SINGULARITIES WITH $\mathbb{G}_m$-ACTION AND THE LOG MINIMAL MODEL PROGRAM FOR $\overline{\mathcal{M}}_g$

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Abstract. We give a precise formulation of the modularity principle for the log canonical models $\overline{\mathrm{M}}_g(\alpha) := \operatorname{Proj} \bigoplus_{d\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_g, \lfloor d(K_{\overline{\mathcal{M}}_g} + \alpha\delta)\rfloor)$ of the moduli space of stable curves. We define a new invariant of Gorenstein curve singularities with $\mathbb{G}_m$-action which can be used to predict the critical α-value at which a singularity first arises in the modular interpretation of $\overline{\mathrm{M}}_g(\alpha)$. We compute these critical α-values for large classes of singularities with $\mathbb{G}_m$-action, including all ADE, toric planar, and unibranch Gorenstein singularities, and use these results to give a conjectural outline of the log MMP for $\overline{\mathcal{M}}_g$.

Contents

1. Introduction

In [Has05], Hassett and Keel initiated a program to give modular interpretations to the log canonical models

$$\overline{\mathrm{M}}_g(\alpha) := \operatorname{Proj} \bigoplus_{d\geq 0} \mathrm{H}^0(\overline{\mathcal{M}}_g, \lfloor d(K_{\overline{\mathcal{M}}_g} + \alpha\delta)\rfloor), \tag{1}$$

for all $\alpha \in [0,1] \cap \mathbb{Q}$ such that $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is effective. Subsequent contributions to the Hassett-Keel program have established the modularity of various log canonical models by realizing them as GIT quotients of Hilbert schemes of pluricanonically embedded curves [HH09, HH13, HL10]. These works have established the modularity of the log canonical models $\overline{\mathrm{M}}_g(\alpha)$ for all α when $g = 2, 3$, and for $\alpha \geq 7/10 - \epsilon$ when $g \geq 4$. For additional partial results when $g = 4, 5, 6$, see [Fed12, CMJL12a, CMJL12b, FS13b, Mül13]. Unfortunately, the GIT approach is limited by the absence of natural GIT quotient descriptions of $\overline{\mathrm{M}}_g(\alpha)$ for low values of α, as well as by the increasing difficulty of the GIT stability analysis even in cases where quotient descriptions do exist. It is therefore natural to seek an abstract formulation of the goal of the Hassett-Keel program.

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1.1. **Modularity Principle for $\overline{\mathrm{M}}_g(\alpha)$.** Informally, the Hassett-Keel program suggests the existence of stacks of singular curves whose moduli spaces are the log canonical models $\overline{\mathrm{M}}_g(\alpha)$. In order to make this precise, we must introduce some terminology. First, we let $\mathcal{U}_g$ denote the stack of complete Gorenstein[1] curves of arithmetic genus g with ample dualizing line bundle. We let $\pi\colon\ \mathcal{C}_g \to \mathcal{U}_g$ denote the universal curve and consider the following line bundles on $\mathcal{U}_g$:

$$\begin{aligned} \lambda_m &:= c_1(\pi_*\omega_\pi^m), \quad \text{for } m \geq 1, \\ K &:= 2\lambda_2 - 13\lambda_1, \\ \delta &:= 13\lambda_1 - \lambda_2. \end{aligned} \tag{2}$$

Remark 1.1. A Grothendieck-Riemann-Roch computation shows that the restriction of K and δ to $\overline{\mathcal{M}}_g$ is the usual canonical class and boundary divisor class, respectively; see [HM82, Section 2] and [HM98, Section 3.E] for details of this computation.

Recall from [Alp08] that a morphism $\phi\colon\ \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if $\phi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ and $\phi_*\colon\ \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(X)$ is an exact functor. The formal properties of good moduli spaces will not play a significant role in this paper, but we encourage the reader to keep in mind that the canonical example of a good moduli space is the morphism $[X^{ss}/G] \to X^{ss} /\!\!/ G$ from a GIT quotient stack to the GIT quotient.

Finally, since $\overline{\mathrm{M}}_g(\alpha)$ is defined as the Proj of the graded section ring of the $\mathbb{Q}$-line bundle $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ on $\overline{\mathcal{M}}_g$, there exists a natural $\mathbb{Q}$-line bundle $\mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)$ on $\overline{\mathrm{M}}_g(\alpha)$ such that $f_*(K_{\overline{\mathcal{M}}_g} + \alpha\delta) = \mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)$ under the rational map $f\colon\ \overline{\mathcal{M}}_g \dashrightarrow \overline{\mathrm{M}}_g(\alpha)$. With all this notation, we may consider the goal of the Hassett-Keel program as the verification of the following *modularity principle*.

Principle 1.2 (Modularity principle for the log MMP for $\overline{\mathrm{M}}_g$)**.** *For $\alpha \in [0,1] \cap \mathbb{Q}$ such that $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is effective, there exists an open normal substack $\overline{\mathcal{M}}_g(\alpha) \subseteq \mathcal{U}_g$ and a morphism $\phi\colon\ \overline{\mathcal{M}}_g(\alpha) \to \overline{\mathrm{M}}_g(\alpha)$ satisfying:*

(1) ϕ is a good moduli space.
(2) $\phi^*\mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)|_{\overline{\mathcal{M}}_g(\alpha)\cap\overline{\mathcal{M}}_g} \simeq \big(K_{\overline{\mathcal{M}}_g} + \alpha\delta\big)|_{\overline{\mathcal{M}}_g(\alpha)\cap\overline{\mathcal{M}}_g}$.

To verify the modularity principle in this abstract setting, one would like a natural recipe for defining the open substacks $\overline{\mathcal{M}}_g(\alpha) \subset \mathcal{U}_g$. One step in this recipe should be a method for predicting which singular curves arise at a given value of α. In this paper, we construct a new invariant of curve singularities with $\mathbb{G}_m$-action, namely the *α-invariant*, which is tailored to answer this question. We compute the α-invariant for several natural classes of singularities, and use our computations to give a conjectural outline of the log MMP down to $\alpha = 5/9$. In fact, these computations have already been deployed as a useful guiding heuristic in recent works on the log minimal model program [CMJL12b, AFSvdW13].

1.2. **Curves with $\mathbb{G}_m$-action in the log MMP for $\overline{\mathrm{M}}_g$.** Before defining the α-invariant, we should explain why singularities *with $\mathbb{G}_m$-action* play such a special role in our analysis. In fact, there is good reason to expect that *all reduced curve singularities arising in the Hassett-Keel*

[1] Our decision to focus on Gorenstein curves is motivated in part by experience (we have yet to see a non-Gorenstein curve appear in the Hassett-Keel program) and in part by technical convenience (the Hodge line bundles which play a central role in this paper are naturally defined over the locus of Gorenstein curves).

program admit a $\mathbb{G}_m$*-action.* Indeed, at each 'threshold' value of α, we expect a diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_g(\alpha+\epsilon) \arrow[r, hook, "i_+"] \arrow[d] & \overline{\mathcal{M}}_g(\alpha) \arrow[dd] & \overline{\mathcal{M}}_g(\alpha-\epsilon) \arrow[l, hook', "i_-"'] \arrow[d] \\
\overline{\mathrm{M}}_g(\alpha+\epsilon) \arrow[dr, "j_+"] & & \overline{\mathrm{M}}_g(\alpha-\epsilon) \arrow[dl, "j_-"'] \\
& \overline{\mathrm{M}}_g(\alpha) &
\end{tikzcd}$$

where i_+, i_- are open immersions, the vertical maps are good moduli spaces, and j_+, j_- are projective morphisms induced by i_+, i_-. Crucially, the *closed* points of

$$\overline{\mathcal{M}}_g(\alpha) - \big(\overline{\mathcal{M}}_g(\alpha+\epsilon) \cup \overline{\mathcal{M}}_g(\alpha-\epsilon)\big)$$

must all have infinite stabilizers, i.e. must correspond to curves admitting a $\mathbb{G}_m$-action.[2] Indeed, the only way an open immersion of stacks can induce a projective morphism of good moduli spaces is for all new points of the larger stack to be identified with certain points of the smaller stack, and this requires the new closed points of the larger stack to have infinite stabilizers.

There are two reasons for believing that each transition of the Hassett-Keel program should entail a diagram of this kind. First, we believe the log MMP for $\overline{\mathcal{M}}_g$ should be modeled after a variation of GIT problem, and VGIT problems always give rise to such diagrams [DH98, Tha96]. Indeed, as one varies the polarization in VGIT, the semistable locus 'jumps' at certain walls, and the corresponding open inclusions of the semistable loci induce projective morphisms of the corresponding quotients. Second, the work of Hassett-Hyeon and Hyeon-Morrison produces precisely such a picture at the first two critical thresholds. Indeed, at $\alpha = 9/11$, the points of $\overline{\mathcal{M}}_g(\alpha) - \big(\overline{\mathcal{M}}_g(\alpha+\epsilon) \cup \overline{\mathcal{M}}_g(\alpha-\epsilon)\big)$ correspond to curves with rational cuspidal tails (Figure 1) which contain all elliptic tails and cuspidal curves in their basins of attraction [HM10]; see [HH13, Definition 5.2] for the definition of the basins of attraction. At $\alpha = 7/10$, the points of $\overline{\mathcal{M}}_g(\alpha) - \big(\overline{\mathcal{M}}_g(\alpha+\epsilon) \cup \overline{\mathcal{M}}_g(\alpha-\epsilon)\big)$ correspond to curves with rational tacnodal bridges, and contain all elliptic bridges and tacnodal curves in their basins of attraction [HH13].

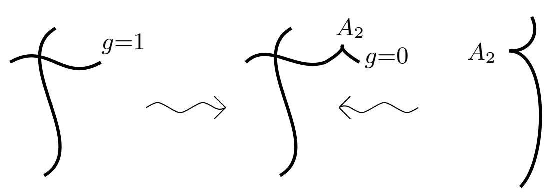

FIGURE 1. Rational cuspidal tail is an isotrivial degeneration of both elliptic tails and cuspidal curves.

[2]Since these curves correspond to closed points of a stack with a good moduli space, their stabilizers must be linearly reductive. But if C is any reduced curve with positive dimensional linearly reductive automorphism group $\mathrm{Aut}(C)$, the connected component of the identity of $\mathrm{Aut}(C)$ is a non-trivial torus, i.e. C admits a $\mathbb{G}_m$-action.

1.3. **α-invariants of curves with $\mathbb{G}_m$-action.** In moduli theory, the presence of objects with infinite automorphism groups is usually considered bad news, but the key idea of this paper is to make a virtue of necessity by exploiting this feature of the log MMP to obtain *a priori* information on the definition of the stacks $\overline{\mathcal{M}}_g(\alpha)$. To see how this works, recall that if $[C] \in \mathcal{U}_g$ is any point and $\mathcal{L}$ is a line bundle defined in a neighborhood of $[C]$, then the natural action of $\mathrm{Aut}(C)$ on the fiber $\mathcal{L}|_{[C]}$ induces a character $\mathrm{Aut}(C) \to \mathbb{G}_m$. If $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ is any one-parameter subgroup, then there is an induced character $\mathbb{G}_m \to \mathbb{G}_m$ which is necessarily of the form $z \mapsto z^n$ for some integer $n \in \mathbb{Z}$. For a given curve C, a non-trivial one-parameter subgroup $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$, and a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{U}_g)$, we denote this integer by $\chi_{\mathcal{L}}(C, \eta)$. If $\mathrm{Aut}(C) \simeq \mathbb{G}_m$, we write simply $\chi_{\mathcal{L}}(C)$, where $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ is understood to be the identity. Note that $\chi_{\mathcal{L}}(C)$ is only defined up to sign due to the choice of isomorphism $\mathrm{Aut}(C) \simeq \mathbb{G}_m$, but the ratios $\chi_{\mathcal{L}}(C)/\chi_{\mathcal{M}}(C)$, for any two line bundles $\mathcal{L}$ and $\mathcal{M}$, are well-defined and this is all we ultimately use. If $\mathcal{L} = \lambda_m$, we write simply $\chi_m(C, \eta)$ or $\chi_m(C)$ if η is understood. These characters are connected to the modularity principle by the following observation.

Proposition 1.3 (Key Observation)**.** *Assume the modularity principle holds. Suppose in addition that the locus of worse-than-nodal curves in $\overline{\mathcal{M}}_g(\alpha)$ has codimension at least 2. Then for any $[C] \in \overline{\mathcal{M}}_g(\alpha)$ and any one-parameter subgroup $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$, we have*

$$\chi_{(K+\alpha\delta)}(C, \eta) = 0.$$

Proof. If the modularity principle holds, then

$$\phi^*\mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g} \simeq \big(K_{\overline{\mathcal{M}}_g} + \alpha\delta\big)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g},$$

where $\phi\colon \overline{\mathcal{M}}_g(\alpha) \to \overline{\mathrm{M}}_g(\alpha)$ is a good moduli space. On the other hand, since the codimension of $\overline{\mathcal{M}}_g(\alpha) \setminus \big(\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g\big)$ in $\overline{\mathcal{M}}_g(\alpha)$ is at least 2 and $\overline{\mathcal{M}}_g(\alpha)$ is normal, the line bundle $K + \alpha\delta = (2-\alpha)\lambda_2 + (13\alpha - 13)\lambda_1$ is the unique locally-free extension of $(K_{\overline{\mathcal{M}}_g} + \alpha\delta)|_{\overline{\mathcal{M}}_g(\alpha) \cap \overline{\mathcal{M}}_g}$ to $\overline{\mathcal{M}}_g(\alpha)$. Here we use the fact that the Hodge line bundles λ_m are defined over all of $\mathcal{U}_g$ and hence over $\overline{\mathcal{M}}_g(\alpha)$. It follows that $K + \alpha\delta \simeq \phi^*\mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)$. However, for any point $[C] \in \overline{\mathcal{M}}_g(\alpha)$, the induced action of $\mathrm{Aut}(C)$ on the fiber of $\phi^*\mathcal{O}_{\overline{\mathrm{M}}_g(\alpha)}(1)$ is necessarily trivial. This finishes the proof. $\square$

What makes this observation useful is the fact that the character $\chi_{(K+\alpha\delta)}(C, \eta)$ can be computed in terms of the intrinsic geometry of C, without knowing the definition of $\overline{\mathcal{M}}_g(\alpha)$ *a priori*. Indeed, by definition

$$\chi_{(K+\alpha\delta)}(C, \eta) = (2-\alpha)\chi_2(C, \eta) + (13\alpha - 13)\chi_1(C, \eta).$$

Now the characters on the right are manifestly computable in terms of the intrinsic geometry of (C, η), indeed $\chi_m(C, \eta)$ is simply the determinant of the action of η on $\mathrm{H}^0(C, \omega_C^m)$. Using this, we may define the α-invariant of a curve with $\mathbb{G}_m$-action to be the unique rational number $\alpha(C, \eta)$ such that this character is zero.

Definition 1.4 (α-invariant)**.** The *α-invariant* of a curve $C \in \mathcal{U}_g$ with $\mathbb{G}_m$-action $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$ is the rational number

$$\alpha(C, \eta) := \frac{13 - 2\left(\dfrac{\chi_2(C, \eta)}{\chi_1(C, \eta)}\right)}{13 - \left(\dfrac{\chi_2(C, \eta)}{\chi_1(C, \eta)}\right)}.$$

If $\chi_1(C,\eta)=0$, the α-invariant is undefined. If $\mathrm{Aut}(C)\simeq\mathbb{G}_m$, we write $\alpha(C)$ for the α-invariant $\alpha(C,\eta)$ where $\eta\colon\ \mathbb{G}_m\to\mathrm{Aut}(C)$ is understood to be the identity.

We conclude that, as long as the α-invariant of a curve with $\mathbb{G}_m$-action is defined, it is the *only* α-value at which the curve can show up in $\overline{\mathcal{M}}_g(\alpha)$.

