta_3+12\Delta_4+11\Delta_5$	C_1, C_2, C_5	$f^*_{10,2}(10\lambda-\delta_{\mathrm{irr}}-2\delta_{\mathrm{red}})$ $\sim 50\lambda_{10,5}(1)-\delta_{\mathrm{irr}}$	$\mathbb{D}(\mathfrak{sl}_2, 3, 1^{10})$
$2\Delta_2+3\Delta_3+3\Delta_4+5\Delta_5$	C_3, C_4, C_7, C_8	$\lambda_{10,5}(2)\sim\lambda_{10,10}(4)$	$\mathbb{D}(\mathfrak{sl}_{10}, 1, w_4^{10})$
$\Delta_2+3\Delta_3+3\Delta_4+4\Delta_5$	C_1, C_3, C_8	$f^*_{10,2}(12\lambda-\delta_{\mathrm{irr}})$	$\mathbb{D}(\mathfrak{sl}_2, 2, 1^{10})$
$\Delta_2+3\Delta_3+6\Delta_4+10\Delta_5$	C_1, C_2, C_3, C_4	$\lambda_{10,5}(1)\sim\lambda_{10,10}(2)$	$\mathbb{D}(\mathfrak{sl}_{10}, 1, w_2^{10})$

TABLE 1. Extremal rays of $\mathrm{Nef}(\widetilde{M}_{0,10})$ via the cyclic covering morphisms.

6. Computing degrees of eigenbundles $\mathbb{E}_j$

In this section, we collect main technical results concerning the eigenbundles $\mathbb{E}_j$ defined in Section 4. These results are well-known. The eigenbundle decomposition of the Hodge bundle over a family of cyclic covers of $\mathbb{P}^1$ with 4 branch points has been considered in [BM10, Section 3] and [EKZ10].

We have decided to include these computations for two reasons. One reason is completeness: with this section the paper becomes essentially self-contained. The second reason is that we work exclusively in the algebraic category, and so all of the results continue to hold in sufficiently high positive characteristic.

6.1. Weight j forms on μ_p-covers of $\mathbb{P}^1$ with 3 branch points.

Definition 6.1 ((Branched covers of a 3-pointed $\mathbb{P}^1$))**.** We define $C(a,b)$ to be the normalization of the curve defined by the equation $y^p = x^a(x-1)^b$. The resulting branched cover[4] $\pi\colon C(a,b)\to\mathbb{P}^1$ has branch points at 0, 1, and ∞. Set $c = \langle p-a-b\rangle_p$. We consider the reduced divisors $[0] := \pi^{-1}(0)$, $[1] := \pi^{-1}(1)$, and $[\infty] := \pi^{-1}(\infty)$. Evidently, $\deg[0] = \gcd(a,p)$, $\deg[1] = \gcd(b,p)$, and $\deg[\infty] = \gcd(c,p)$. Note that by symmetry $C(a,b) = C(a,c) = C(b,c)$.

Lemma 6.2. *Let $C = C(a,b)$ be as in Definition 6.1. The weight spaces of $\mathrm{H}^0(C,\omega_C)$ with respect to the μ_p-action are computed as follows. Consider the unique integers k and ℓ satisfying*

$$aj-\gcd(a,p) = kp + \langle aj-\gcd(a,p)\rangle_p,$$
$$bj-\gcd(b,p) = \ell p + \langle bj-\gcd(b,p)\rangle_p,$$

[4] No divisibility conditions on a and b are imposed; in particular, $C(a,b)$ can be disconnected.

and define $\omega := y^j\, dx/x^{k+1}(x-1)^{\ell+1}$.

(a) *If* $\langle aj\rangle_p = 0$ *or* $\langle bj\rangle_p = 0$, *then* $\mathrm{H}^0(C,\omega_C)_j = (0)$.
(b) *If* $\langle aj\rangle_p + \langle bj\rangle_p \geq p$, *then* $\mathrm{H}^0(C,\omega_C)_j = (0)$.
(c) *If* $\langle aj\rangle_p\langle bj\rangle_p > 0$ *and* $\langle aj\rangle_p + \langle bj\rangle_p \leq p-1$, *then* $\mathrm{H}^0(C,\omega_C)_j = \mathrm{span}\{\omega\}$.

Proof. Every rational weight j form on C looks like $y^j\, dx/g(x)$. We begin by writing down the relevant divisors on $C(a,b)$:

$$\begin{aligned}
(y) &= \frac{a}{\gcd(a,p)}[0] + \frac{b}{\gcd(b,p)}[1] - \left(\frac{a}{\gcd(a,p)} + \frac{b}{\gcd(b,p)}\right)[\infty],\\
(dx) &= \frac{p-\gcd(a,p)}{\gcd(a,p)}[0] + \frac{p-\gcd(b,p)}{\gcd(b,p)}[1] - \left(\frac{p}{\gcd(c,p)} + 1\right)[\infty],\\
(x) &= \frac{p}{\gcd(a,p)}[0] - \frac{p}{\gcd(a,p)}[\infty],\\
(x-1) &= \frac{p}{\gcd(b,p)}[1] - \frac{p}{\gcd(b,p)}[\infty].
\end{aligned}$$