Corollary 1.5. *Assume the modularity principle holds and let $[C]\in\overline{\mathcal{M}}_g(\alpha)$ be any point. If C admits a $\mathbb{G}_m$-action η such that $\chi_1(C,\eta)\neq 0$, then $\alpha=\alpha(C,\eta)$.*

Proof. By Proposition 1.3, we must have

$$\chi_{(K+\alpha\delta)}(C,\eta)=(2-\alpha)\chi_2(C,\eta)+(13\alpha-13)\chi_1(C,\eta)=0.$$

Solving for α, we obtain the desired statement. □

While the α-invariant $\alpha(C,\eta)$ can be computed for any explicitly given curve with $\mathbb{G}_m$-action, certain classes of curves play a distinguished role in 'announcing' critical thresholds of the log minimal model program. In this paper, we isolate three classes of complete curves with $\mathbb{G}_m$-action, namely $\widehat{\mathcal{O}}$*-atoms,* $\widehat{\mathcal{O}}^S$*-atoms*, and $A_{i/j}$*-atoms*. These will be defined carefully in Section 2 (Definitions 2.6, 2.9, and 2.12), but we give here an informal description of these curves and of their geometric significance in the context of the log minimal model program.

We have already seen that $\overline{\mathcal{M}}_g(9/11)$ contains curves of the form $C_0\cup E$, where E is a rational cuspidal tail, and that $\overline{\mathcal{M}}_g(7/10)$ contains curves of the form $C_0\cup E$, where E is a rational tacnodal bridge. The $\widehat{\mathcal{O}}$-atom is simply a generalization of this construction to an arbitrary Gorenstein curve singularity $\widehat{\mathcal{O}}$ with $\mathbb{G}_m$-action. Indeed, if the singularity $\widehat{\mathcal{O}}$ has b branches and δ-invariant $\delta(p)$, an $\widehat{\mathcal{O}}$-atom is simply a curve of the form

$$C=E_1\cup\ldots\cup E_b\cup C_0,$$

where C_0 is any smooth curve of genus $g-\delta(p)-b+1$ and $E_1,\ldots,E_b$ are rational curves attached to C_0 nodally and meeting in a singularity analytically isomorphic to $\widehat{\mathcal{O}}$ (see Figure 2 for an example of J_{10}-atom).

The $\mathbb{G}_m$-action on $\widehat{\mathcal{O}}$ extends to define a one-parameter subgroup $\eta\colon\ \mathbb{G}_m\to\mathrm{Aut}(C)$, and we define the α-invariant of the $\widehat{\mathcal{O}}$-atom to be $\alpha(C,\eta)$. Using this definition, the α-invariant of a cusp and a tacnode are $9/11$ and $7/10$ respectively, and in general we expect the α-invariant associated to a singularity to be the first α-value at which the *generic form* of this singularity appears in the modular interpretation of $\overline{\mathrm{M}}_g(\alpha)$. (We discuss non-generic forms of singularities in the next paragraph.) We compute the α-invariants of a broad range of singularities and the results are collected in Table 1 of Section 3.4.

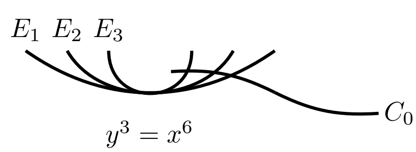

FIGURE 2. An atom of the $y^3=x^6$ singularity (also called J_{10}-atom).

There are two natural generalizations of $\widehat{\mathcal{O}}$-atoms which we expect to play an important role in future stages of the log minimal model program for $\overline{\mathcal{M}}_g$. To motivate these constructions, let us consider which loci on $\overline{\mathcal{M}}_g$ must be modified at the critical value $\alpha=19/29$. The divisor Δ_2 is covered by curves intersecting $K+\frac{19}{29}\delta$ trivially, and we expect curves with genus 2 tails to be replaced by 'dangling oscnodes,' using the blow-up/blow-down procedure pictured in Figure 3. The distinguished curve with $\mathbb{G}_m$-action to which this dangling oscnode isotrivially specializes is the curve pictured in Figure 4.

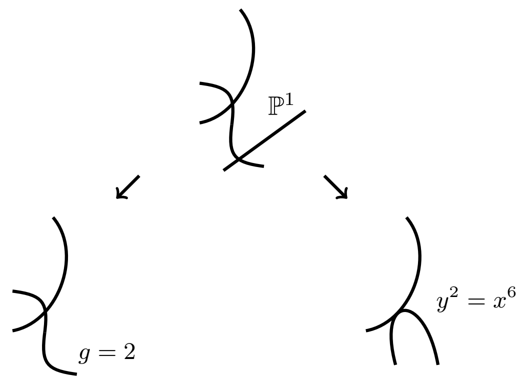

FIGURE 3. Given a smoothing of a curve with a genus 2 tail attached at an arbitrary point p, after blowing up the conjugate point of p and contracting the genus 2 curve, one obtains a dangling $\mathbb{P}^1$ attached at an oscnode.

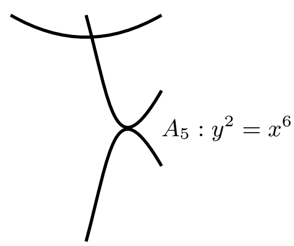

FIGURE 4. Dangling A_5-atom.

Note that this curve is not an oscnodal atom because one of the branches of the oscnode 'dangles,' i.e. is not nodally attached to the rest of the curve. This motivates a generalization of the construction of the $\widehat{\mathcal{O}}$-atom, where some subset S of the branches of $\widehat{\mathcal{O}}$ are allowed to dangle, and we call such curves $\widehat{\mathcal{O}}^S$-atoms. The α-invariant of an $\widehat{\mathcal{O}}^S$-atom depends on S, and this gives rise to important subtleties in the unfolding of the log minimal model program. For example, while the α-invariant of the dangling oscnodal atom is $19/29$, the α-invariant of the standard oscnodal atom is $17/28$, reflecting the fact that genus 2 bridges attached at conjugate Weierstrass points are not replaced until $\alpha = 17/28$. The α-invariants for a wide selection of $\widehat{\mathcal{O}}^S$-atoms are displayed in Table 2 of Section 3.4.

A second locus that must be modified at $\alpha = 19/29$ is the locus of genus two tails attached *tacnodally* at a Weierstrass point. Of course, this locus does not appear in $\overline{\mathcal{M}}_g$, but it appears in $\overline{\mathcal{M}}_g(7/10)$ and subsequent models, and one can check that it is covered by curves intersecting $K + \frac{19}{29}\delta$ trivially. We expect these curves to be replaced by tacnodally attached ramphoid tails. The corresponding distinguished curve with $\mathbb{G}_m$-action to which such curves specialize is pictured in Figure 5. This motivates consideration of $A_{i/j}$-atoms constructed by concatenating

A_i and A_j-singularities along rational branches (Definition 2.12). The α-invariants associated to $A_{i/j}$-atoms are listed in Table 2.

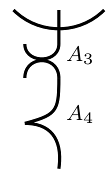

FIGURE 5. $A_{3/4}$-atom.

In summary, using the computations of α-invariants of $\widehat{\mathcal{O}}$-atoms, $\widehat{\mathcal{O}}^S$-atoms, and $A_{i/j}$-atoms, we can assemble a fairly comprehensive set of predictions for the log minimal model program. While there are innumerable further variations on these constructions, we expect that these three varieties of singular curves with $\mathbb{G}_m$-action are sufficient to give a skeletal outline of the log minimal model program for $\alpha \geq 5/9$.

1.4. **The case of pointed curves.** We note that there is a simple way to generalize our results to the case of pointed curves. Namely, let $\mathcal{U}_{g,n}$ denote the stack of Gorenstein curves C of arithmetic genus g with smooth marked points $p_1, \dots, p_n \in C$ such that $\omega_C\left(\sum_{i=1}^n p_i\right)$ is ample. Let $\pi: \mathcal{C}_{g,n} \to \mathcal{U}_{g,n}$ denote the universal curve and $\sigma_i: \mathcal{U}_{g,n} \to \mathcal{C}_{g,n}$ denote the universal sections. Then we can define

$$\begin{aligned} \lambda_m &:= c_1\Big(\pi_*\omega_\pi\big(\sum_{i=1}^n \sigma_i\big)^m\Big), \quad \text{for } m \geq 1, \\ \psi_i &:= \sigma_i^*(\omega_\pi), \quad \psi := \sum_{i=1}^n \psi_i, \\ K &:= 2\lambda_2 - 13\lambda_1 - \psi, \quad \delta := 13\lambda_1 - \lambda_2 + \psi. \end{aligned} \tag{3}$$

With this notation, we will be able to define and study the α-invariants of pointed curves with $\mathbb{G}_m$-action, and relate them to the α-invariants of unmarked curves.

Outline of the paper. Let us now give a roadmap for the rest of the paper. In Section 2, we discuss singularities with $\mathbb{G}_m$-action, and define $\widehat{\mathcal{O}}$-atoms, $\widehat{\mathcal{O}}^S$-atoms, and $A_{i/j}$-atoms, and the associated α-invariants. In Section 3, we explain how to compute these α-invariants: In 3.1, we compute the α-invariants of a wide range of singularities, including ADE, toric planar, and unibranch Gorenstein singularities. In 3.2, we give an alternative method for computing χ_K and χ_δ, under additional assumptions on the deformation space of the curve. While the results of these calculations are ultimately derivable from the calculations in 3.1, they help to illuminate the geometric significance of α-invariants. We close Section 3 by summarizing our calculations of the α-invariants in Tables 1 and 2. We use them to generate predictions for the log MMP, and give a complete outline of the transitions of the log MMP for $\alpha \geq 5/9$ in Table 3.

Finally, in Sections 4 and 5, we explore the connection between α-invariants and important invariants coming from intersection theory and geometric invariant theory. In Section 4, we explain how the α-invariant is related to slopes of families in the locus of stable limits of a singularity. In Section 5, we explain how Hilbert-Mumford indices are related to the characters. We use this connection to prove some new instability results for rational ribbons of Bayer–Eisenbud. These computations were a key inspiration for the results of [AFS13].

Notation. We work over an algebraically closed field $\Bbbk$ of characteristic 0 and denote by $\mathbb{G}_m$ the multiplicative group of units in $\Bbbk$. A *curve* is a connected finite type scheme over $\Bbbk$ of pure dimension 1.

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2. Curve singularities with $\mathbb{G}_m$-action

In this section, we collect definitions and preliminary results concerning reduced curve singularities with $\mathbb{G}_m$-action. We begin by recalling the definition of the dualizing sheaf of a singular curve in Section 2.1. We use this description in Section 2.2 to show that the analytic isomorphism type of a Gorenstein curve singularity with $\mathbb{G}_m$-action uniquely determines an affine curve with $\mathbb{G}_m$-action. This leads to a precise definition in Section 2.3 of an $\widehat{\mathcal{O}}$-atom for any Gorenstein singularity $\widehat{\mathcal{O}}$ with $\mathbb{G}_m$-action.

2.1. Dualizing sheaf of a singular curve.

We recall duality theory on singular curves as in [AK70, VIII], which treats reduced curves over any field. The summary of this theory can also be found in [BG80, p.244], as well as in [Ser88, Ch.IV.9] (at least for irreducible curves) and in [BHPVdV04, Prop.6.2] for reduced curves on smooth surfaces.

Given a reduced curve C, denote by $\nu\colon \widetilde{C} \to C$ the normalization of C and consider the sheaf $\mathcal{M}_{\widetilde{C}}$ of rational differentials on $\widetilde{C}$. Then by [AK70, VIII, Prop.1.16] the dualizing sheaf of C can be defined as the subsheaf $\omega_C \subset \nu_*\mathcal{M}_{\widetilde{C}}$ of the rational differentials ω satisfying the following condition: For every $p \in C$ and every $f \in \mathcal{O}_{C,p}$

$$\sum_{q \in \nu^{-1}(p)} \operatorname{Res}_q (f\omega) = 0. \tag{4}$$

Differentials ω satisfying (4) are called *Rosenlicht differentials*. A singularity $p \in C$ is called *Gorenstein* if $\omega_{C,p}$ is a free $\mathcal{O}_{X,p}$-module of rank 1.

Suppose $p \in C$ is a curve singularity with b branches. Let $t_1, \dots, t_b$ be the uniformizers of the branches of the normalization $\widetilde{\mathcal{O}}_{C,p}$. If $\mathcal{O}_{C,p}$ is Gorenstein and

$$\operatorname{Ann}(\widetilde{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p}) = (t_1^{m_1}) \times (t_2^{m_2}) \times \cdots \times (t_b^{m_b})$$

is the conductor ideal, then $\omega_{C,p}$ is generated by a Rosenlicht differential of the form

$$\left(u_1 \frac{dt_1}{t_1^{m_1}}, u_2 \frac{dt_2}{t_2^{m_2}}, \dots, u_b \frac{dt_b}{t_b^{m_b}} \right), \tag{5}$$

where u_i is a unit on the i^{th} branch; see [Ser88, Ch.IV,11].

For every pair of branches $i \neq j \in \{1, \dots, b\}$, we have a Rosenlicht differential at p defined by

$$\omega(i,j) := \left(\cdots 0, \frac{dt_i}{t_i}, 0 \cdots 0, -\frac{dt_j}{t_j}, 0 \cdots \right) \in \omega_{C,p}. \tag{6}$$

In addition, if the branch with the uniformizer t_i is singular, then $\dfrac{dt_i}{t_i^2} \in \omega_{C,p}$.

Recall from [Ste96, Def.2.1, p.385] that a curve singularity $p \in C$ is *decomposable* if C is the union of two curves C_1 and C_2 which lie in smooth spaces intersecting each other transversely in p. Algebraically, this means that

$$\widehat{\mathcal{O}}_{C,p} \simeq \Bbbk[[x_1, \dots, x_n, y_1, \dots, y_m]]/(I, y_1, \dots, y_m) \cap (J, x_1, \dots, x_n), \tag{7}$$

where $I \subset (x_1, \dots, x_n) \cap \Bbbk[x_1, \dots, x_n]$, and $J \subset (y_1, \dots, y_m) \cap \Bbbk[y_1, \dots, y_m]$ are the ideals of C_1 and C_2.

Proposition 2.1. *Every Gorenstein curve singularity, with the exception of the ordinary node $xy = 0$, is indecomposable.*

Proof. This follows from a more general result of [Ste96, p.385], but for the reader's convenience we include a self-contained proof. Suppose $\widehat{\mathcal{O}}_{C,p}$ is a decomposable Gorenstein curve singularity as in (7). Let $\Bbbk[[t_1]] \times \cdots \times \Bbbk[[t_b]]$ be the normalization of $\widehat{\mathcal{O}}_{C,p}$. By the decomposability assumption, there are two branches, say those with uniformizers t_1 and t_2, such that $(x_1, \dots, x_n) \mapsto 0 \in \Bbbk[[t_1]]$ and $(y_1, \dots, y_m) \mapsto 0 \in \Bbbk[[t_2]]$. Recall the definition of $\omega(i,j)$ from (6). Note that $\omega(1,2) \in \omega_{C,p}$, but $\omega(1,2) \notin (x_1, \dots, x_n)\omega_{C,p} + (y_1, \dots, y_m)\omega_{C,p}$ because $(dt_1/t_1, 0 \cdots) \notin \omega_{C,p}$. We conclude that $\omega_{C,p}$ must be generated by $\omega(1,2)$. If $b \geq 3$, then $\omega(1,3)$ is not a multiple of $\omega(1,2)$ and we arrive at a contradiction. If $b = 2$, then the fact that dt_i/t_i^2 (for $i = 1,2$) is not a multiple of $\omega(1,2)$, hence not in $\omega_{C,p}$, implies that both branches of p are smooth. Since they intersect transversally, $p \in X$ is a node. □

2.2. **Curve singularities with $\mathbb{G}_m$-action.** Observe that if $C \in \mathcal{U}_g$ is a curve with $\mathbb{G}_m$-action and $p \in C$ is a singular point, then there is an open affine $\mathbb{G}_m$-invariant subcurve $X \subset C$ such that $p \in X$.[3] For this reason, we focus our attention on singular affine curves with $\mathbb{G}_m$-action.

Given an affine curve X with a distinguished singular point $p \in X$ and $\mathbb{G}_m$-action on X fixing p, we can choose coordinates so that:

1. $X \simeq \operatorname{Spec} \Bbbk[x_1, \dots, x_n]/I$ where p corresponds to the maximal ideal $(x_1, \dots, x_n)$, and
2. $\mathbb{G}_m = \operatorname{Spec} \Bbbk[\lambda, \lambda^{-1}]$ acts by $x_i \mapsto \lambda^{d_i} x_i$ and I is invariant under this action.

The completion $\widehat{\mathcal{O}}_{X,p}$ is called a *singularity with $\mathbb{G}_m$-action* [OW71, Pin74]. The $\mathbb{G}_m$-action is *good* if the weights d_i are either all negative or all positive and $\gcd(d_1, \dots, d_n) = 1$. In this situation, the completion $\widehat{\mathcal{O}}_{X,p}$ is called a *quasi-homogeneous singularity.* Note that if the $\mathbb{G}_m$-action on (X,p) is good, then $\mathbb{G}_m$ acts freely on $X - p$. Our first goal is to show that the pair (X,p) is uniquely determined from the isomorphism type of $\widehat{\mathcal{O}}_{X,p}$, at least when $\widehat{\mathcal{O}}_{X,p}$ is Gorenstein.

Evidently, if (X,p) is an affine curve with good $\mathbb{G}_m$-action, then the induced action on the normalization $\widetilde{X}$ is non-trivial on every branch and hence $\widetilde{X} = \operatorname{Spec} \Bbbk[t_1] \times \cdots \times \Bbbk[t_b]$, where b is the number of the branches of $p \in X$. In particular, we can describe $\Gamma(X, \mathcal{O}_X)$ as a $\mathbb{G}_m$-invariant subring of $\Bbbk[t_1] \times \cdots \times \Bbbk[t_b]$. The following proposition says that a $\mathbb{G}_m$-action on an indecomposable curve singularity must be good.

Lemma 2.2. *Suppose (X,p) is an indecomposable singular affine curve with faithful $\mathbb{G}_m$-action. Then the $\mathbb{G}_m$-action is good.*

Proof. Suppose $X \simeq \operatorname{Spec} \Bbbk[x_1, \dots, x_n, y_1, \dots, y_m]/I$ is an affine reduced curve with $\mathbb{G}_m$-action fixing the origin such that $\mathbb{G}_m$ acts with positive weights on $(x_1, \dots, x_n)$, and with non-positive weights on $(y_1, \dots, y_m)$. The $\mathbb{G}_m$-action lifts to the normalization of X at p. Let $t_1, \dots, t_k$ be

[3] As the example of a nodal plane cubic illustrates, the condition that ω_C is ample cannot be omitted.

the uniformizers of the branches on which $\mathbb{G}_m$ acts with positive weight, and let $t_{k+1}, \dots, t_b$ be the uniformizers of the remaining branches. Then $(x_1, \dots, x_n)$ maps to 0 in $\Bbbk[t_{k+1}] \times \cdots \times \Bbbk[t_b]$ and $(y_1, \dots, y_m)$ maps to 0 in $\Bbbk[t_1] \times \cdots \times \Bbbk[t_k]$. It follows that $x_i y_j = 0$ on the normalization, for all i and j. Hence $(x_1, \dots, x_n)(y_1, \dots, y_m) \subset I$, which in turn implies that X is decomposable. A contradiction. $\square$

Corollary 2.3. *A faithful $\mathbb{G}_m$-action on a Gorenstein non-nodal singularity is necessarily good.*

Greuel, Martin, and Pfister gave a numerical characterization of quasi-homogeneous Gorenstein curve singularities, generalizing a quasi-homogeneity criterion for hypersurface singularities due to Saito [Sai71]. Namely, a Gorenstein curve singularity is quasi-homogeneous if and only if the Milnor number equals to the Deligne number [GMP85, Theorem 2.1], both of which are effectively computable [GMP85, Eqns.(1.1),(1.4)].

In particular, one can determine whether a Gorenstein curve singularity $\widehat{\mathcal{O}}$ is the completion of $\mathcal{O}_{X,p}$ for *some* affine curve (X, p) with good $\mathbb{G}_m$-action from numerical invariants of $\widehat{\mathcal{O}}$ itself. Our next result shows that in fact a quasi-homogeneous Gorenstein curve singularity $\widehat{\mathcal{O}}$ comes from a unique affine curve (X, p) with $\mathbb{G}_m$-action.

Proposition 2.4. *Suppose (X, p) and (Y, q) are affine curves with faithful $\mathbb{G}_m$-actions such that $\widehat{\mathcal{O}}_{X,p}$ and $\widehat{\mathcal{O}}_{Y,q}$ are isomorphic Gorenstein complete local rings, which are not nodes. Then there is a $\mathbb{G}_m$-equivariant isomorphism $(X, p) \simeq (Y, q)$ of pointed affine schemes.*

Proof. Note that $\mathbb{G}_m$-actions on (X, p) and (Y, q) are good by Corollary 2.3. Let $\widetilde{X} = \operatorname{Spec} \Bbbk[t_1] \times \cdots \times \Bbbk[t_b]$ and $\widetilde{Y} = \operatorname{Spec} \Bbbk[s_1] \times \cdots \times \Bbbk[s_b]$ be the $\mathbb{G}_m$-equivariant normalizations of (X, p) and (Y, q), respectively. Then the $\mathbb{G}_m$-actions are given by $\lambda \cdot t_i = \lambda^{\alpha_i} t_i$ and $\lambda \cdot s_i = \lambda^{\beta_i} s_i$, with $\alpha_i, \beta_i > 0$. (We are using the fact that the number of branches is an analytic invariant of a singularity.)

The isomorphism $\widehat{\mathcal{O}}_{X,p} \simeq \widehat{\mathcal{O}}_{Y,q}$ induces an isomorphism

$$f : \ \Bbbk[[t_1]] \times \cdots \times \Bbbk[[t_b]] \to \Bbbk[[s_1]] \times \cdots \times \Bbbk[[s_b]]$$

of the normalizations satisfying $f(\widehat{\mathcal{O}}_{X,p}) = \widehat{\mathcal{O}}_{Y,q}$. After renumbering the branches and scaling, we can assume that

$$f(t_i) = s_i + c_{i,2} s_i^2 + c_{i,3} s_i^3 + \cdots , \quad \text{where } c_{i,j} \in \Bbbk.$$

First, we show that the weights $(\alpha_i)_{i=1}^b$ and $(\beta_i)_{i=1}^b$ are the same. Since $\omega_{X,p}$ is a free rank one $\mathcal{O}_{X,p}$-module, we can choose a $\mathbb{G}_m$-semi-invariant generator of $\omega_{X,p}$ of the form

$$\left(\frac{dt_1}{t_1^{m_1}}, \dots, \frac{dt_b}{t_b^{m_b}} \right),$$

where $(t_1^{m_1}) \times (t_2^{m_2}) \times \cdots \times (t_b^{m_b})$ is the conductor of $\mathcal{O}_{X,p}$. By the $\mathbb{G}_m$-semi-invariance, $\alpha_1(m_1 - 1) = \cdots = \alpha_b(m_b - 1)$. Since $\mathcal{O}_{X,p}$ is not a node, at least one $m_i \geq 2$. Hence all $m_i \geq 2$. It follows that $(\alpha_1, \dots, \alpha_b)$ are determined by $(m_1, \dots, m_b)$, which are determined by the conductor ideal of $\mathcal{O}_{X,p}$, and hence form an analytic invariant. We conclude that $(\alpha_i)_{i=1}^b = (\beta_i)_{i=1}^b$.

For any $\lambda \in \mathbb{G}_m$, consider the following isomorphism

$$f_\lambda := \lambda \circ f \circ \lambda^{-1} : \ \Bbbk[[t_1]] \times \cdots \times \Bbbk[[t_b]] \to \Bbbk[[s_1]] \times \cdots \times \Bbbk[[s_b]].$$

We have $f(\widehat{\mathcal{O}}_{X,p}) = \widehat{\mathcal{O}}_{Y,q}$. Since $\widehat{\mathcal{O}}_{X,p}$ and $\widehat{\mathcal{O}}_{Y,q}$ are $\mathbb{G}_m$-invariant subrings of $\Bbbk[[t_1]] \times \cdots \times \Bbbk[[t_b]]$ and $\Bbbk[[s_1]] \times \cdots \times \Bbbk[[s_b]]$, respectively, we conclude that f_λ also satisfies $f_\lambda(\widehat{\mathcal{O}}_{X,p}) = \widehat{\mathcal{O}}_{Y,q}$ for any $\lambda \neq 0$.

Since λ acts with the same weight α_i on t_i and s_i, we compute

$$f_\lambda(t_i) = s_i + c_{i,2}\lambda^{\alpha_i} s_i^2 + c_{i,3}\lambda^{2\alpha_i} s_i^3 + \cdots . \tag{8}$$

Next, let $f_0 : \ s_i \mapsto t_i$ be the standard isomorphism of the normalizations. Set $\mathfrak{m} = (t_0) \times \cdots \times (t_b)$ and $\mathfrak{n} = (s_0) \times \cdots \times (s_b)$. Then for any $N \geq 0$, f_λ induces an isomorphism

$$f_\lambda^N : \ \Bbbk[[t_1]] \times \cdots \times \Bbbk[[t_b]]/\mathfrak{m}^N \to \Bbbk[[s_1]] \times \cdots \times \Bbbk[[s_b]]/\mathfrak{n}^N$$

satisfying $f_\lambda^N\left(\widehat{\mathcal{O}}_{X,p}/\widehat{\mathcal{O}}_{X,p} \cap \mathfrak{m}^N\right) = \widehat{\mathcal{O}}_{Y,q}/\widehat{\mathcal{O}}_{Y,q} \cap \mathfrak{n}^N$ for $\lambda \neq 0$. Equation (8) then implies that by letting $\lambda \to 0$, we must have

$$f_0^N\left(\widehat{\mathcal{O}}_{X,p}/\widehat{\mathcal{O}}_{X,p} \cap \mathfrak{m}^N\right) = \widehat{\mathcal{O}}_{Y,q}/\widehat{\mathcal{O}}_{Y,q} \cap \mathfrak{n}^N.$$

By letting $N \to \infty$ and taking the inverse limit, we conclude that $f_0(\widehat{\mathcal{O}}_{X,p}) = \widehat{\mathcal{O}}_{Y,q}$. The proposition now follows from the equalities $\Gamma(X, \mathcal{O}_X) = \widehat{\mathcal{O}}_{X,p} \cap (\Bbbk[t_1] \times \cdots \times \Bbbk[t_b])$ and $\Gamma(Y, \mathcal{O}_Y) = \widehat{\mathcal{O}}_{Y,p} \cap (\Bbbk[s_1] \times \cdots \times \Bbbk[s_b])$. $\square$

Remark 2.5. Proposition 2.4 can be regarded as (a much easier) analog of the following result due independently to Dolgachev and Pinkham [Dol75, Pin77]: Suppose Spec $\oplus_{k\geq 0} A_k$ is a normal surface with an isolated singularity and a $\mathbb{G}_m$-action given by the grading. Then the analytic isomorphism type of the singularity determines $\oplus_{k\geq 0} A_k$. In particular, A_k can be read off from the minimal resolution of the singularity [Pin77, Theorem 5.1].

2.3. **Atoms associated to curve singularities with $\mathbb{G}_m$-action.** Suppose $\widehat{\mathcal{O}}$ is a Gorenstein non-nodal curve singularity with non-trivial $\mathbb{G}_m$-action. By Proposition 2.4, there exists a unique pointed affine curve (X, p) with good $\mathbb{G}_m$-action such that $\widehat{\mathcal{O}} \simeq \widehat{\mathcal{O}}_{X,p}$. The affine scheme X admits a canonical compactification $\bar{X}$, constructed as follows: Let $\{E_i\}_{i=1}^b$ be the branches of X at p, where the enumeration is chosen once and for all. Since $\widetilde{E}_i \simeq \mathbb{A}^1$ and the $\mathbb{G}_m$-action on E_i has a unique fixed point p, we can compactify $\widetilde{E}_i$ to $\mathbb{P}^1$ by adding a single point p_i at infinity. Compactifying each branch of X, we obtain a complete pointed curve $(\bar{X}; \{p_i\}_{i=1}^b)$ whose irreducible components have normalization isomorphic to $\mathbb{P}^1$, and such that $\mathbb{G}_m$ acts on each E_i by scaling with fixed points p and p_i. We call $(\bar{X}; \{p_i\}_{i=1}^b)$ a *$\mathbb{G}_m$-equivariant compactification* of $\widehat{\mathcal{O}}$.

Definition 2.6 ($\widehat{\mathcal{O}}$-atoms)**.** Suppose $\widehat{\mathcal{O}}$ is a Gorenstein curve singularity with $\mathbb{G}_m$-action and $\mathbb{G}_m$-equivariant compactification $(\bar{X}; \{p_i\}_{i=1}^b)$. Let $\delta(p)$ be the δ-invariant. An *$\widehat{\mathcal{O}}$-atom* of genus g is any curve C obtained by nodally attaching a smooth curve C_0 of genus $g - \delta(p) - b + 1$ to each of the b branches of $\bar{X}$ at the points $\{p_1, \dots, p_b\}$.

Let $\eta : \ \mathbb{G}_m \to \mathrm{Aut}(C)$ be the unique $\mathbb{G}_m$-action on the $\widehat{\mathcal{O}}_{X,p}$-atom C extending the $\mathbb{G}_m$-action on X. Then by the construction of C and Proposition 2.4, the characters $\chi_1(C, \eta)$ and $\chi_2(C, \eta)$ (defined in Section 1.3) depend only on the complete local ring $\widehat{\mathcal{O}}_{X,p}$. When the singularity $\widehat{\mathcal{O}}_{X,p}$ has a name, e.g. $E_6 : y^3 - x^4 = 0$, we use that name to denote the corresponding atom, e.g. 'E_6-atom.'

Definition 2.7. Suppose $\widehat{\mathcal{O}}$ is a Gorenstein curve singularity with $\mathbb{G}_m$-action. The *α-invariant* $\alpha(\widehat{\mathcal{O}})$ is defined to be the α-invariant $\alpha(C, \eta)$ of any $\widehat{\mathcal{O}}$-atom C.

Remark 2.8. It would have been arguably more natural to define the α-invariant of $\widehat{\mathcal{O}}_{X,p}$-singularity in terms of the characters of the complete curve $\bar{X}$. We chose the above definition so that the α-invariant agrees with the α-values at which A_2 and A_3-singularities first appear in $\overline{\mathcal{M}}_g(\alpha)$ for $g \geq 3$.

There is a natural variant of the construction of the $\widehat{\mathcal{O}}$-atom, which we expect to play an important role in the future stages of the program.

Definition 2.9 ($\widehat{\mathcal{O}}^S$-atoms)**.** Suppose $\widehat{\mathcal{O}}$ is a Gorenstein curve singularity with $\mathbb{G}_m$-action and $\mathbb{G}_m$-invariant compactification $(\bar{X}; \{p_i\}_{i=1}^b)$. Let $\delta(p)$ be the δ-invariant. For any subset $S \subset \{1, \dots, b\}$ of the branches of $\widehat{\mathcal{O}}$ we define an $\widehat{\mathcal{O}}^S$*-atom* of genus g to be any curve C^S obtained by nodally attaching a smooth curve C_0 of genus $g - \delta(p) - |S| + 1$ to the points $\{p_i\}_{i \in S^c}$. (Here, $S^c := \{1, \dots, b\} \setminus S$.)

In this construction, we think of S as indexing the branches that 'dangle', and we will say that a curve with an $\widehat{\mathcal{O}}$-singularity in which several of the branches are dangling rational components to have a dangling $\widehat{\mathcal{O}}$-singularity. We may naturally define a collection of modified α-invariants associated to these dangling singularities.

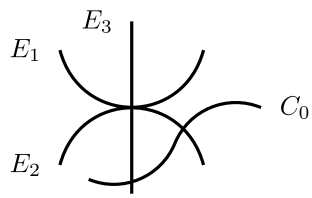

FIGURE 6. Dangling $D_6^{\{1\}}$-singularity.

Definition 2.10. The *α-invariant* $\alpha(\widehat{\mathcal{O}}^S)$ is defined to be the α-invariant $\alpha(C^S, \eta)$ of any $(\widehat{\mathcal{O}}^S, \eta)$-atom (C^S, η).

Remark 2.11. An $\widehat{\mathcal{O}}^S$-atom C^S with $\{p_i\}_{i \in S}$ can be regarded as a point in $\mathcal{U}_{g,|S|}$, and we can define the corresponding characters $\chi_{\psi_i}(C^S; \{p_i\}_{i \in S})$; see Section 1.4 for the definition of ψ_i and Corollary 3.6 for the computation of these characters.