The key observation is that $\omega = y^j\, dx/x^{k+1}(x-1)^{\ell+1}$ is a rational form of weight j which is regular on $\mathbb{P}^1 \smallsetminus \infty$ and has the least possible orders of vanishing along $[0]$ and $[1]$. Namely, $\gcd(a,p)\operatorname{ord}_0(\omega) = \langle aj-\gcd(a,p)\rangle_p$ and $\gcd(b,p)\operatorname{ord}_1(\omega) = \langle bj-\gcd(b,p)\rangle_p$. Note that $\deg\omega = p-\gcd(a,p)-\gcd(b,p)-\gcd(c,p)$. If $\langle aj\rangle_p = 0$ or $\langle bj\rangle_p = 0$, then we see immediately that $\operatorname{ord}_\infty(\omega) < 0$. It follows that $\mathrm{H}^0(C,\omega_C)_j = (0)$. If $\langle aj\rangle_p + \langle bj\rangle_p \geq p$, then $\langle aj-\gcd(a,p)\rangle_p + \langle bj-\gcd(b,p)\rangle_p \geq p-\gcd(a,p)-\gcd(b,p)$. Thus $\operatorname{ord}_\infty(\omega) < 0$ and so $\mathrm{H}^0(C,\omega_C)_j = (0)$.

Finally, if $\langle aj\rangle_p\langle bj\rangle_p > 0$ and $\langle aj\rangle_p + \langle bj\rangle_p \leq p-1$, then we have

$$\begin{aligned}
\gcd(c,p)\operatorname{ord}_\infty(\omega) &= p-\gcd(a,p)-\gcd(b,p)-\gcd(c,p)\\
&\quad - \langle aj-\gcd(a,p)\rangle_p - \langle bj-\gcd(b,p)\rangle_p \geq 1-\gcd(c,p).
\end{aligned}$$

Since $\gcd(c,p)\operatorname{ord}_\infty(\omega)$ is divisible by $\gcd(c,p)$, it follows that $p-1 \geq \gcd(c,p)\operatorname{ord}_\infty(\omega) \geq 0$. We conclude that ω is a unique (up to scaling) regular form of weight j. ☐

Remark 6.3. In the situation of Lemma 6.2 (c), we have $k = \lfloor aj/p\rfloor$ and $\ell = \lfloor bj/p\rfloor$.

6.2. Weight j forms on μ_p-covers of $\mathbb{P}^1$ with 4 branch points.

Definition 6.4 ((Branched covers of a 4-pointed $\mathbb{P}^1$))**.** Suppose $\lambda \neq 0,1,\infty$. We define $C(a,b,c)$ to be the normalization of the curve defined by the equation

$$y^p = x^a(x-1)^b(x-\lambda)^c.$$

We consider the resulting branched cover[5] $\pi\colon C(a,b,c) \to \mathbb{P}^1$ with branch points 0, 1, λ, ∞. Set $d = \langle p-a-b-c\rangle_p$. We consider the reduced divisors $[0] := \pi^{-1}(0)$, $[1] := \pi^{-1}(1)$, $[\lambda] = \pi^{-1}(\lambda)$, and $[\infty] := \pi^{-1}(\infty)$. Evidently, $\deg[0] = \gcd(a,p)$, $\deg[1] = \gcd(b,p)$, $\deg(\lambda) = \gcd(c,p)$, and $\deg[\infty] = \gcd(d,p)$. Note that by symmetry $C(a,b,c) = C(a,c,d) = C(a,b,d) = C(b,c,d)$.

[5] No divisibility conditions on a, b, and c are imposed; in particular, $C(a,b,c)$ can be disconnected.

Lemma 6.5. *Let $C = C(a,b,c)$ be as in Definition 6.4. The weight spaces of $\mathrm{H}^0(C,\omega_C)$ with respect to the μ_p-action are as follows. Consider the unique integers k, ℓ, and m satisfying*

$$\begin{aligned} aj - \gcd(a,p) &= kp + \langle aj - \gcd(a,p)\rangle_p, \\ bj - \gcd(b,p) &= \ell p + \langle bj - \gcd(b,p)\rangle_p, \\ cj - \gcd(c,p) &= mp + \langle cj - \gcd(c,p)\rangle_p, \end{aligned}$$

and define $\omega := y^j dx/x^{k+1}(x-1)^{\ell+1}(x-\lambda)^{m+1}$. If $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p = 0$, then $\mathrm{H}^0(C,\omega_C)_j = (0)$. In the case $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p > 0$, we have

(a) *If $\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \geq 2p$, then $\mathrm{H}^0(C,\omega_C)_j = (0)$.*
(b) *If $p \leq \langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \leq 2p-1$, then $\mathrm{H}^0(C,\omega_C)_j = \mathrm{span}\{\omega\}$.*
(c) *If $\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \leq p-1$, then $\mathrm{H}^0(C,\omega_C) = \mathrm{span}\{x\omega, (x-1)\omega\}$.*