In general, the invariants $\alpha^S(\widehat{\mathcal{O}})$ will depend on the subset S, which reflects the fact that curves C^S for different S may appear in the moduli stack $\overline{\mathcal{M}}_g(\alpha)$ at different values of α. The relationship between the characters of $\widehat{\mathcal{O}}$ and $\widehat{\mathcal{O}}^S$-atoms are explained in Corollaries 3.3 and 3.6. In Table 2, we list the α-invariants for all dangling ADE singularities. Note that since branches of any A_k or toric planar singularity are isomorphic, the only relevant feature of the subset $S \subset \{1, \dots, n\}$ is the size. For D_{2k+1} singularities, we use the labeling "1" for the singular branch and "2" for the smooth branch, and for D_{2k+2}-singularities, we use "1,2" for the tangent branches and "3" for the smooth branch with unique tangent direction. Similarly for the E_7 singularity, we use the labeling "1" for the singular branch and "2" for the smooth branch.

The final construction which we expect to play an important role in the log minimal model program is an atom made of two type A singularities. While it is possible to make this construction in a much greater generality, we will focus on the simple cases of $A_{2i+1/2j+1}$ and $A_{2i+1/2j}$-atoms, as these are the only configurations we expect to play a role for $\alpha \geq 5/9$.

Definition 2.12 ($A_{2i+1/2j+1}$, $A_{2i+1/2j}$-atoms)**.** We say that a genus g curve C is an $A_{2i+1/2j+1}$-*atom* (resp. $A_{2i+1/2j}$-*atom*) if the following two conditions are satisfied: (1) C is of the form

$$C = C_0 \cup E_1 \cup E_2 \cup E_3 \qquad \text{(resp. } C = C_0 \cup E_1 \cup E_2\text{)},$$

where C_0 is a genus $g-i-j$ curve, each E_k is a smooth rational curve, E_1 meets C_0 at a node, E_2 meets E_1 at an A_{2i+1}-singularity, and E_3 meets E_2 at an A_{2j+1}-singularity (resp. E_2 has an A_{2j}-singularity); (2) There is a $\mathbb{G}_m$-action on C which restricts to a non-trivial action on each E_k. See Figure 7.

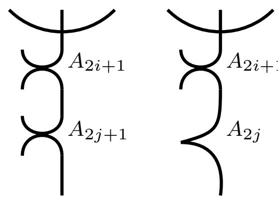

FIGURE 7. $A_{2i+1/2j+1}$ and $A_{2i+1/2j}$-atoms.

Definition 2.13. The α-invariants $\alpha(A_{2i+1/2j+1})$ and $\alpha(A_{2i+1/2j})$ are defined to be the α-invariants $\alpha(C,\eta)$ of an $A_{2i+1/2j+1}$-atom and $A_{2i+1/2j}$-atom respectively.

3. Character calculations

In this section, we compute the α-invariants of all ADE, toric planar, and monomial unibranch Gorenstein singularities. We also compute the α-invariants for the canonical rational ribbons with $\mathbb{G}_m$-action using results of Bayer and Eisenbud [BE95]. These computations are summarized in Tables 1 and 2 in Section 3.4, where we use our results to give an outline of the log MMP for $\overline{\mathcal{M}}_g$ for $\alpha \geq 5/9$.

In Section 3.1, we explain how to compute the α-invariant of a curve C with $\mathbb{G}_m$-action using Rosenlicht differentials to write down diagonalized bases for the vector spaces $\mathrm{H}^0(C, \omega_C^m)$. Under certain hypotheses on C, there is a somewhat more geometric method of computing α-invariants. Recall from (2) that K and δ are formally defined by the equations

$$K := 2\lambda_2 - 13\lambda_1,$$
$$\delta := 13\lambda_1 - \lambda_2,$$

and that these classes restrict to the usual canonical class and boundary divisor on $\overline{\mathcal{M}}_g$. These functorial interpretations cannot be extended to all of $\mathcal{U}_g$, but they do extend under additional assumptions on the deformation space of C. In Section 3.2, we will see that if C has a smooth deformation space, then the fiber $K_{[C]}$ can be identified with $\det(T^1(C))^\vee \otimes \det(\mathfrak{g})$, where $\mathfrak{g}$ is the adjoint representation of $\mathrm{Aut}(C)$, and we will use this identification to compute $\chi_K(C)$ for $A_{i/j}$-atoms. Similarly, we will see that if the locus of singular deformations of C is cut out formally locally by a single equation and generically parameterizes nodal curves, then $\chi_\delta(C)$ can be computed from the weight of this equation, and we use this to compute $\chi_\delta(C)$ for $A_{i/j}$-atoms.

It is clear that any two of the characters $\chi_1(C)$, $\chi_2(C)$, $\chi_K(C)$, $\chi_\delta(C)$ determine the others, and are therefore sufficient to determine the α-invariant. Roughly speaking, the characters

$\chi_i(C)$ are easier to compute for singularities described intrinsically, i.e. in terms of the subalgebra of functions that descend from the normalization. By contrast, $\chi_K(C)$ and $\chi_\delta(C)$ are easier to compute for singularities described by extrinsic equations, from which the deformation space and discriminant can be explicitly described.[4]

3.1. **Computing characters χ_i.** In this section, we use Rosenlicht differentials to compute α-invariants for several classes of Gorenstein singularities. We begin with the ADE singularities, which we encounter in the initial stages of the Hassett-Keel program. Next, we compute the α-invariant for an arbitrary unibranch Gorenstein singularity, and then do a multi-branch example of the elliptic m-fold points, which play a prominent role in the log MMP for $\overline{\mathcal{M}}_{1,n}$ [Smy11].

We begin by explaining how to algorithmically compute the characters $\chi_i(C)$ in the cases when C is an $\widehat{\mathcal{O}}^S$-atom.

3.1.1. *The setup.* We keep the notation of Section 2.3. Suppose $\widehat{\mathcal{O}}_{X,p}$ is a Gorenstein curve singularity with $\mathbb{G}_m$-action and $\mathbb{G}_m$-equivariant compactification $(\bar{X}; \{p_i\}_{i=1}^b)$. Recall that for $S \subset \{1, \dots, b\}$, we define the $\widehat{\mathcal{O}}_{X,p}^S$-atom as the union $C^S = \bar{X} \cup C_0$, where $\bar{X}$ is nodally attached at p_i to a point $q_i \in C_0$ for $i \in S^c$.

Suppose $X = \operatorname{Spec} \Bbbk[x_1, \dots, x_n]/I$ and $\widetilde{X} = \operatorname{Spec} \Bbbk[t_1] \times \cdots \times \Bbbk[t_b]$. Then the normalization map $\nu\colon\ \widetilde{X} \to X$ is given by $x_j \mapsto \big(f_{1j}(t_1), \dots f_{bj}(t_b)\big)$, for some monomials $f_{ij}(t_i)$. In what follows, we also let $s_i = 1/t_i$ to be the local coordinate around p_i on $\bar{X}$.

The $\mathbb{G}_m$-action on X extends to $\widetilde{X}$ as $\eta\colon\ \lambda \cdot t_i = \lambda^{-\alpha_i} t_i$ for some integers $\alpha_i \geq 1$. The weight of a semi-invariant function (or differential) f with respect to this action will be called η-weight and denoted $\operatorname{wt}_\eta(f)$. Note that, $\alpha_i \deg f_{ij} = \operatorname{wt}_\eta(x_j)$ for all i and j.

Since X is a Gorenstein curve with $\mathbb{G}_m$-action, we can find a $\mathbb{G}_m$-semi-invariant generator of $\omega_{X,p}$, which we denote by $\omega_0(X)$, or simply ω_0. After rescaling t_i we have

$$\omega_0 = \left(\frac{dt_1}{t_1^{m_1}}, \frac{dt_2}{t_2^{m_2}}, \dots, \frac{dt_b}{t_b^{m_b}}\right),$$

where $\prod_{i=1}^b (t_i^{m_i})$ is the conductor ideal. Note that $\sum_{i=1}^b m_i = 2\delta(p)$ by [Ser88, Ch.IV,11]. We make the following observation:

Lemma 3.1. $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ *is spanned by* $\mathbb{G}_m$*-semi-invariant differentials of the form* $f\omega_0$*, where* $f \in \Bbbk[x_1, \dots, x_n]$ *ranges over all monomials satisfying* $\operatorname{wt}_\eta(f\omega_0) > 0$.

Proof. This is clear. For example, to prove the weight condition, note that if $f\omega_0$ has a local equation dt_i/t_i^a around $0 \in \widetilde{E}_i$, then its equation around p_i is $-s_i^{a-2} ds_i$. It follows that $f\omega_0$ is regular at p_i if and only if $a \geq 2$ if and only if the η-weight of $f\omega_0$ is positive. □

Lemma 3.2. *There is a* $\mathbb{G}_m$*-invariant decomposition* $\mathrm{H}^0(C^S, \omega_{C^S}) \simeq \mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) \oplus W_0$*, where* W_0 *is the space of* η*-weight* 0 *differentials and* $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ *are regular differentials on* $\bar{X}$ *extended by zero to* C_0.

Proof. By definition,

$$\mathrm{H}^0\left(C^S, \omega_{C^S}\right) = \{(w,u) \in \mathrm{H}^0\Big(\bar{X}, \omega_{\bar{X}}\big(\sum_{i \in S^c} p_i\big)\Big) \oplus \mathrm{H}^0\Big(C_0, \omega_{C_0}\big(\sum_{i \in S^c} q_i\big)\Big) : \operatorname{Res}_{p_i} w + \operatorname{Res}_{q_i} u = 0\}.$$

[4] A final point of interest: Using χ_K and χ_δ, one can define α-invariants for *non-Gorenstein curves*, provided they have a smooth deformation space and the locus of worse-than-nodal deformations has codimension at least 2. This may be of interest for later stages in the log minimal model program.

A global section of $\mathrm{H}^0\big(\bar{X}, \omega_{\bar{X}}(\sum_{i \in S^c} p_i)\big)$ is a Rosenlicht differential on X with at worst simple poles at $\{p_i\}_{i\in S^c}$. By Lemma 3.1, $\mathbb{G}_m$-semi-invariant elements in $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ have positive η-weight. On the other hand, any semi-invariant differential in $\mathrm{H}^0\big(\bar{X}, \omega_{\bar{X}}(\sum_{i \in S^c} p_i)\big)$ with a simple pole at some p_i has local equation $ds_i/s_i = -dt_i/t_i$ and hence has η-weight 0. □

Corollary 3.3 (Computing λ_1). *For any $S \subset \{1, \dots, b\}$, the character $\lambda_1(C^S)$ is the sum of the weights of the $\mathbb{G}_m$-action on the vector space $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$. In other words,*

$$\lambda_1(C^S) = \lambda_1(\bar{X}).$$

In the same vein, we can algorithmically compute $\lambda_2(C^S)$.

Lemma 3.4. *Suppose $p_a(\bar{X}) \geq 2$. Then there is a $\mathbb{G}_m$-invariant decomposition*

$$\mathrm{H}^0(C^S, \omega_{C^S}^2) \simeq \mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) \oplus U_+ \oplus U_0,$$

where U_0 is the space of η-weight 0 quadratic differentials and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ are regular quadratic differentials on $\bar{X}$ extended by zero to C_0. Moreover, the weights of the $\mathbb{G}_m$-action on U_+ are precisely $\{\alpha_i\}_{i\in S^c}$.

Remark 3.5. The Gorenstein singular curves $\bar{X}$ with $p_a(\bar{X}) = 1$ are precisely *elliptic m-fold points*; see [Smy11]. The cases of $m = 1$ and $m = 2$ (the cusp and the tacnode) are considered in 3.1.2 and 3.1.3, and the case of $m \geq 3$ is considered separately in Section 3.1.9.

Proof of Lemma 3.4. Note that $p_a(\bar{X}) \geq 2$ implies that $m_i \geq 3$ for some $i \in \{1, \dots, b\}$. Clearly, $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ is a $\mathbb{G}_m$-invariant subspace of $\mathrm{H}^0(C^S, \omega_{C^S}^2)$ consisting of quadratic differentials whose local equation at $0 \in \widetilde{E}_i$ is $(dt_i)^2/t_i^{a_i}$ for some $a_i \geq 4$. The quadratic differentials of weight 0 are exactly those that have local equation $c(dt_i)^2/t_i^2$ around $0 \in \widetilde{E}_i$ (with c possibly zero) for each $i \in C^S$. It follows that $\mathrm{H}^0(C^S, \omega_{C^S}^2)/\big(\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) \oplus U_0\big)$ is spanned by $\mathbb{G}_m$-semi-invariant quadratic differentials which have local equation $(dt_i)^2/t_i^3$ on some $\widetilde{E}_i$ with $i \in S^c$. It remains to show that there are precisely $|S^c|$ linearly independent quadratic differentials of this form and to compute their weight. In doing so, we repeatedly use the differentials $\omega(i,j)$ introduced in Equation (6).

If $b = 1$, then $\omega_0 = dt/t^{m+1}$, where m is the largest gap of the semigroup of vanishing orders in $\widehat{\mathcal{O}}_{X,p}$. Since $m \geq 3$, $t^{2m-1} \in \widehat{\mathcal{O}}_{X,p}$ and we conclude that $(dt)^2/t^3 = t^{2m-1}\omega_0^2 \in \omega_{X,p}$. If $S = \varnothing$, then $(dt)^2/t^3$ spans U_+ and has weight α_1, as desired.

Consider now the case $b \geq 2$. First, we note that if E_i is a singular branch, then $dt_i/t_i^2 \in \omega_{X,p}$. It follows that $u(i) := (dt_i/t_i^2)\omega(i, i+1) = (dt_i)^2/t_i^3 \in \omega_{X,p}^2$. We conclude that for every singular branch E_i with $i \in S^c$, the differential $u(i)$ extended by zero to C^S lies in $\mathrm{H}^0(C^S, \omega_{C^S}^2)$. In particular, we are done if all the branches E_i with $i \in S^c$ are singular.

Suppose that there is a smooth branch E_i with $i \in S^c$, but not all branches are smooth and pairwise tangent. If $m_i \geq 3$, take a $\mathbb{G}_m$-semi-invariant function f on X whose local equation along E_i is exactly t_i and which vanishes to order 2 or higher along a branch E_j for some $j \neq i$ (in particular, either $\alpha_i > \alpha_j$ or $f \equiv 0$ on E_j). Then

$$u(i) := f^{m_i - 2} \cdot \omega_0 \omega(i,j) = (\dots, (dt_i)^2/t_i^3, \dots, c(dt_j)^2/(t_j^d), \dots),$$

where $d \geq 4$ and c is possibly zero. If $m_i = 2$, take j such that $m_j \geq 3$ and define

$$u(i) := \omega_0 \omega(i,j) = (\dots, (dt_i)^2/t_i^3, \dots, -(dt_j)^2/(t_j^{m_j+1}), \dots).$$

Clearly, the constructed differentials $\{u(i)\}_{i\in S^c} \subset \mathrm{H}^0(C^S, \omega_{C^S}^2)$ are linearly independent and have weights $\{\alpha_i\}_{i\in S^c}$.

Finally, suppose all branches of $p \in X$ are smooth and pairwise tangent. Then modulo $(t_1^2) \times \cdots \times (t_b^2)$ the algebra of regular function on X is generated by $f = (t_1, \dots, t_b)$. In particular, $\alpha_1 = \cdots = \alpha_b$, and so $m_1 = \cdots = m_b \geq 3$. One sees at once that

$$f^{m_i-2}\omega_0\omega(i,j) = (\cdots 0, (dt_i)^2/t_i^3, 0 \cdots 0, -(dt_j)^2/t_j^3, 0 \cdots), \quad \text{and}$$
$$f(\cdots 0, dt_i/t_i^2, 0 \cdots 0, -dt_j/t_j^2, 0 \cdots)^2 = (\cdots 0, (dt_i)^2/t_i^3, 0 \cdots 0, (dt_j)^2/t_j^3, 0 \cdots),$$

where $i \neq j \in S^c$, span U_+. Therefore U_+ has dimension $|S^c|$ and the claim follows. □

Corollary 3.6 (Computing λ_2)**.** *Suppose $p_a(\bar{X}) \geq 2$. For any $S \subset \{1, \dots, b\}$, the character $\lambda_2(C^S)$ is the sum of the weights of the $\mathbb{G}_m$-action on the vector space $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ and $\sum_{i \in S^c} \alpha_i$. The character of the $|S|$-pointed curve $(C^S; \{p_k\}_{k \in S})$ computed with respect to the cotangent line bundle ψ_i is*

$$\chi_{\psi_i}(C^S; \{p_k\}_{k\in S}) = \alpha_i.$$

In particular,

$$\lambda_2(C^S) = \lambda_2(\bar{X}) + \sum_{i \in S^c} \alpha_i = \lambda_2(C) - \sum_{i\in S} \chi_{\psi_i}(C^S; \{p_k\}_{k\in S}).$$

Proof. The first assertion follows directly from Lemma 3.2. The second assertion follows from the observation that the fiber of ψ_i at $(C^S; \{p_k\}_{k\in S})$ is $ds_i = -dt_i/t_i^2$ and so $\mathbb{G}_m$ acts on it with weight α_i. □

3.1.2. *A_{2k}-singularity.* Let $C = C_0 \cup \bar{X}$, where $X \simeq \operatorname{Spec} \Bbbk[x,y]/(y^2 - x^{2k+1})$, be an A_{2k+1}-atom. The $\mathbb{G}_m$-action on C is $\eta:\ \lambda \cdot (x,y) = (\lambda^{-2}x, \lambda^{-(2k+1)}y)$. The normalization map is given by $(x,y) \mapsto (t^2, t^{2k+1})$ and the $\mathbb{G}_m$-action extends as $t \mapsto \lambda^{-1}t$.

It is easy to see that $\omega_0 := dt/t^{2k}$ is a $\mathbb{G}_m$-semi-invariant generator for ω_X at the cusp, of η-weight $2k-1$. By Lemma 3.1, a basis of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ diagonalizing the $\mathbb{G}_m$-action is $\langle x^i w_0 \mid 0 \leq i \leq k-1 \rangle$. The η-weights of this basis are $(2k-1), (2k-3), \dots, 1$. Thus by Corollary 3.3, the character χ_1 is

$$\chi_1(A_{2k}) = \sum_{i=1}^{k} (2i-1) = k^2.$$

Similarly, if $k \geq 2$, a basis of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-action is

$$\langle x^i\omega_0^2,\ x^j y \omega_0^2 \mid 0 \leq i \leq 2k-2,\ 0 \leq j \leq k-3 \rangle.$$

Thus, by Corollary 3.6 the character χ_2 is given by

$$\chi_2(A_{2k}) = \sum_{i=0}^{2k-2} (4k-2-2i) + \sum_{j=0}^{k-3} (2k-3-2j) + 1 = 5k^2 - 4k + 1.$$

If $k = 1$, then the only weight space of $\mathrm{H}^0(C, \omega_C^2)$ with non-zero weight is spanned by $(dt)^2/t^4$, hence $\chi_2(A_2) = 2$.

3.1.3. *A_{2k+1}-singularity.* Next, we consider an A_{2k+2}-atom $C = C_0 \cup \bar{X}$, where $X \simeq \operatorname{Spec} \Bbbk[x,y]/(y^2 - x^{2k+2})$. The $\mathbb{G}_m$-action on C is $\eta:\ \lambda \cdot (x,y) = (\lambda^{-1}x, \lambda^{-(k+1)}y)$. The normalization map is $(x,y) \mapsto ((t_1,t_2), (t_1^{k+1}, -t_2^{k+1}))$ and the $\mathbb{G}_m$-action extends as $\lambda \cdot (t_1,t_2) = (\lambda^{-1}t_1, \lambda^{-1}t_2)$. It is easy to see that $\omega_0 := \left(dt_1/t_1^{k+1}, -dt_2/t_2^{k+1}\right)$ is a $\mathbb{G}_m$-semi-invariant generator for ω_X at the tacnode, of η-weight k. If $k \geq 2$, we may write down bases of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-action as

$$\begin{aligned}\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) &= \langle x^i\omega_0 \mid 0 \leq i \leq k-1\rangle,\\ \mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) &= \langle x^i w_0^2,\ x^j y w_0^2 \mid 0 \leq i \leq 2k-2,\ 0 \leq j \leq k-3\rangle.\end{aligned}$$

It follows by Corollaries 3.3 and 3.6 that

$$\begin{aligned}\chi_1(A_{2k+1}) &= \sum_{i=0}^{k-1}(k-i) = \frac{k^2+k}{2},\\ \chi_2(A_{2k+1}) &= \sum_{i=0}^{2k-2}(2k-i) + \sum_{j=0}^{k-3}(k-1-j) + (1+1) = \frac{5k^2+k}{2}.\end{aligned}$$

(N.B. If $k = 1$, we need to modify the computation of χ_2. Namely, we observe that the only weight spaces of $\mathrm{H}^0(C, \omega_C^2)$ with non-zero weights are spanned by $((dt_1)^2/t_1^4, (dt_2)^2/t_2^4)$ and $((dt_1)^2/t_1^3, (dt_2)^2/t_2^3)$. Hence $\chi_2(A_3) = 3$.)

3.1.4. *D_{2k+1}-singularity.* Here, $X \simeq \operatorname{Spec} \Bbbk[x,y]/x(y^2 - x^{2k-1})$ and the normalization map is given by $(x,y) \mapsto \big((t_1^2, 0), (t_1^{2k-1}, t_2)\big)$ with the $\mathbb{G}_m$-action on the normalization being $\lambda \cdot (t_1, t_2) = (\lambda^{-1}t_1, \lambda^{-(2k-1)}t_2)$. A $\mathbb{G}_m$-semi-invariant generator for ω_X at the singularity is $\omega_0 := \big(dt_1/t_1^{2k}, -dt_2/t_2^2\big)$, of η-weight $2k-1$. It follows that the bases of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-action are

$$\begin{aligned}\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) &= \langle x^i\omega_0 \mid 0 \leq i \leq k-1\rangle,\\ \mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) &= \langle x^i w_0^2,\ x^j y w_0^2 \mid 0 \leq i \leq 2k-2,\ 1 \leq j \leq k-2\rangle.\end{aligned}$$

Therefore, by Corollaries 3.3 and 3.6

$$\begin{aligned}\chi_1(D_{2k+1}) &= \sum_{i=0}^{k-1}(2k-1-2i) = k^2,\\ \chi_2(D_{2k+1}) &= \sum_{i=0}^{2k-2}(4k-2-2i) + \sum_{j=1}^{k-2}(2k-1-2j) + (1+2k-1) = 5k^2-2k.\end{aligned}$$

3.1.5. *D_{2k+2}-singularity.* Consider a D_{2k+2}-atom $C = C_0 \cup \bar{X}$, where we take $X \simeq \Bbbk[x,y]/x(y^2 - x^{2k})$. The normalization map is given by $(x,y) \mapsto \big((t_1, t_2, 0), (t_1^k, -t_2^k, t_3)\big)$. The $\mathbb{G}_m$-action is given by $\lambda \cdot (t_1, t_2, t_3) = (\lambda^{-1}t_1, \lambda^{-1}t_2, \lambda^{-k}t_3)$ and the $\mathbb{G}_m$-semi-invariant generator of ω_X at the singularity is $\omega_0 := \big(dt_1/t_1^{k+1}, -dt_2/t_2^{k+1}, -2dt_3/t_3^2\big)$, of η-weight k. The bases of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-action are

$$\begin{aligned}\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) &= \langle x^i\omega_0 \mid 0 \leq i \leq k-1\rangle,\\ \mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) &= \langle x^i w_0^2,\ x^j y w_0^2 \mid 0 \leq i \leq 2k-2,\ 1 \leq j \leq k-2\rangle.\end{aligned}$$

Therefore, by Corollaries 3.3 and 3.6

$$\begin{aligned}\chi_1(D_{2k+2}) &= \sum_{i=0}^{k-1}(k-i) = \frac{k^2+k}{2},\\ \chi_2(D_{2k+2}) &= \sum_{i=0}^{2k-2}(2k-2i) + \sum_{j=1}^{k-2}(k-j) + (1+1+k) = \frac{5k^2+3k}{2}.\end{aligned}$$

3.1.6. *Exceptional simple singularities: E_6, E_7, E_8.* To complete a list of the χ_i characters of all ADE singularities, it remains to consider the cases of E_6, E_7, E_8 singularities. Among these, $E_6 : y^3 - x^4 = 0$ and $E_8 : y^3 - x^5 = 0$ are unibranch. We compute the characters of all unibranch curve singularities with $\mathbb{G}_m$-action in Section 3.1.7, but we list here the characters of E_6 and E_8-atoms for the reader's convenience:

$$\chi_1(E_6) = 8,$$
$$\chi_2(E_6) = 33.$$

$$\chi_1(E_8) = 14,$$
$$\chi_2(E_8) = 63.$$

Let $C = C_0 \cup \bar{X}$ be an E_7-atom, where $X \simeq \operatorname{Spec} \Bbbk[x, y]/y(y^2 - x^3)$. The normalization map is given by $(x, y) \mapsto \big((t_1^2, t_2), (t_1^3, 0)\big)$ and the $\mathbb{G}_m$-action by $\lambda \cdot (t_1, t_2) = (\lambda^{-1} t_1, \lambda^{-2} t_2)$. The generator of ω_X at the singularity is $\omega_0 := \big(dt_1/t_1^5, -dt_2/t_2^3\big)$, of η-weight 4. The bases of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-action are

$$\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) = \langle \omega_0, x\omega_0, y\omega_0 \rangle,$$
$$\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) = \langle \omega_0^2, x\omega_0^2, x^2\omega_0^2, y\omega_0^2, xy\omega_0^2, y^2\omega_0^2 \rangle$$

It follows that

$$\chi_1(E_7) = 4 + 2 + 1 = 7,$$
$$\chi_2(E_7) = (8 + 6 + 4 + 5 + 3 + 2) + 1 + 2 = 31.$$

3.1.7. *Monomial unibranch singularities.* Suppose $\Gamma = \mathbb{Z}_{\geq 0} \smallsetminus \{b_1, \dots, b_g\}$ is a semigroup containing 0. Let $\bar{X}$ be the $\mathbb{G}_m$-equivariant compactification of $X \simeq \operatorname{Spec} \Bbbk[t^n \colon n \in \Gamma]$, with the $\mathbb{G}_m$-action given by $\eta\colon\ \lambda \cdot t = \lambda^{-1} t$. It is easy to check that $p_a(\bar{X}) = g$, the normalization of $\bar{X}$ is $\mathbb{P}^1$, and $\bar{X}$ has an isolated monomial unibranch singularity at $t = 0$ with the gap sequence $\{b_1, \dots, b_g\}$. From now on we assume that $\bar{X}$ is Gorenstein, which by [Kun70] is equivalent to Γ being a *symmetric* semigroup:

$$n \in \{b_1, \dots, b_g\} \Leftrightarrow 2g - 1 - n \notin \{b_1, \dots, b_g\}.$$

In particular, $b_g = 2g-1$. Evidently, a $\mathbb{G}_m$-semi-invariant generator for $\omega_{\bar{X}}$ in a neighborhood of zero is given by dt/t^{b_g+1}. Therefore, we can write down the bases of $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ and $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2)$ diagonalizing the $\mathbb{G}_m$-actions as

$$\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}) = \left\langle \frac{dt}{t^{b_1+1}}, \frac{dt}{t^{b_2+1}}, \dots, \frac{dt}{t^{b_g+1}} \right\rangle,$$
$$\mathrm{H}^0(\bar{X}, \omega_{\bar{X}}^2) = \left\langle \frac{(dt)^2}{t^{2b_g+2-j}} \quad : \quad j \in \{0, \dots, 2b_g - 2\} \smallsetminus \{b_1, \dots, b_g\} \right\rangle.$$

From this, we compute

$$\chi_1(\bar{X}) = \sum_{i=1}^{g} b_i\ , \tag{9}$$

$$\chi_2(\bar{X}) = \sum_{j=0}^{2b_g-2} (2b_g - j) - \sum_{i=1}^{g} (2b_g - b_i) = (2g-1)^2 + \sum_{i=1}^{g} b_i - 1. \tag{10}$$

If $C = C_0 \cup \bar{X}$ is an $\widehat{\mathcal{O}}_{X,p}$-atom, then by Corollaries 3.3 and 3.6, we have

$$\begin{aligned} \chi_1(C) &= \sum_{i=1}^{g} b_i, \\ \chi_2(C) &= (2g-1)^2 + \sum_{i=1}^{g} b_i. \end{aligned} \tag{11}$$

Set $R(b_1, \dots, b_g) := (2g-1)^2/\left(\sum_{i=1}^{g} b_i\right)$. Then by Definition 1.4, the α-invariant of C is

$$\alpha(C, \eta) = \frac{11 - 2R(b_1, \dots, b_g)}{12 - R(b_1, \dots, b_g)}.$$

From the point of view of the log MMP for $\overline{\mathcal{M}}_g$, it is an interesting problem to determine possible values of the α-invariants of unibranch Gorenstein singularities.

Remark 3.7. In a symmetric gap sequence, one necessarily has $b_i \leq 2i - 1$, so that $\sum_{i=1}^{g} b_i \leq g^2$, with the equality achieved only for the A_{2g+1}-singularity. A similarly elementary argument shows that the only symmetric gap sequences with $\sum_{i=1}^{g} b_i > g^2 - g$, or equivalently $\alpha(\widehat{\mathcal{O}}_{X,p}) > 3/8$, are precisely the unibranch planar singularities of types A and E, and the unique exception $y^3 = x^7$.

In 3.1.8, we compute a closed form formula for the α-invariants of unibranch monomial singularities of embedding dimension 2. In particular, it is evident from this calculation that there are Gorenstein unibranch singularities with negative α-invariant. This raises a question[5] of what unibranch Gorenstein singularities have non-negative α-invariant (i.e. are expected to arise in the Hassett-Keel program). Since we do not know the answer, we state this question as an open problem:

Problem 3.8. *Classify (symmetric) numerical semigroups with gap sequences $\{b_1, \dots, b_g\}$ such that*

$$R(b_1, \dots, b_g) = \frac{(2g-1)^2}{\left(\sum_{i=1}^{g} b_i\right)} \geq \frac{11}{2}.$$

We also note that one can associate a sub-semigroup of $\mathbb{Z}_{\geq 0}^b$ to an arbitrary curve singularity with b branches and Gorenstein singularities are characterized as those whose semigroups are symmetric [DdlM88]. It would be interesting to understand how the α-invariant of an arbitrary Gorenstein singularity can be computed in terms of its semigroup.

3.1.8. *Unibranch planar singularities.* For embedding dimension 2, the gap sequence of a monomial unibranch singularity and hence the α-invariant is easily computed. Such singularities are defined by $x^p = y^q$, or as $X = \operatorname{Spec} \Bbbk[t^{pi+qj} \colon i, j \in \mathbb{Z}_{\geq 0}]$, where p and q are coprime. The gap sequence $\{b_1, \dots, b_g\}$ is the set of positive integers that cannot be expressed as $pi + qj$ with $i, j \geq 0$. The study of this sequence, e.g. finding its cardinality and the largest element is classically known in elementary number theory as the *Frobenius problem* [RA05]. It is well-known that the largest gap is $b_g = pq - p - q$. It is also easy to see that the gap sequence is symmetric: n is a gap if and only if $pq - p - q - n$ is not a gap. It follows that the genus of the singularity $x^p = y^q$ is $g = (p-1)(q-1)/2$. By [BS93] (see also [Rød94]), the sum of gaps is

$$\sum_{n=1}^{g} b_i = (p-1)(q-1)(2pq - p - q - 1)/12. \tag{12}$$

[5] The same question can be posed for arbitrary monomial unibranch singularities.

It follows from Equations (9)–(10) that

$$\begin{aligned} \chi_1(\bar{X}) &= \frac{1}{12}(p-1)(q-1)(2pq-p-q-1), \\ \chi_2(\bar{X}) &= (pq-p-q)^2 + \frac{1}{12}(p-1)(q-1)(2pq-p-q-1) - 1. \end{aligned} \tag{13}$$

Remarkably, intersection theory calculations of Proposition 4.3 give an independent algebro-geometric proof of the highly nontrivial combinatorial Formula (12); see Corollary 4.4 in Section 4 below.