Proof. As in the proof of Lemma 6.5, we begin by computing

$$\begin{aligned} (y) &= \frac{a}{\gcd(a,p)}[0] + \frac{b}{\gcd(b,p)}[1] + \frac{c}{\gcd(c,p)}[\lambda] - \left(\frac{a}{\gcd(a,p)} + \frac{b}{\gcd(b,p)} + \frac{c}{\gcd(c,p)}\right)[\infty], \\ (dx) &= \frac{p-\gcd(a,p)}{\gcd(a,p)}[0] + \frac{p-\gcd(b,p)}{\gcd(b,p)}[1] + \frac{p-\gcd(c,p)}{\gcd(c,p)}[0] - \left(\frac{p}{\gcd(d,p)} + 1\right)[\infty], \\ (x) &= \frac{p}{\gcd(a,p)}[0] - \frac{p}{\gcd(a,p)}[\infty], \\ (x-1) &= \frac{p}{\gcd(b,p)}[1] - \frac{p}{\gcd(b,p)}[\infty], \\ (x-\lambda) &= \frac{p}{\gcd(c,p)}[\lambda] - \frac{p}{\gcd(c,p)}[\infty]. \end{aligned}$$

Evidently, $\omega = y^j dx/x^{k+1}(x-1)^{\ell+1}(x-\lambda)^{m+1}$ is a form of weight j which is regular on $\mathbb{P}^1 \smallsetminus \infty$ and has the least possible orders of vanishing along [0], [1], and $[\lambda]$. Namely, we have $\gcd(a,p)\operatorname{ord}_0(\omega) = \langle aj - \gcd(a,p)\rangle_p$, $\gcd(b,p)\operatorname{ord}_1(\omega) = \langle bj - \gcd(b,p)\rangle_p$, and $\gcd(c,p)\operatorname{ord}_\lambda(\omega) = \langle cj - \gcd(c,p)\rangle_p$. Note that $\deg\omega = 2p - \gcd(a,p) - \gcd(b,p) - \gcd(c,p) - \gcd(d,p)$.

If $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p = 0$, then $\operatorname{ord}_\infty(\omega) < 0$. It follows that $\mathrm{H}^0(C,\omega_C)_j = (0)$.

If $\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \geq 2p$, then $\langle aj - \gcd(a,p)\rangle_p + \langle bj - \gcd(b,p)\rangle_p + \langle cj - \gcd(c,p)\rangle_p \geq 2p - \gcd(a,p) - \gcd(b,p) - \gcd(c,p)$. Thus $\operatorname{ord}_\infty(\omega) < 0$ and so $\mathrm{H}^0(C,\omega_C)_j = (0)$.

If $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p > 0$ and $p \leq \langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \leq 2p-1$, then $0 \leq \gcd(d,p)\operatorname{ord}_\infty(\omega) \leq p-1$. It follows that ω is a unique (up to scaling) regular form of weight j.

Finally, if $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p > 0$ and $0 \leq \langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p \leq p-1$, then we have that $p \leq \gcd(d,p)\operatorname{ord}_\infty(\omega) \leq 2p-1$. It follows that any other regular form of weight j looks like $g(x)\omega$, where $g(x)$ is a rational function with at worst a single pole at ∞ and no other poles. The statement follows. □

6.3. **Universal μ_p-cover over an F-curve.** We briefly recall the construction of the family of stable μ_p-covers over $\overline{M}_{0,4}$ completing the family of smooth μ_p-covers given by

$$C_\lambda: \quad y^p = x^a(x-1)^b(x-\lambda)^c, \quad \lambda \in \mathbb{P}^1 \smallsetminus \{0,1,\infty\}. \tag{6.1}$$

The construction parallels the global construction outlined in Section 2.1.

Construction 6.6. Begin with a trivial family $\mathcal{X} := \mathbb{P}^1_x \times \mathbb{P}^1_\lambda$ over $\mathbb{P}^1_\lambda$ with 4 sections
$\Sigma_0 : \{x = 0\}$, $\Sigma_1 : \{x = 1\}$, $\Sigma_\infty : \{x = \infty\}$, and $\Sigma_\lambda : \{x = \lambda\}$. Now perform the following steps:

(1) Blow up 3 points where sections intersect; set $\mathcal{X}' = \mathrm{Bl}\,\mathcal{X}$.
(2) Make a base change $B \to \mathbb{P}^1_\lambda$ of degree p totally ramified over $\lambda = 0, 1, \infty$. Set $\mathcal{Y} := \mathcal{X}' \times_{\mathbb{P}^1_\lambda} B$ and $f\colon \mathcal{Y} \to \mathcal{X}$.
(3) Take the degree p branched cover of $\mathcal{Y}$ ramified over $f^*(a\Sigma_0 + b\Sigma_1 + c\Sigma_\lambda + d\Sigma_\infty)$.
(4) Normalize the total space to obtain a family of stable curves $\mathcal{Z} \to B$.

Let $g\colon \mathcal{Z} \to \mathcal{X}$ be the resulting morphism. The strict transforms on $\mathcal{Z}$ of the fibers of $\mathcal{X}$ over $\lambda = 0, 1, \infty$ are denoted F_0, F_1, and F_∞, respectively. The exceptional divisors of g lying over $\lambda = 0, 1, \infty$ are denoted by E_0, E_1, and E_∞. Note that $F_0 = C(a + c, b)$ and $E_0 = C(a, c)$, etc.