3.1.9. *Elliptic m-fold points.* Let $m \geq 3$. An elliptic m-fold point X is a Gorenstein union of m general lines through a point in $\mathbb{A}^{m-1}$ [Smy11]. Every such singularity is isomorphic to the cone over the points $p_1 = (1, 0, \dots, 0)$, $p_2 = (0, 1, \dots, 0)$, $\dots$, $p_{m-1} = (0, 0, \dots, 1)$, and $p_m = (1, \dots, 1)$, with the vertex at $p = (0, \dots, 0) \in \mathbb{A}^{m-1}$. If $(x_1, \dots, x_{m-1})$ are coordinates centered at the vertex then the normalization map from m copies of $\mathbb{A}^1$ to X is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{m-1} \end{pmatrix} \to \begin{pmatrix} t_1 & 0 & \dots & 0 & t_m \\ 0 & t_2 & \ddots & \vdots & t_m \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & t_{m-1} & t_m \end{pmatrix}.$$

The $\mathbb{G}_m$-action η is defined by $\lambda \cdot (t_1, \dots, t_m) = (\lambda^{-1}t_1, \dots, \lambda^{-1}t_m)$. We let C be the singular curve obtained by attaching $\bar{X}$ to a smooth curve C_0 nodally at the points $p_1, \dots, p_m$. We note that $\mathrm{H}^0(\bar{X}, \omega_{\bar{X}})$ is one-dimensional and is generated by

$$\omega_0 = \left(\frac{dt_1}{t_1^2}, \dots, \frac{dt_{m-1}}{t_{m-1}^2}, -\frac{dt_m}{t_m^2} \right).$$

Thus, $\chi_1(C, \eta) = 1$ by Corollary 3.6. A generator for $\omega_{\bar{X}}^2$ in the neighborhood of the m-fold point is

$$\omega_0^2 = \left(\frac{(dt_1)^2}{t_1^4}, \dots, \frac{(dt_{m-1})^2}{t_{m-1}^4}, \frac{(dt_m)^2}{t_m^4} \right),$$

and the only weight spaces of $\mathrm{H}^0(C, \omega_C^2)$ with non-zero η-weights are spanned by

$$(\omega_0^2, 0), (x_1\omega_0^2, 0), \dots, (x_{m-1}\omega_0^2, 0).$$

It follows that $\chi_2(C) = 2 + (m-1) = m+1$. Thus, the α-invariant of the elliptic m-fold atom is $\alpha = \dfrac{11-2m}{12-m}$.

3.2. **Computing χ_K and χ_δ.** In this section, we explain how to compute the characters $\chi_K(C, \eta)$ and $\chi_\delta(C, \eta)$ for a curve C with $\mathbb{G}_m$-action $\eta \colon \mathbb{G}_m \to \mathrm{Aut}(C)$ by studying the induced action on the deformation space of C. Throughout this section, we make the following additional assumptions:

(A1) The miniversal deformation space of C is smooth.
(A2) The locus of worse-than-nodal deformations of C has codimension at least 2.
(A3) The locus $\Delta \subset \mathcal{U}_g$ of singular deformations of C is a divisor in a neighborhood of $[C]$.

First, we explain how to compute the character of $\chi_K(C, \eta)$ by observing that the smoothness hypothesis gives a functorial interpretation to the canonical divisor. Indeed, since $\mathcal{U}_g$ is a smooth stack in a neighborhood of $[C]$, it admits a canonical line bundle $K_{\mathcal{U}_g}$, defined as $\det L_{\mathcal{U}_g}$, where $L_{\mathcal{U}_g}$ is the cotangent complex of $\mathcal{U}_g$ [Ols07]. Using this fact, it is easy to see that $\chi_{K_{\mathcal{U}_g}}(C, \eta)$

can be computed as the determinant of the action of η on $T^1(C)^\vee$, the dual of the space of first-order deformations.

Lemma 3.9. *Let (C,η) be a curve with $\mathbb{G}_m$-action satisfying (A1). Then there is a canonical identification*

$$K_{\mathcal{U}_g,[C]} = \det(T^1(C))^\vee \otimes \det(\mathfrak{g}),$$

where $\mathfrak{g}$ is the adjoint representation of $\operatorname{Aut}(C)$. In particular, if $\operatorname{Aut}(C)^\circ$ is abelian, then $\chi_{K_{\mathcal{U}_g}}(C,\eta)$ is simply the weight of the action of η on $\det(T^1(C))^\vee$.

Proof. Choose a miniversal deformation space $p\colon\ M \to \mathcal{U}_g$ around $[C]$ with M smooth; that is, $M \to \mathcal{U}_g$ is a smooth morphism and there is a point $m \in M$ above $[C]$ inducing an isomorphism $T_{M,m} \to T^1(C)$ of tangent spaces. Moreover, we may assume that there is a cartesian diagram

$$\begin{tikzcd} \operatorname{Spec}\Bbbk \arrow[r, "i'"] \arrow[d, "p'"] & M \arrow[d, "p"] \\ B\operatorname{Aut}(C) \arrow[r, "i"] & \mathcal{U}_g \end{tikzcd}$$

where $B\operatorname{Aut}(C) \to \mathcal{U}_g$ is the residual gerbe associated to $[C]$. The cotangent complex $L_{M/\mathcal{U}_g}$ is supported in degree 0 by a locally free sheaf which we denote by $\Omega_{M/\mathcal{U}_g}$. There is an exact triangle

$$p^*L_{\mathcal{U}_g} \to L_M \to L_{M/\mathcal{U}_g} \to p^*L_{\mathcal{U}_g}[1]$$

inducing an exact sequence

$$0 \to \mathrm{H}^0(p^*L_{\mathcal{U}_g}) \to \Omega_M \to \Omega_{M/\mathcal{U}_g} \to \mathrm{H}^1(p^*L_{\mathcal{U}_g}) \to 0$$

of quasi-coherent sheaves on M. It follows that $p^*K_{\mathcal{U}_g} = K_M \otimes (\det\Omega_{M/\mathcal{U}_g})^\vee$. The restriction $i'^*\Omega_{M/\mathcal{U}_g} = \Omega_{\operatorname{Spec} CC/B\operatorname{Aut}(C)}$ is canonically identified with $\mathfrak{g}^\vee$. Therefore, $i^*K_{\mathcal{U}_g}$ corresponds to the natural action of $\operatorname{Aut}(C)$ on $\det(T^1(C))^\vee \otimes \det(\mathfrak{g})$. ☐

Corollary 3.10. *Suppose that $C \in \mathcal{U}_g$ is a curve with $\mathbb{G}_m$-action $\eta\colon\ \mathbb{G}_m \to \operatorname{Aut}(C)$ satisfying Assumptions (A1)-(A2). Then $\chi_K(C,\eta)$ is the weight of the action of η on $\det(T^1(C))^\vee$.*

Proof. Since $K_{\mathcal{U}_g}$ and K are line bundles defined in a neighborhood of $[C] \in \mathcal{U}_g$ that agree away from a codimension 2 locus by Remark 1.1, we have $K \simeq K_{\mathcal{U}_g}$. The claim now follows from Lemma 3.9. ☐

Next, we explain how to compute the character $\chi_\delta(C,\eta)$. The key observation is that the character $\chi_{\mathcal{O}(D)}(C,\eta)$ corresponding to an effective $\mathbb{G}_m$-invariant Cartier divisor D in the miniversal deformation space of C is determined by the weight of the defining equation of D.

Lemma 3.11. *Let $C \in \mathcal{U}_g$ be a curve with $\mathbb{G}_m$-action $\eta\colon\ \mathbb{G}_m \to \operatorname{Aut}(C)$. Take $\operatorname{Spf}(A)$ to be a formal miniversal deformation space of C. Note that η induces a $\mathbb{G}_m$-action on $\operatorname{Spf}(A)$. Suppose D is a Cartier divisor defined on an étale neighborhood of $[C] \in \mathcal{U}_g$ and that $D|_{\operatorname{Spf} A} = \{f = 0\}$, where $f \in A$. Then*

$$\chi_{\mathcal{O}(D)}(C,\eta) = -d,$$

where $f \mapsto \lambda^d f$ under the induced action of $\mathbb{G}_m = \operatorname{Spec}\Bbbk[\lambda,\lambda^{-1}]$ on $\operatorname{Spf} A$.

Proof. Denote by $\sigma : \ A \to \Bbbk[\lambda, \lambda^{-1}]\widehat{\otimes} A$ the dual action of $\mathbb{G}_m$ on $\operatorname{Spf} A$. The exact sequence $0 \to \mathcal{O}(-D) \to \mathcal{O}_{\mathcal{U}_g} \to \mathcal{O}_D \to 0$ restricted to $\operatorname{Spf} A$ corresponds to the exact sequence

$$0 \to A_\eta \xrightarrow{f} A \to A/f \to 0$$

where A_η is the $\mathbb{G}_m$-A-module corresponding to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$; that is, A_η is A as an A-module with coaction $a \mapsto \lambda^d \sigma(a)$. Therefore $\mathcal{O}(-D)|_{B\mathbb{G}_m}$ corresponds to the character $\mathbb{G}_m \xrightarrow{d} \mathbb{G}_m$ and $\chi_{\mathcal{O}(-D)}(C, \eta) = d$. $\square$

Corollary 3.12. *Suppose that $C \in \mathcal{U}_g$ is a curve with $\mathbb{G}_m$-action $\eta : \ \mathbb{G}_m \to \operatorname{Aut}(C)$ satisfying Assumptions (A1)-(A3). Then Δ is formally locally cut out by a $\mathbb{G}_m$-semi-invariant of weight $-\chi_\delta(C, \eta)$.*

Proof. By (A3), the locus Δ of singular deformations is a divisor, which is clearly $\mathbb{G}_m$-invariant. Since Δ is Cartier by (A1), it is cut out by a $\mathbb{G}_m$-semi-invariant. We have $\mathcal{O}_{\mathcal{U}_g}(\Delta)|_{\overline{\mathcal{M}}_g} \simeq \delta|_{\overline{\mathcal{M}}_g}$ by Remark 1.1 and the line bundles $\mathcal{O}_{\mathcal{U}_g}(\Delta)$ and δ are defined in a neighborhood of $[C]$, this isomorphism extends to a neighborhood of $[C] \in \mathcal{U}_g$ by (A2) and the claim follows. $\square$

In what follows, we use Corollaries 3.10 and 3.12 to present an alternative computation of the characters of A-singularities and to compute the characters of $A_{i/j}$-atoms.

3.2.1. *A_{2k}-singularity.* For $k \geq 2$, let $C = C_0 \cup \bar{X}$ be an A_{2k}-atom as in 3.1.2, with $p \in X$ being the A_{2k}-singularity. Let $q = C_0 \cap \bar{X}$ be the node. The space of first-order deformations $T^1(C)$ can be decomposed as:

$$T^1(C) = T^1(C_0, q) \times T^1(\widehat{\mathcal{O}}_{C,p}) \times T^1(\widehat{\mathcal{O}}_{C,q}) \times \operatorname{Cr}^1(\widehat{\mathcal{O}}_{C,p}),$$

where $T^1(C_0, q)$ are the first-order deformations of the pointed curve (C_0, q), $T^1(\widehat{\mathcal{O}}_{C,p})$ and $T^1(\widehat{\mathcal{O}}_{C,q})$ are the first-order deformations of the A_{2k}-singularity and the node respectively, and $\operatorname{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ are the first-order "crimping" deformations of C that preserve the A_{2k}-singularity at p and the normalization of C at p; see [vdW10] for a detailed treatment of crimping deformations. Choose coordinates $a_0, \dots, a_{2k-1}$ on $T^1(\widehat{\mathcal{O}}_{C,p})$ and n on $T^1(\widehat{\mathcal{O}}_{C,q})$ corresponding to the miniversal deformations

$$\begin{aligned} y^2 &= x^{2k+1} + a_{2k-1}x^{2k-1} + \cdots + a_1 x + a_0, \\ zw &= n, \end{aligned}$$

of the A_{2k}-singularity and the node, respectively. Here, we choose z to be the uniformizer of $\bar{X}$ at q. The action of $\mathbb{G}_m$ is given by

$$\begin{aligned} \lambda \cdot (x, y, a_i) &= (\lambda^{-2}x,\ \lambda^{-(2k+1)}y,\ \lambda^{-(4k+2-2i)}a_i), \\ \lambda \cdot (z, w, n) &= (\lambda z,\ w,\ \lambda n). \end{aligned}$$

Thus, the induced $\mathbb{G}_m$-action on $\det(T^1(\widehat{\mathcal{O}}_{C,p}))$ has weight $\sum_{i=0}^{2k-1}(4k+2-2i) = (4k^2+6k)$, while the induced action on $T^1(\widehat{\mathcal{O}}_{C,q})$ has weight -1. (N.B. The tangent space to the miniversal deformation $\widehat{\mathcal{O}}_{C,p}$ is $T^1(\widehat{\mathcal{O}}_{C,p}) = \big((a_0, \dots, a_{2k-1})/(a_0, \dots, a_{2k-1})^2\big)^\vee$, hence the reversal of signs).

The $\mathbb{G}_m$-action on the normalization of X is $\lambda \cdot t = \lambda^{-1}t$. By [vdW10, Example 1.111], the weights of the induced $\mathbb{G}_m$-action on $\operatorname{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ are $-1, -3, \dots, -(2k-3)$. We note that this can be seen from the fact that we may choose coordinates $c_1, \dots, c_k$ on $\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ corresponding to the crimping deformation of the A_{2k}-singularity as given by

$$(t^2 + c_1 t^3 + c_2 t^5 + \ldots + c_k t^{2k-1}, t^{2k+1}) \subset \Bbbk[t].$$

Thus, the induced character on $\det(\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p}))$ is $-\sum_{i=1}^{k-1}(2i-1) = -(k-1)^2$. Finally, since the $\mathbb{G}_m$-action on $T^1(C_0, q)$ is trivial, we conclude by Corollary 3.10 that

$$\chi_K(A_{2k}) = -\big((4k^2+6k) - 1 - (k-1)^2\big) = -3k^2 - 8k + 2. \tag{14}$$

Let $\Delta(a_0, \dots, a_{2k-1})$ denote the discriminant of $x^{2k+1} + a_{2k-1}x^{2k-1} + \cdots + a_1 x + a_0$. Then the locus of singular deformations of C is cut out formally locally by the polynomial $n\Delta(a_0, \dots, a_{2k-1})$. Since, $\Delta(a_0, \dots, a_{2k-1})$ is a semi-invariant of η-weight $-4k(2k+1)$, we see that $\mathbb{G}_m$ acts on $n\Delta(a_0, \dots, a_{2k-1})$ with weight $1 - 4k(2k+1)$. Thus, by Corollary 3.12, we have

$$\chi_\delta(A_{2k}) = 8k^2 + 4k - 1. \tag{15}$$

Remark 3.13. Note that we have $\chi_K(A_{2k}) = 13\chi_1(A_{2k}) - 2\chi_\delta(A_{2k})$ (cf. Table 1), as guaranteed by Equation (2).

3.2.2. *A_{2k+1}-singularity.* Let $C = C_0 \cup \bar{X}$ be an A_{2k+1}-atom as in 3.1.3, with $p \in X$ being the A_{2k+1}-singularity. Let $\{q_1, q_2\} = C_0 \cap \bar{X}$ be the nodes. The space of first-order deformations $T^1(C)$ can be decomposed as:

$$T^1(C) = T^1(C_0, q_1, q_2) \times \mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p}) \times T^1(\widehat{\mathcal{O}}_{C,p}) \times T^1(\widehat{\mathcal{O}}_{C,q_1}) \times T^1(\widehat{\mathcal{O}}_{C,q_2}),$$

where $\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ are the first-order deformations crimping deformations of C which preserve the A_{2k+1}-singularity at p and the normalization of C at p; see [vdW10]. Choose coordinates $a_0, \dots, a_{2k}$ on $T^1(\widehat{\mathcal{O}}_{C,p})$ and n_j on $T^1(\widehat{\mathcal{O}}_{C,q_j})$ corresponding to the miniversal deformations

$$\begin{aligned} y^2 &= x^{2k+2} + a_{2k}x^{2k} + \cdots + a_1 x + a_0, \\ z_1 w &= n_1, \\ z_2 w &= n_2. \end{aligned}$$

of the A_{2k+1}-singularity and the nodes. Here, we choose z_j to be the uniformizer of $q_j \in \bar{X}$. The action of $\mathbb{G}_m$ is given by

$$\begin{aligned} \lambda \cdot (x, y, a_i) &= (\lambda^{-2}x,\ \lambda^{-(k+1)}y,\ \lambda^{-(2k+2-i)}a_i), \\ \lambda \cdot (z, w, n_j) &= (\lambda z,\ w,\ \lambda n_j). \end{aligned}$$

Therefore, the induced $\mathbb{G}_m$-action on $\det(T^1(\widehat{\mathcal{O}}_{C,p}))$ has weight $\sum_{i=0}^{2k}(2k+2-i) = (2k^2+5k+2)$, while the induced action on $T^1(\widehat{\mathcal{O}}_{C,q_j})$ has weight -1. The $\mathbb{G}_m$-action on $T^1(C_0, q)$ is trivial.