We call the family of stable curves obtained in Construction 6.6 the *universal μ_p-cover of type (a, b, c, d) over $\overline{M}_{0,4}$*. It is precisely the moving component of the pullback, by the cyclic covering morphism $f_{n,p}\colon \overline{M}_{0,n} \to \overline{\mathcal{M}}_g$, of the universal family over $\overline{\mathcal{M}}_g$ to an F-curve of type (a, b, c, d).

Proposition 6.7. *Let $\mathbb{E}$ be the Hodge bundle of the universal cyclic μ_p-cover of type (a, b, c, d) over $\overline{M}_{0,4}$. Then:*

(1) *The eigenbundle $\mathbb{E}_j$ has rank 0 for all $j \in \{0, 1, \dots, p-1\}$ such that*
$$\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p + \langle dj\rangle_p = 3p.$$
(2) *The eigenbundle $\mathbb{E}_j$ has rank 2 and $\deg(\mathbb{E}_j) = 0$ for all $j \in \{0, 1, \dots, p-1\}$ such that*
$$\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p + \langle dj\rangle_p = p.$$
(3) *The eigenbundle $\mathbb{E}_j$ has rank 1 for all $j \in \{0, 1, \dots, p-1\}$ such that*
$$\langle aj\rangle_p + \langle bj\rangle_p + \langle cj\rangle_p + \langle dj\rangle_p = 2p.$$
In this case, we have
$$\deg(\mathbb{E}_j) = \frac{1}{p} min\{\langle aj\rangle_p, \langle bj\rangle_p, \langle cj\rangle_p, \langle dj\rangle_p, \langle -aj\rangle_p, \langle -bj\rangle_p, \langle -cj\rangle_p, \langle -dj\rangle_p\}.$$

Proof. If $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p\langle dj\rangle_p = 0$, the statement follows immediately from Lemma 6.5. From now on, we assume that $\langle aj\rangle_p\langle bj\rangle_p\langle cj\rangle_p\langle dj\rangle_p > 0$.

We consider the unique positive integers k, ℓ, m satisfying

$$\begin{aligned} aj - \gcd(a,p) &= kp + \langle aj - \gcd(a,p)\rangle_p, \\ bj - \gcd(b,p) &= \ell p + \langle bj - \gcd(a,p)\rangle_p, \\ cj - \gcd(c,p) &= mp + \langle cj - \gcd(a,p)\rangle_p. \end{aligned}$$

Set $f(x) := x^{k+1}(x-1)^{\ell+1}(x-\lambda)^{m+1}$ and $\omega := y^j dx/f(x)$.

Proof of (1). By Lemma 6.5, the fiber of $\mathbb{E}_j$ at a point of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is empty. The statement follows.

Proof of (2). By Lemma 6.5, the fiber of $\mathbb{E}_j$ at a point of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is spanned by $\omega_0 := x\omega$ and $\omega_1 := (x-1)\omega$. We extend $\omega_0 \wedge \omega_1$ to a global rational section of $\det \mathbb{E}_j$ and compute its zeros and poles.

At $\lambda = 0$, we have that $\omega_0 = x\omega$ is a regular form of weight j on F_0 by Lemma 6.5. We now compute the extension of $\omega_0 - \omega_1 = \omega$ to E_0. Set $x = \overline{x}\lambda$ and $\lambda = \eta^p$. Then $\overline{x}$ and η are local coordinates near the generic point of E_0. The local equation of the branched cover in Step (3) of Construction 6.6 is

$$y^p = x^a(x-1)^b(x-\lambda)^c = \lambda^{a+c}\overline{x}^a(\overline{x}\lambda-1)^b(\overline{x}-1)^c = \eta^{p(a+c)}\overline{x}^a(\overline{x}-1)^c(\overline{x}\lambda-1)^b.$$

It follows that after the normalization in Step (4) of Construction 6.6, E_0 has equation

$$z^p = \overline{x}^a(\overline{x}-1)^c(\overline{x}\lambda-1)^b,$$

where $z = y/\eta^{a+c}$. It follows that, modulo $d\eta$,

$$\begin{aligned} \omega = y^j dx/f(x) = \eta^{p+j(a+c)} z^j d\overline{x} / \big(\eta^{p(k+m+2)}\overline{x}^{k+1}(\lambda\overline{x}-1)^{\ell+1}(\overline{x}-1)^{m+1}\big) \\ = \eta^{j(a+c)-p(k+m)-p}\omega', \end{aligned}$$

where ω' restricts to a generator of $\mathrm{H}^0(E_0, \omega_{E_0})_j$ by Lemma 6.2. We conclude that $\omega_0 \wedge \omega_1$ vanishes to order $j(a+c) - p(k+m+1)$ at $\lambda = 0$. Similarly, $\omega_0 \wedge \omega_1$ vanishes to order $j(b+c) - p(\ell+m+1)$ at $\lambda = 1$.

Finally, we compute the vanishing order of $\omega_0 \wedge \omega_1$ at $\lambda = \infty$. In terms of the local coordinate η in the neighborhood of ∞ on B we have $\lambda = 1/\eta^p$. The equation of the branched cover in Step (3) of Construction 6.6 becomes $(y\eta^c)^p = x^a(x-1)^b(\eta^p x - 1)^c$. After the normalization in Step (4) of Construction 6.6, the equation of $\mathcal{Z}$ in the neighborhood of F_∞ is $z^p = x^a(x-1)^b$, where $z = y\eta^c$. We compute that

$$\omega_0 - \omega_1 = \omega = \eta^{p+pm-cj} z^j dx / x^{k+1}(x-1)^{\ell+1}(\eta^p x - 1)^{m+1} = \eta^{p(m+1)-cj}\omega',$$

where ω' restricts to a generator of $\mathrm{H}^0(F_\infty, \omega_{F_\infty})_j$ by Lemma 6.2.