The $\mathbb{G}_m$-action on the normalization of X is $\lambda \cdot (t_1, t_2) = (\lambda^{-1}t_1, \lambda^{-1}t_2)$. By [vdW10, Example 1.112], the weights of the induced $\mathbb{G}_m$-action on $\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ are $-1, -2, \dots, -(k-1)$. This can be seen from the fact that we may choose coordinates $c_1, \dots, c_{k-1}$ on $\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p})$ corresponding to the crimping deformation of the A_{2k+1}-singularity as given by

$$\Bbbk[(t_1, t_2 + c_1 t_2^2 + c_2 t_2^3 + \cdots + c_{k-1}t_2^k), (t_1^{k+1}, t_2^{k+1})] \subset \Bbbk[t_1] \oplus \Bbbk[t_2].$$

Therefore, the induced character on $\det(\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p}))$ is $-\sum_{i=1}^{k-1} i = -(k-1)k/2$. By Corollary 3.10, we conclude

$$\chi_K(A_{2k+1}) = -\big((2k^2+5k+2) - 2 - (k-1)k/2\big) = -(3k^2+11k)/2. \tag{16}$$

Let $\Delta(a_0,\dots,a_{2k})$ denote the discriminant of the polynomial $x^{2k+2}+a_{2k}x^{2k}+\cdots+a_1x+a_0$. Then the locus of singular deformations of C is cut out formally locally by the polynomial $n_1n_2\Delta(a_0,\dots,a_{2k})$ and $\mathbb{G}_m$ acts on this polynomial with weight $2-(2k+1)(2k+2)$. Thus, by Corollary 3.12, we have

$$\chi_\delta(A_{2k+1}) = 4k^2+6k. \tag{17}$$

In the following two examples, we build on Sections 3.2.1 and 3.2.2 to compute the characters of $A_{i/j}$-atoms from Definition 2.12.

3.2.3. *$A_{2i+1/2j}$-atoms.* Let $C=C_0\cup E_1\cup E_2$ be an $A_{2i+1/2j}$-atom. Let $q=E_1\cap C_0$ denote the node, $p_1=E_1\cap E_2$ denote the A_{2i+1}-singularity, and $p_2\in E_2$ denote the A_{2j}-singularity. The space of first-order deformations $T^1(C)$ is decomposed as:

$$T^1(C)=T^1(C_0,q)\times T^1(\widehat{\mathcal{O}}_{C,q})\times T^1(\widehat{\mathcal{O}}_{C,p_1})\times \mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p_1})\times T^1(\widehat{\mathcal{O}}_{C,p_2})\times \mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p_2}).$$

Choose coordinates $(a_0,\dots,a_{2j-1})$, $(b_0,\dots,b_{2j})$, and n on $T^1(\widehat{\mathcal{O}}_{C,p_2})$, $T^1(\widehat{\mathcal{O}}_{C,p_1})$, and $T^1(\widehat{\mathcal{O}}_{C,q})$, respectively, corresponding to the miniversal deformations

$$\begin{aligned} y^2 &= x^{2j+1}+a_{2j-1}x^{2j-1}+\cdots+a_1x+a_0,\\ w^2 &= z^{2i+2}+b_{2i}z^{2i}+\cdots+b_1z+b_0,\\ uv &= n. \end{aligned}$$

The $\mathbb{G}_m$-action is given by $\lambda\cdot(x,y)=(\lambda^{-2}x,\lambda^{-(2j+1)}y)$, $\lambda\cdot(z,w)=(\lambda z,\lambda^{i+1}w)$, and $\lambda\cdot u=\lambda^{-1}u$. From 3.2.1, the characters of $\det(T^1(\widehat{\mathcal{O}}_{C,p_2}))$ and $\det(\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p_2}))$ are $4j^2+6j$ and $-(j-1)^2$, respectively. From 3.2.2, the characters of $\det(T^1(\widehat{\mathcal{O}}_{C,p_1}))$ and $\det(\mathrm{Cr}^1(\widehat{\mathcal{O}}_{C,p_1}))$ are $-(2i^2+5i+2)$ and $(i^2-i)/2$ (note that the present action on the A_{2i+1}-singularity has sign opposite from that in 3.2.2). Finally, the characters of $T^1(\widehat{\mathcal{O}}_{C,q})$ and $T^1(C_0,q)$ are 1 and 0, respectively. By Lemma 3.9, we conclude

$$\chi_K(A_{2i+1}/A_{2j})=2i^2+5i+2-\frac{(i^2-i)}{2}-(4j^2+6j)+(j-1)^2-1=\frac{3i^2+11i}{2}-(3j^2+8j)+2.$$

If $\Delta(a_0,\dots,a_{2j-1})$ is the discriminant of $x^{2j+1}+a_{2j-1}x^{2j-1}+\cdots+a_0$ and $\Delta(b_0,\dots,b_{2i})$ is the discriminant of $z^{2i+2}+b_{2i}z^{2i}+\cdots+b_0$, then the locus of singular deformations of C is cut out formally locally by the polynomial $n\Delta(a_0,\dots,a_{2j-1})\Delta(b_0,\dots,b_{2i})$. Using (15) and (17), and applying Corollary 3.12, we obtain

$$\chi_\delta(A_{2i+1}/A_{2j})=1+(8j^2+4j)-(4i^2+6i+2).$$

3.2.4. *$A_{2i+1/2j+1}$-atoms.* The computation of χ_K and χ_δ for the $A_{2i+1/2j+1}$-atom follows from the computation in 3.2.2 using the strategy of 3.2.3. For the reader's convenience we list these characters here:

$$\begin{aligned} \chi_K(A_{2i+1}/A_{2j+1}) &= (2i^2+5i+2)-\frac{(i^2-i)}{2}-(2j^2+5j+2)+\frac{(j^2-j)}{2}-2,\\ \chi_\delta(A_{2i+1}/A_{2j+1}) &= (4j^2+6j+2)-(4i^2+6i+2)+1. \end{aligned}$$

3.3. **Non-reduced curves: A case study of ribbons.** The character theory is particularly suited to the study of non-reduced Gorenstein schemes. Here, we treat the case of *canonical rational ribbons* with $\mathbb{G}_m$-action. Ribbons occur as certain flat limits (in the Hilbert scheme) of canonically embedded smooth curves degenerating to hyperelliptic curves [Fon93]. Our exposition is self-contained but we refer the reader to [BE95] for a more systematic study of ribbons.

A rational ribbon is a scheme obtained by gluing together two copies of a doubled affine line. Precisely, let $U_1 = \operatorname{Spec} \Bbbk[x, \varepsilon]/(\varepsilon^2)$ and $U_2 = \operatorname{Spec} \Bbbk[y, \eta]/(\eta^2)$, and let $(U_1)_x$ and $(U_2)_y$ be the corresponding open affine subschemes. Then a rational ribbon of genus g is given by a gluing isomorphism $\varphi\colon (U_1)_y \to (U_2)_x$ defined by

$$x \mapsto y^{-1} - y^{-2} f(y)\eta,$$
$$\varepsilon \mapsto y^{-g-1}\eta,$$

where $f(y) = f_1 y^{-1} + \cdots + f_{g-2} y^{-(g-2)} \in \dfrac{\Bbbk[y, y^{-1}]}{\Bbbk[y] + y^{-g+1}\Bbbk[y^{-1}]}$; see [BE95, p. 733] for more details.

We focus here on non-split ribbons that admit $\mathbb{G}_m$-action. There are $g-2$ such ribbons, each given by $f(y) = y^{-\ell}$, for $\ell \in \{1, \dots, g-2\}$. Denote the ribbon corresponding to $f(y) = y^{-\ell}$ by C_ℓ. Then the $\mathbb{G}_m$-action on C_ℓ is given by $\lambda \cdot (x, y, \varepsilon, \eta) = (\lambda x, \lambda^{-1} y, \lambda^{g-\ell}\varepsilon, \lambda^{-\ell-1}\eta)$.

By adjunction, the sections of ω_{C_ℓ} over U_1 are identified with restrictions to U_1 of 2-forms $f(x,\varepsilon)\frac{dx\wedge d\varepsilon}{\varepsilon^2}$ on $\operatorname{Spec}\Bbbk[x,\varepsilon]$, and the sections of ω_{C_ℓ} over U_2 are identified with restrictions to U_2 of 2-forms $f(y,\eta)\frac{dy\wedge d\eta}{\eta^2}$ on $\operatorname{Spec}\Bbbk[y,\eta]$. With this in mind, we can write down g linearly independent global sections of ω_{C_ℓ}:

For $k = 0, \dots, g-\ell-2$, take

$$\omega_k = x^k \frac{dx \wedge d\varepsilon}{\varepsilon^2} = -(y^{g-1-k} + (g-\ell-k-1)y^{g-\ell-k-2}\eta)\frac{dy\wedge d\eta}{\eta^2},$$

for $k = g-\ell-1, \dots, g-1$, take

$$\omega_k = (x^k + (\ell+k+1-g)x^{\ell+k-1}\varepsilon)\frac{dx\wedge d\varepsilon}{\varepsilon^2} = -y^{g-1-k}\frac{dy\wedge d\eta}{\eta^2}.$$

It follows that $\{\omega_i\}_{i=0}^{g-1}$ form the basis of $\mathrm{H}^0(C_\ell, \omega_{C_\ell})$ diagonalizing the $\mathbb{G}_m$-action. Note that we recover the second part of [BE95, Theorem 5.1], namely the identification of the sections of $\mathrm{H}^0(C_\ell, \omega_{C_\ell})$ with functions

$$1, y, y^2, \dots, y^\ell, y^{\ell+1} + \eta, y^{\ell+2} + 2y\eta, \dots, y^{g-1} + (g-\ell-1)y^{g-\ell-2}\eta,$$

under a trivialization of ω_{C_ℓ} on U_2.

We now proceed with character computations. Under the $\mathbb{G}_m$-action above, we have that

$$\lambda \cdot \omega_k = \lambda^{k-g+\ell+1}\omega_k.$$

Summing up the weights of the $\mathbb{G}_m$-action on the basis $\{\omega_i\}_{i=0}^{g-1}$, we obtain the character of λ_1:

$$\chi_1(C_\ell) = \sum_{k=0}^{g-1}(k-g+\ell+1) = g(\ell+1-g) + g(g-1)/2 = g\left(\ell - \frac{g-1}{2}\right).$$

It remains to compute the weights of the $\mathbb{G}_m$-action on $\mathrm{H}^0(C_\ell, \omega_{C_\ell}^2)$ and the corresponding character $\chi_2(C_\ell)$. Since $h^0(C_\ell, \omega_{C_\ell}^2) = 3g-3$, it suffices to exhibit $3g-3$ linearly independent $\mathbb{G}_m$-semi-invariant sections. One such choice is presented by

$$1, y, y^2, \dots, y^{2\ell}, y^{2\ell+1} + y^\ell \eta, \dots, y^{\ell+g-1} + (g-\ell-1)y^{g-2}\eta,$$
$$y^{\ell+g} + (g-\ell)y^{g-1}, \dots, y^{2g-2} + (2g-2\ell-2)y^{2g-\ell-3}, \eta, y\eta, \dots, y^{g-3}\eta.$$

In particular, taking into account that the weight of $\left(\dfrac{dy \wedge d\eta}{\eta^2}\right)^2$ is 2ℓ, we see that the weights of the $\mathbb{G}_m$-action on $\mathrm{H}^0(C_\ell, \omega_{C_\ell}^2)$ are

$$2\ell, 2\ell-1, \dots, 2\ell-(2g-2),\ \ell-1, \dots, \ell-g+2.$$

Summing up these weights, we obtain the character of λ_2:

$$\begin{aligned}\chi_2(C_\ell) &= 2(2g-1)\ell - (g-1)(2g-1) + (g-2)\ell - (g-2)(g-1)/2 \\ &= (5g-4)(\ell - \frac{g-1}{2}) = (5g-4)\chi_1(C_\ell).\end{aligned}$$

Using $\delta = 13\lambda_1 - \lambda_2$, we deduce that the character of δ is $\chi_\delta(C_\ell) = (8g+4)(\ell - \frac{g-1}{2})$. In particular, if $\ell \neq (g-1)/2$, then all three characters $\chi_1(C_\ell), \chi_2(C_\ell), \chi_\delta(C_\ell)$ are non-zero, and we have

$$\frac{\chi_\delta(C_\ell)}{\chi_1(C_\ell)} = \frac{8g+4}{g}.$$

Remark 3.14. Generalizing the computations above, Anand Deopurkar computed characters of Gorenstein n-ribbons[6] with $\mathbb{G}_m$-action and verified that always

$$\frac{\chi_\delta}{\chi_1} = \frac{12(2g+n-1)n}{n^2+(4g-3)n+2-2g}.$$

This recovers the ratio $\frac{8g+4}{g}$ for 2-ribbons, gives the ratio $\frac{36(g+1)}{5g+1}$ for 3-ribbons (see Corollary 4.5 and the subsequent discussion for the significance of this slope), and more generally gives the same ratio χ_δ/χ_1 as that of the toric singularity $y^n = x^{2g/(n-1)+1}$ (note that the arithmetic genus of an n-ribbon always satisfies $n-1 \mid 2g$).

3.4. **Outline of the log MMP for $\overline{\mathrm{M}}_g$.** We now summarize the outcome of our computations. We begin by collecting the character and α-invariant computations for all ADE, toric planar, and Gorenstein unibranch singularities in Table 1. The characters of all ADE atoms are computed in 3.1.2 – 3.1.6, the characters of the unibranch Gorenstein singularities are computed in 3.1.7, and the characters of ribbons are computed in Section 3.3. Note that our predictions for when type A singularities arise agree with the computations of Hyeon, who uses different heuristics [Hye10]. The characters of dangling singularities in Table 2 are obtained from the characters in Table 1 using the comparison results of Corollaries 3.3 and 3.6. Finally, the characters of $A_{i/j}$-atoms are computed in Sections 3.2.3 and 3.2.4.

Because the polarization on the GIT quotient of the Chow variety of genus g canonical curves is proportional to $K + \frac{3g+8}{8g+4}\delta$ (cf. Proposition 5.1 and Remark 1.1), it is an interesting question as to which singularities have α-invariant greater than $3/8$. As the table shows, all ADE singularities as well as $y^3 = x^6$ and $y^3 = x^7$ have this property. We do not know any other examples. Note that by Remark 3.7 any such example would necessarily be non-unibranch.

[6] An n-ribbon is a non-reduced scheme supported on $\mathbb{P}^1$ and locally isomorphic to $U \times \operatorname{Spec} \Bbbk[\varepsilon]/(\varepsilon^n)$, where $U \subset \mathbb{P}^1$ is affine.

Singularity type	λ_1	λ_2	δ	α-invariant	slope $\frac{\delta}{\lambda_1}$
$A_{2k}: y^2 - x^{2k+1}$	k^2	$5k^2 - 4k + 1$	$8k^2 + 4k - 1$	$\frac{3k^2+8k-2}{8k^2+4k-1}$	$\frac{8k^2+4k-1}{k^2}$
$A_{2k+1}: y^2 - x^{2k+2}$	$\frac{k^2+k}{2}$	$\frac{5k^2+k}{2}$	$4k^2 + 6k$	$\frac{3k+11}{8k+12}$	$\frac{8k+12}{k+1}$
$D_{2k+1}: x(y^2 - x^{2k-1})$	k^2	$5k^2 - 2k$	$8k^2 + 2k$	$\frac{3k+4}{8k+2}$	$\frac{8k+2}{k}$
$D_{2k+2}: x(y^2 - x^{2k})$	$\frac{k^2+k}{2}$	$\frac{5k^2+3k}{2}$	$4k^2 + 5k$	$\frac{3k+7}{8k+10}$	$\frac{8k+10}{k+1}$
$E_6: y^3 - x^4$	8	33	71	38/71	71/8
$E_7: y(y^2 - x^3)$	7	31	60	29/60	60/7
$E_8: y^3 - x^5$	14	63	119	8/17	17/2
$y^3 - x^6$	7	34	57	23/57	57/7
$y^3 - x^7$	31	152	251	99/251	251/31
$y^3 - x^8$	42	211	335	124/335	335/42
$T_{p,q}: x^p - y^q$	See Corollary 4.4 for character values				
monomial unibranch with gaps $\{b_1, \dots, b_g\}$	$\sum b_i$	$(2g-1)^2 + \sum b_i$	$12 \sum b_i - (2g-1)^2$	$\frac{11 \sum b_i - 2(2g-1)^2}{12 \sum b_i - (2g-1)^2}$	$12 - \frac{(2g-1)^2}{\sum b_i}$
Ribbon C_ℓ	$g\left(\ell - \frac{g-1}{2}\right)$	$(5g-4)(\ell - \frac{g-1}{2})$	$(8g+4)(\ell - \frac{g-1}{2})$	$\frac{3g+8}{8g+4}$	$8 + \frac{4}{g}$

TABLE 1. Character values of some Gorenstein singular curves.