In the neighborhood of E_∞, we choose local coordinates (η, u) such that $\lambda = 1/\eta^p$ and $x = u/\eta^p$. The local equation of the branched cover in Step (3) of Construction 6.6 becomes $y^p = \frac{1}{\eta^{p(a+b+c)}}(u-\eta^p)^b(u-1)^c u^a$.

It follows that after the normalization in Step (4) of Construction 6.6, the equation of $\mathcal{Z}$ in the neighborhood of E_∞ is $z^p = u^a(u-\eta^p)^b(u-1)^c$, where $z = y\eta^{a+b+c}$.

It follows that, modulo $d\eta$,

$$\omega_0 = x\omega = \eta^{p-j(a+b+c)+kp+mp+\ell p} z^j du/\big(u^k(u-\eta^p)^{m+1}(u-1)^{\ell+1}\big).$$

Summarizing, $\omega_0 \wedge \omega_1$ vanishes to order $p(m+1)-cj+p-j(a+b+c)+kp+mp+\ell p = (2m+2)p+kp+\ell p - j(a+b+2c)$ at ∞. We conclude that

$$\deg(\mathbb{E}_j) = j(a+c)-p(k+m+1)+j(b+c)-p(\ell+m+1)+(2m+2)p+kp+\ell p-j(a+b+2c) = 0.$$

Proof of (3). By Lemma 6.5, the fiber of $\mathbb{E}_j$ at a point of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is one-dimensional. Thus $\mathbb{E}_j$ is a line bundle. We compute $c_1(\mathbb{E}_j)$ by writing down a rational section of $\mathbb{E}_j$ and counting its zeros and poles. We begin with the global rational 1-form $\omega = y^j dx/f(x)$ that restricts to a generator of $\mathrm{H}^0(C_\lambda, \omega_{\mathcal{C}_\lambda})_j$ for all $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$. Suppose, without loss of generality, that $0 < \langle aj\rangle_p \leq \langle bj\rangle_p \leq \langle dj\rangle_p \leq \langle cj\rangle_p$. We will treat only the case $\langle cj\rangle_p + \langle aj\rangle_p \geq p$. Then $p - \langle cj\rangle_p = \min\{\langle aj\rangle_p, \langle bj\rangle_p, \langle cj\rangle_p, \langle dj\rangle_p, \langle -aj\rangle_p, \langle -bj\rangle_p, \langle -cj\rangle_p, \langle -dj\rangle_p\}$.

The key observation is that under our assumptions, the global 1-form ω restricts to the generator of $\mathrm{H}^0(C_\lambda, \omega_{C_\lambda})_j$ for every $\lambda \neq \infty$. Indeed, the assumption $\langle cj\rangle_p + \langle aj\rangle_p \geq p$ implies that at $\lambda = 0$, $\langle (a+c)j\rangle_p + \langle bj\rangle_p + \langle dj\rangle_p = \langle aj\rangle_p + \langle cj\rangle_p + \langle bj\rangle_p + \langle dj\rangle_p - p = p$ and so ω restricts to a regular form on F_0 by Lemma 6.2. Similarly, ω restricts to a regular form on F_1.

By Lemma 6.2, ω will restrict to a multiple of the generator of $\mathrm{H}^0(F_\infty, \omega_{F_\infty})_j$. It remains to compute the order of vanishing of ω at ∞. In terms of the local coordinate η in the neighborhood of ∞ on B, we have $\lambda = 1/\eta^p$. The equation of the branched cover in Step (3) of Construction 6.6 becomes $(y\eta^c)^p = x^a(x-1)^b(\eta^p x - 1)^c$. After the normalization in Step (4) of Construction 6.6, the equation of $\mathcal{Z}$ in the neighborhood of F_∞ is $z^p = x^a(x-1)^b$, where $z = y\eta^c$. We compute that

$$\omega = \eta^{p+pm-cj} z^j dx/x^{k+1}(x-1)^{\ell+1}(\eta^p x-1)^{m+1} = \eta^{p(m+1)-cj}\omega',$$

where ω' restricts to a generator of $\mathrm{H}^0(F_\infty, \omega_{F_\infty})_j$ by Lemma 6.2. It follows that ω vanishes to the order $pm + p - cj = p - \langle cj\rangle_p$.

Taking into the account the factor $1/p$ arising from the degree p base change $B \to \mathbb{P}^1$ in Step (2) of Construction 6.6, we conclude the proof. □

Remark 6.8. Proposition 6.7 is also proved in [EKZ10, Theorem 1]. In particular, the computation similar to that of Proposition 6.7 Part (3) appears in [EKZ10, Section 2.4].

Acknowledgements. We would like to thank Aise Johan de Jong and Ian Morrison for helpful conversations.

Appendix A. Class of the Hodge eigenbundle using orbifold Riemann–Roch by Anand Deopurkar

In Section 6, the class of the Hodge eigenbundle $\mathbb{E}_j$ was computed indirectly by calculating it on the F-curves. We describe a more direct method using the Grothendieck–Riemann–Roch theorem for Deligne–Mumford stacks. Throughout,

μ_r denotes the group of rth roots of unity and $\mathbb{C}(i)$ the line bundle on $B\mu_r$ associated to the character $\zeta \mapsto \zeta^i$.

A.1. **Expressing $\mathbb{E}_j$ as a pushforward.** Let $p \geq 2$ be an integer and $(d_1, \dots, d_n)$ a sequence of non-negative integers such that $\sum d_j = mp$ for some integer m. Consider the cyclic cover $\phi\colon C \to \mathbb{P}^1$ given by the regular projective model of

$$y^p = (x - x_1)^{d_1} \dots (x - x_n)^{d_n}.$$

In these coordinates, O_C is generated as an $O_{\mathbb{P}^1}$ module by the functions 1, y/f_1, $\dots$, y^{p-1}/f_{p-1} where $f_j = (x - x_1)^{\lfloor jd_1/p \rfloor} \dots (x - x_n)^{\lfloor jd_n/p \rfloor}$. Note that these generators are μ_p eigenvectors. Letting D be the $\mathbb{Q}$ divisor $D = \sum (d_i/p)x_i$, we thus have the μ_p-equivariant decomposition

$$\phi_* O_C = \bigoplus_{j=0}^{p-1} O_{\mathbb{P}^1}(-jm) \otimes O_{\mathbb{P}^1}(\lfloor jD \rfloor),$$

where the local generator of the jth summand is y^j/f_j. We then get the μ_p-eigenspace decomposition

$$H^1(C, O_C) = \bigoplus_{j=0}^{p-1} H^1\left(\mathbb{P}^1, O_{\mathbb{P}^1}(-jm) \otimes O_{\mathbb{P}^1}(\lfloor jD \rfloor)\right).$$

The above analysis for an individual cover goes through over the moduli stack. Let $\mathcal{M}$ be the stack of p-divisible n-pointed orbicurves with the universal curve $\pi\colon \mathcal{P} \to \mathcal{M}$ with n sections σ_i and let $\mathcal{L}$ be a line bundle on $\mathcal{P}$ with $\mathcal{L}^p = O_{\mathcal{P}}(\sum d_i \sigma_i)$. On $\mathcal{M}$ lives the universal cyclic p cover $\phi\colon \mathcal{C} \to \mathcal{P}$ whose fibers are as above. Setting $D = \sum (d_i/p)\sigma_i$, we get the the μ_p-equivariant decomposition

$$\phi_* O_{\mathcal{C}} = \bigoplus_{j=0}^{p-1} \mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor).$$

Applying $R\pi_*$ gives the μ_p-eigenbundle decomposition

$$R^1\pi_*(O_{\mathcal{C}}) = \bigoplus_{j=0}^{p-1} R^1\pi_*\left(\mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor)\right).$$

Since the left hand side is the dual of the Hodge bundle, we get

$$\mathbb{E}_j = R^1\pi_*(\mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor))^*. \tag{A.1}$$

A.2. **Applying Grothendieck–Riemann–Roch.** Section A.1 reduces the computation of $c_1(\mathbb{E}_j)$ to the familiar problem of computing the Chern classes of a (derived) push-forward. Since the push forward is from a stack, we have to use a bit of technology. For less cluttered notation, we calculate in the following setting: Let B be a smooth projective curve, $\pi\colon \mathcal{P} \to B$ a generically smooth family of rational orbinodal curves with $\mathcal{P}$ smooth, and $\mathcal{F}$ a line bundle on $\mathcal{P}$. For an orbinode $x \in \mathcal{P}$ with stabilizer μ_{r_x}, let $\mathcal{F}|_x \cong \mathbb{C}(i_x)$, where $0 \leq i_x < r_x$. Denote by ω the class of the relative dualizing sheaf of $\mathcal{P} \to B$.

Proposition A.1. *In the above setup, we have*

$$c_1(R\pi_*\mathcal{F}) = \frac{c_1(\mathcal{F})^2}{2} - \frac{c_1(\mathcal{F})\cdot\omega}{2} - \sum_{Orbi\ x\in\mathcal{P}} \frac{i_x(r_x-i_x)}{2r_x}$$

Proof. By the GRR for Deligne–Mumford stacks [?, Corollary 5.3], we have

$$\operatorname{ch} R\pi_*\mathcal{F} = \pi_*\left(\operatorname{ch}\left(t\left(\frac{f^*\mathcal{F}}{\lambda_{-1}N_f^*}\right)\right)\frac{\operatorname{td}(I\mathcal{P})}{\operatorname{td}(B)}\right), \tag{A.2}$$

where $f\colon I\mathcal{P}\to\mathcal{P}$ is the inertia, N_f^* is the conormal bundle of f, the operator λ_{-1} on $K(I\mathcal{P})$ is the alternating sum of wedge powers $1-\Lambda^1+\Lambda^2+\dots$ and the operator t on $K(I\mathcal{P})\otimes\mathbb{C}$ is the 'twisting operator' [?, Section 4]. We evaluate these objects in our setup. First, note that the coarse space map $\mathcal{P}\to P$ is an isomorphism except possibly at the singular points of $P\to B$, where the local picture is

$$[\operatorname{Spec} k[u,v]/(uv-t)/\mu_r] \to \operatorname{Spec} k[x,y]/(xy-t^r), \tag{A.3}$$

with μ_r acting by $u\mapsto\zeta u$ and $v\mapsto\zeta^{-1}v$. Since the orbistructure is at isolated points, the inertia stack is

$$I\mathcal{P} = \mathcal{P}\sqcup\bigsqcup_{\text{Orbi } x\in\mathcal{P}}(\mu_{r_x}\setminus\{\mathrm{id}\})\times B\mu_{r_x}.$$

The normal bundle N_f is trivial on the $\mathcal{P}$ component and is $\mathbb{C}(1)\oplus\mathbb{C}(-1)$ on the $\{\zeta\}\times B\mu_{r_x}$ component. The twisting operator acts trivially on the $\mathcal{P}$ component and sends $\mathbb{C}(i)$ to $\zeta^i\mathbb{C}(i)$ on the $\{\zeta\}\times B\mu_{r_x}$ component. Thus, Equation A.2 becomes

$$\operatorname{ch} R\pi_*\mathcal{F} = \pi_*\left(1+c_1(\mathcal{F})+\frac{c_1^2(\mathcal{F})}{2}\right)\left(1-\frac{c_1(\Omega_{\mathcal{P}/B})}{2}+\frac{c_1^2(\Omega_{\mathcal{P}/B})+c_2(\Omega_{\mathcal{P}/B})}{12}\right)+\sum_{\substack{\text{Orbi } x\\ 1\neq\zeta\in\mu_{r_x}}}\frac{\zeta^{i_x}}{r_x(2-\zeta-\zeta^{-1})}.$$

Let $\omega_{\mathcal{P}/B}$ be the relative dualizing sheaf. Let $\delta_{\mathcal{P}/B}$ the class of the support of $\operatorname{coker}(\Omega_{\mathcal{P}/B}\to\omega_{\mathcal{P}/B})$ (with multiplicities) so that $c_2(\Omega_{\mathcal{P}/B})=\delta_{\mathcal{P}/B}$. Define $\omega_{P/B}$ and $\delta_{P/B}$ analogously. Identifying the rational Chow groups of $\mathcal{P}$ and P, we have $c_1(\omega_{\mathcal{P}/B})=c_1(\omega_{P/B})=\omega$. The value of δ, however, is different for $\mathcal{P}\to B$ and $P\to B$: the numerical contribution of the orbinode in Equation A.3 is $1/r$ whereas that of the node is r. With these simplifications, we get

$$c_1(R\pi_*\mathcal{F}) = \frac{c_1(\mathcal{F})^2}{2}-\frac{c_1(\mathcal{F})\cdot\omega}{p}+\frac{\omega^2+\delta_{P/B}}{12}+\sum_{\text{Orbi } x}\frac{1/r_x-r_x}{12}+\left(\sum_{1\neq\zeta\in\mu_{r_x}}\frac{\zeta^{i_x}}{r_x(2-\zeta-\zeta^{-1})}\right). \tag{A.4}$$

Since $P\to B$ is a family of rational curves, we have $\omega^2+\delta_{P/B}=0$. The following is a nice exercise

$$\sum_{1\neq\zeta\in\mu_r}\frac{\zeta^i}{2-\zeta-\zeta^{-1}} = \frac{i(i-r)}{2}+\frac{r^2-1}{12}\quad(\text{for } 0\le i<r).$$

Substituting in Equation A.4 gives the result. ∎

Corollary A.2. *The class of the Hodge eigenbundle is given by*

$$c_1(\mathbb{E}_j) = \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p \langle p - jd_i\rangle_p \psi_i - \sum_{I,J} \langle jd(I)\rangle_p \langle jd(J)\rangle_p \Delta_{I,J}\right).$$

Proof. Take a smooth curve B and a map $B \to \mathcal{M}$ transverse to the boundary. Let $\pi\colon \mathcal{P} \to B$ and $\mathcal{L}$ be pullbacks from $\mathcal{M}$. Set $\mathcal{F} = \mathcal{L}^{-j} \otimes O_{\mathcal{P}}(\lfloor jD \rfloor)$. Then $c_1(\mathcal{F}) = -\sum(\langle jd_i\rangle_p/p) \cdot \sigma_i$. Let $x \in \mathcal{P}$ be an orbinode corresponding to the boundary divisor $\Delta_{I,J}$. Set $d = \gcd(p, jd(I))$. Then the stabilizer at x has order $r = p/d$ and $\mathcal{F}|_x \cong \mathbb{C}(i)$ for $i = \langle jd(I)/d\rangle_r$. Also, x contributes r towards $\Delta_{I,J}$. Using Equation A.1 and Theorem A.1, we conclude that

$$c_1(\mathbb{E}_j) = c_1(R\pi_*\mathcal{F}) = \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p^2 \sigma_i^2 + p\langle jd_i\rangle_p \sigma_i \cdot \omega - \sum_{\text{Orbi } x} \frac{p^2 \langle jd(I)/d\rangle_r \langle jd(J)/d\rangle_r}{r}\right)$$

$$= \frac{1}{2p^2}\left(\sum_i \langle jd_i\rangle_p \langle p - jd_i\rangle_p \psi_i - \sum_{I,J} \langle jd(I)\rangle_p \langle jd(J)\rangle_p \Delta_{I,J}\right).$$

□

References

[AGS10] Valery Alexeev, Angela Gibney, and David Swinarski, *Conformal blocks divisors on $\overline{M}_{0,n}$ from sl_2*, `arXiv:1011.6659 [math.AG]`. 2, 20

[AGSS10] Maxim Arap, Angela Gibney, James Stankewicz, and David Swinarski, *sl_n level 1 conformal blocks divisors on $\overline{M}_{0,n}$*, `arXiv:1009.4664 [math.AG]`. 2, 17, 20

[Avi] D. Avis, `lrs` *homepage, 2011*,
Available from `lrs` homepage http://cgm.cs.mcgill.ca/∼avis/C/lrs.html. 20

[BM10] Irene I. Bouw and Martin Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. (2) **172** (2010), no. 1, 139–185. MR 2680418 1, 12, 21

[Che10] D. Chen, *Covers of the projective line and the moduli space of quadratic differentials*, `arXiv:1005.3120 [math.AG]`, 2010. 1

[EKZ10] A. Eskin, M. Kontsevich, and A. Zorich, *Lyapunov spectrum of square-tiled cyclic covers*, `arXiv:1007.5330v2 [math.DS]`. 1, 12, 21, 26

[Fab96] Carel Faber, *Intersection-theoretical computations on $\overline{\mathcal{M}}_g$*, Parameter spaces (Warsaw, 1994), Banach Center Publ., vol. 36, Polish Acad. Sci., Warsaw, 1996, pp. 71–81. MR 1481481 (98j:14033) 1, 19

[Fak09] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, `arXiv:0904.2918v3 [math.AG]`. 2, 13, 14

[Fed10] Maksym Fedorchuk, *Moduli spaces of hyperelliptic curves with A and D singularities*, `arXiv:1007.4828 [math.AG]`, 2010. 6

[FG03] Gavril Farkas and Angela Gibney, *The Mori cones of moduli spaces of pointed curves of small genus*, Trans. Amer. Math. Soc. **355** (2003), no. 3, 1183–1199 (electronic). MR MR1938752 (2003m:14043) 1, 18, 19

[Fon09] Claudio Fontanari, *Positive divisors on quotients of $\overline{M}_{0,n}$ and the Mori cone of $\overline{M}_{g,n}$*, J. Pure Appl. Algebra **213** (2009), no. 4, 454–457. MR MR2483830 (2010h:14045) 1

[Gia10] Noah Giansiracusa, *Conformal blocks and rational normal curves*, `arXiv:1012.4835 [math.AG]`. 2

[Gib09] Angela Gibney, *Numerical criteria for divisors on $\overline{M}_g$ to be ample*, Compos. Math. **145** (2009), no. 5, 1227–1248. MR 2551995 1

[GKM02] Angela Gibney, Sean Keel, and Ian Morrison, *Towards the ample cone of* $\overline{M}_{g,n}$, J. Amer. Math. Soc. **15** (2002), no. 2, 273–294 (electronic). MR MR1887636 (2003c:14029) 1, 19, 20

[HK00] Yi Hu and Seán Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR MR1786494 (2001i:14059) 8

[HM82] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by William Fulton. MR MR664324 (83i:14018) 5

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR MR1631825 (99g:14031) 5

[KM96] Seán Keel and James McKernan, *Contractible extremal rays on* $\overline{M}_{0,n}$, 1996, arXiv:9607.009. 1, 5, 8, 18, 19

[Kol90] János Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268. MR MR1064874 (92e:14008) 12, 13

[Lar09] P. Larsen, *Fulton's conjecture for* $\overline{M}_{0,7}$, `arXiv:0912.3104 [math.AG]`, 2009. 1

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR MR2095471 (2005k:14001a) 6

[McM] Curtis T. McMullen, *Braid groups and Hodge theory*, Math. Ann., to appear. 1

[MO11] Ben Moonen and Frans Oort, *The Torelli locus and special subvarieties*, Handbook of Moduli (G. Farkas and I. Morrison, eds.), to appear, 2011. 1

[Moo10] Ben Moonen, *Special subvarieties arising from families of cyclic covers of the projective line*, Doc. Math. **15** (2010), 793–819. MR 2735989 1

[Pro10] Yuri G. Prokhorov, *Fields of invariants of finite linear groups*, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston Inc., Boston, MA, 2010, pp. 245–273. MR 2605172 (2011d:14017) 4

[Rul06] William F. Rulla, *Effective cones of quotients of moduli spaces of stable* n*-pointed curves of genus zero*, Trans. Amer. Math. Soc. **358** (2006), no. 7, 3219–3237 (electronic). MR 2216265 (2007b:14053) 8

[SF00] Zvezdelina E. Stankova-Frenkel, *Moduli of trigonal curves*, J. Algebraic Geom. **9** (2000), no. 4, 607–662. MR MR1775309 (2001h:14031) 16

[Swi] David Swinarski, Private communication, 2011. 2

E-mail address: `mfedorch@math.columbia.edu`

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027