Dangling type	λ_1	λ_2	δ	α-invariant	slope $\frac{\delta}{\lambda_1}$
$A_{2k}^{\{1\}}: y^2 - x^{2k+1}$	k^2	$5k^2 - 4k$	$8k^2 + 4k$	$\frac{3k+8}{8k+4}$	$\frac{8k+4}{k}$
$A_{2k+1}^{\{1,2\}}: y^2 - x^{2k+2}$	$\frac{k^2+k}{2}$	$\frac{5k^2+k-4}{2}$	$4k^2 + 6k + 2$	$\frac{3k+8}{8k+4}$	$\frac{8k+4}{k}$
$A_{2k+1}^{\{1\}}: y^2 - x^{2k+2}$	$\frac{k^2+k}{2}$	$\frac{5k^2+k-2}{2}$	$4k^2 + 6k + 1$	$\frac{3k^2+11k+4}{8k^2+12k+2}$	$\frac{8k^2+12k+2}{k^2+k}$
$D_{2k+1}^{\{1\}}: x(y^2 - x^{2k-1})$	k^2	$5k^2 - 2k - 1$	$8k^2 + 2k + 1$	$\frac{3k^2+4k+2}{8k^2+2k+1}$	$\frac{8k^2+2k+1}{k^2}$
$D_{2k+2}^{\{1,2\}}: x(y^2 - x^{2k})$	$\frac{k^2+k}{2}$	$\frac{5k^2+3k-4}{2}$	$4k^2 + 5k + 2$	$\frac{3k^2+7k+8}{8k^2+10k+4}$	$\frac{8k^2+10k+4}{k^2+k}$
$D_{2k+2}^{\{1\}}: x(y^2 - x^{2k})$	$\frac{k^2+k}{2}$	$\frac{5k^2+3k-2}{2}$	$4k^2 + 5k + 1$	$\frac{3k^2+7k+4}{8k^2+10k+2}$	$\frac{8k^2+10k+2}{k^2+k}$
$E_6^{\{1\}}: y^3 - x^4$	8	32	72	5/9	9
$E_7^{\{1\}}: y(y^2 - x^3)$	7	30	61	31/61	61/7
$E_8^{\{1\}}: y^3 - x^5$	14	62	120	29/60	60/7
$A_{i/j}$-atoms	λ_1		δ		
$A_{2i+1/2j}$	$j^2 - (\frac{i^2+i}{2})$		$8j^2 + 4j - 4i^2 - 6i - 1$		
$A_{2i+1/2j+1}$	$\frac{j^2+j-i^2-i}{2}$		$4j^2 + 6j - 4i^2 - 6i + 1$		

TABLE 2. Character values for dangling ADE singularities

Problem 3.15. *Classify all Gorenstein curve singularities with α-invariant above* $3/8$.

Using heuristics provided by both intersection theory and character computations, we offer predictions in Table 3 for modular interpretations of $\overline{\mathcal{M}}_g(\alpha)$ for $\alpha \geq 5/9$. (For small g these predictions have to be modified. For example, in $\overline{\mathcal{M}}_3$ Weierstrass genus 2 tails are also elliptic tails and so $\alpha = 2/3$ is not a threshold value, as can be seen from [HL10]; similarly, in genus 4 the $A_{2/3}$-atom is dangling, so one needs to add 1 to the character of λ_2, see [CMJL12b].) We note that the first two lines of Table 3 have been verified in the work of Hassett and Hyeon [HH09, HH13], where the reader can find a precise definition of elliptic tails and bridges. For the definition and extended discussion of all other tails, bridges, and triboroughs featured in the rightmost column of Table 3 we refer the reader to [FS13a, Section 4.4].

α-invariant	Singularity type added at α	Locus removed at $\alpha - \epsilon$
9/11	A_2	elliptic tails attached nodally
7/10	A_3	elliptic bridges attached nodally
2/3	A_4	genus 2 tails attached nodally at a Weierstrass point
19/29	$A_5^{\{1\}}$	genus 2 tails attached nodally
	$A_{3/4}$	genus 2 tails attached tacnodally at a Weierstrass point
12/19	$A_{3/5}$	genus 2 tails attached tacnodally
17/28	A_5	genus 2 bridges attached nodally at conjugate points
49/83	A_6	hyperelliptic genus 3 tails attached nodally at a Weierstrass point
32/55	$A_7^{\{1\}}$	hyperelliptic genus 3 tails attached nodally
42/73	$A_{3/6}$	hyperelliptic genus 3 tails attached tacnodally at a Weierstrass point
5/9	D_4	elliptic triboroughs attached nodally
	D_5	genus 2 bridges attached nodally at a Weierstrass and free point
	$D_6^{\{1\}}$	genus 2 bridges attached nodally at two free points
	$A_{3/7}$	hyperelliptic genus 3 tails attached tacnodally
	$A_{5/7}$	hyperelliptic genus 3 tails attached oscnodally
	$A_{5/6}$	hyperelliptic genus 3 tails attached oscnodally at a Weierstrass point
	A_7	hyperelliptic genus 3 bridges attached nodally at conjugate points

TABLE 3. Predictions for the log MMP for $\alpha \geq 5/9$

Giving a complete description of $\overline{\mathcal{M}}_g(\alpha)$ is much more subtle than generally describing the singularities added and the loci removed. For instance, after removing elliptic tails (connected genus 1 subcurves which are attached nodally) at $\alpha = 9/11-\epsilon$, in each subsequent moduli stack parameterizing curves with additional singularities, one needs to redefine what is meant by an elliptic tail by specifying the allowed attaching singularities; see [HH13] and [AFSvdW13] for examples.

Remark 3.16. There are in fact other singular curves with $\mathbb{G}_m$-action whose characters allow them to appear in $\overline{\mathcal{M}}_g(\alpha)$ for $\alpha \in [5/9, 1]$. However, they can be excluded by geometric considerations. Heuristically, a necessary condition for a curve to appear in $\overline{\mathcal{M}}_g(\alpha)$ is that it is an isotrivial specialization of curves in $\overline{\mathcal{M}}_g(\alpha+\epsilon)$. For example, an $A_{5/4}$-atom has α-invariant $9/11$ but obviously is not an isotrivial specialization of any stable curve.

4. Connections to Intersection Theory

In this section, we explore a remarkable connection between the intersection theory on $\overline{\mathcal{M}}_g$ and α-invariants. To see why such a connection should exist, suppose that the modularity principle holds and consider a worse-than-nodal curve $[C] \in \overline{\mathcal{M}}_g(\alpha)$. Let $\mathcal{T}_C \subset \overline{\mathcal{M}}_g$ be the variety of all possible stable limits of the curve $[C]$, as defined in [Has00, Section 3]. Then $\mathcal{T}_C$ should lie in the exceptional locus of the rational map $\overline{\mathcal{M}}_g \dashrightarrow \overline{\mathrm{M}}_g(\alpha)$. Conversely, from the definition of $\overline{\mathrm{M}}_g(\alpha)$ given in Equation (1), one would expect the α-value at which $[C]$ first appears in $\overline{\mathcal{M}}_g(\alpha)$ to be the supremum of α-values for which $\mathcal{T}_C$ lies in the stable base locus of $K_{\overline{\mathcal{M}}_g} + \alpha\delta$. This motivates us to make the following conjecture:

Conjecture 4.1. *Let $\widehat{\mathcal{O}}_{X,p}$ be a Gorenstein curve singularity with $\mathbb{G}_m$-action. If C is an $\widehat{\mathcal{O}}_{X,p}$-atom and $\mathcal{T}_C \subset \overline{\mathcal{M}}_g$ is the variety of all possible stable limits of C, then*

$$\alpha(\widehat{\mathcal{O}}_{X,p}) = \sup \left\{ \alpha \mid \mathcal{T}_C \text{ lies in the stable base locus of } K_{\overline{\mathcal{M}}_g} + \alpha\delta \right\}.$$

In this section, we give some evidence for this conjecture by proving, under certain hypotheses, a relationship between the characters $\chi_{\mathcal{L}}(C,\eta)$ and intersection numbers on $\mathcal{T}_C$.

4.1. **Intersection numbers and characters.** To set the notation, let $C \in \mathcal{U}_g$ be an $\widehat{\mathcal{O}}_{X,p}$-atom of a Gorenstein singularity $\widehat{\mathcal{O}}_{X,p}$ with $\mathbb{G}_m$-action, with the convention that $\mathbb{G}_m$ acts with *positive* weights on the maximal ideal of $\widehat{\mathcal{O}}_{X,p}$. Then C admits an induced $\mathbb{G}_m$-action $\eta\colon \mathbb{G}_m \to \mathrm{Aut}(C)$. Assume that the versal deformation space $\mathrm{Def}(C)$ is normal and $\mathrm{Aut}(C)$ is linearly reductive. Then C admits a versal deformation space with $\mathbb{G}_m$-action, i.e. a pointed affine scheme $0 \in \mathrm{Def}(C)$ with a $\mathbb{G}_m$-action and a smooth morphism $[\mathrm{Def}(C)/\mathbb{G}_m] \to \mathcal{U}_g$ sending 0 to $[C]$; see [Alp10, Thm. 3] and also [Pin74, Prop. 2.3] for the formal case.

According to Pinkham's theory of deformations of varieties with $\mathbb{G}_m$-action [Pin74, Prop. 2.2], the space of infinitesimal deformations of C has a decomposition into weight spaces:

$$T^1(C) = \bigoplus_{\nu=-\infty}^{\infty} T^1(C)(\nu).$$

Following Pinkham [Pin74, Section (3.1)], we define $\mathrm{Def}^-(C)$ to be the closed subscheme of $\mathrm{Def}(C)$ corresponding to *negative deformations*: The tangent space to $\mathrm{Def}^-(C)$ is $\bigoplus_{\nu<0} T^1_C(\nu)$ and the maximal ideal of 0 in the coordinate ring of $\mathrm{Def}^-(C)$ is *positively* graded.

Since every curve in $[(\mathrm{Def}^-(C) - 0)/\mathbb{G}_m]$ admits an isotrivial degeneration to C, we have that $[(\mathrm{Def}^-(C)-0)/\mathbb{G}_m] \cap \overline{\mathcal{M}}_g \subset \mathcal{T}_C$. The precise relationship between intersection theory on $[\mathrm{Def}^-(C)/\mathbb{G}_m]$ and characters is given by the following observation:

Theorem 4.2. *Let B be any complete curve with a non-constant map $B \to [\mathrm{Def}^-(C)/\mathbb{G}_m]$ and let $\mathcal{L}$ be any line bundle defined on a neighborhood of $[C] \in \mathcal{U}_g$. Then*

$$\chi_{\mathcal{L}}(C,\eta) = -\frac{\mathcal{L}\cdot B}{\deg(B)},$$

where $\deg(B)$ *is the degree of* B *with respect to the natural* $\mathcal{O}(1)$ *on* $[\mathrm{Def}^-(C)/\mathbb{G}_m]$*. In particular,*

$$\chi_i(C,\eta) = -\frac{\lambda_i \cdot B}{\deg(B)}.$$

Moreover, if the discriminant divisor $\Delta \subset \mathrm{Def}^-(C)$ *is Cartier and the locus of worse-than-nodal deformations of* C *has codimension at least* 2*, then* $\chi_\delta(C,\eta) = -\dfrac{\Delta \cdot B}{\deg(B)}$*.*

Proof. We can write $\mathrm{Def}^-(C) = \mathrm{Spec}\, A$ with A a non-negatively graded $\Bbbk$-algebra. The line bundle $\mathcal{L}$ corresponds to a graded projective A-module which is free of rank 1 by [Eis95, Theorem 19.2]. It follows that $\mathcal{L} = \widetilde{A(d)} = \mathcal{O}(d)$ for some d. Therefore $\chi_{\mathcal{L}}(C,\eta) = -d$ and $\mathcal{L} \cdot B = \deg_B(\mathcal{L}) = d \deg(B)$. □

4.2. **Toric singularities.** We apply Theorem 4.2 to compute characters of toric planar singularities. Let $X \simeq \mathrm{Spec}\, \Bbbk[x,y]/(x^p - y^q)$, where $p, q \geq 2$. We set $b := \gcd(p,q)$ and $p_0 := p/b, q_0 := q/b$. The $\mathbb{G}_m$-action is given by $\eta\colon\ \lambda \cdot (x,y) = (\lambda^{q_0} x, \lambda^{p_0} y)$. The miniversal deformation of X is given by

$$x^p = y^q + \sum a_{ij} x^i y^j, \quad 0 \leq i \leq p-2,\ 0 \leq j \leq q-2.$$

Let C be an $\widehat{\mathcal{O}}_{X,0}$-atom of genus g. Then $\mathrm{Def}^-(C) \simeq \mathrm{Spec}\, \Bbbk[a_{ij} \colon qi + pj < pq]$. We note that the weighted projective stack $[(\mathrm{Def}^-(C) - 0)/\mathbb{G}_m]$ is identified with the moduli space of curves on $\mathbb{P}(q,p,1)$ defined by the weighted homogeneous equation

$$x^p = y^q + \sum a_{ij} x^i y^j z^{pq-qi-pj}, \quad 0 \leq i \leq p-2,\ 0 \leq j \leq q-2,\ qi + pj < pq. \tag{18}$$

By [Has00, Theorem 6.3], the arithmetic genus of these curves is $h = (pq - p - q - b + 2)/2$. To apply Theorem 4.2, we need to write down a complete non-constant family in $[\mathrm{Def}^-(C)/\mathbb{G}_m]$. We do this via a degeneration argument in Proposition 4.3.

Proposition 4.3. *There exists a complete curve* $E \subset [(\mathrm{Def}^-(C) - 0)/\mathbb{G}_m] \cap \overline{\mathcal{M}}_g$ *satisfying* $\deg(E) = 1$ *and*

$$\lambda_1 \cdot E = \frac{1}{12b}\left((pq-p-q)^2 + pq(pq-p-q+1) - b^2\right),$$
$$\delta \cdot E = \frac{pq}{b}(pq-p-q+1) - b.$$

Proof. Suppose F_0 is a pencil of plane curves of degree $d \gg 0$ containing a curve D with a unique singularity $x^p = y^q$ and such that the total space over F_0 is smooth. Consider a one-parameter deformation $\{F_s\}$ such that F_s is a Lefschetz pencil for $s \neq 0$ and such that the total space of the family $\{F_s\}$ has local equation $x^p = y^q + sxy + t$.

By [CML13], the simultaneous stable reduction of this family is obtained by an alteration of $\mathbb{A}^2_{s,t}$. Let $E \simeq \mathbb{P}(pq - p - q, pq)$ be the exceptional divisor of this alteration. The resulting stable curve over $[s:t] \in E$ is a nodal union of the normalization $\widetilde{D}$ of D and the curve on $\mathbb{P}(q,p,1)$ defined by

$$x^p = y^q + sxyz^{pq-p-q} + tz^{pq} = 0.$$

It is easy to verify that every member of this family is in fact an irreducible at-worst nodal curve. It follows from the description of $[(\mathrm{Def}^-(C) - 0)/\mathbb{G}_m]$ given in Equation (18) that $E \subset [(\mathrm{Def}^-(C) - 0)/\mathbb{G}_m] \cap \overline{\mathcal{M}}_g$ and $\deg(E) = 1$.

We proceed to compute the intersection numbers of E with λ_1 and δ. Let $\mathcal{X}_i$ be the total families of the pencils F_i for $i = 0, 1$. Our first goal is to compare the degrees of δ and

$\kappa = 12\lambda_1 - \delta$ on $\mathcal{X}_0$ and $\mathcal{X}_1$. Since F_1 is a Lefschetz pencil, we have $\delta(\mathcal{X}_1) = 3(d-1)^2$. To find the number of singular fibers in $\mathcal{X}_0 \smallsetminus D$, we observe that the topological Euler characteristic of D is $2 - 2g(D) - (b-1)$, where $g(D) = \binom{d-1}{2} - h - b + 1$ is the geometric genus of D. Since the topological Euler characteristics of $\mathcal{X}_0$ and $\mathcal{X}_1$ are the same, we have that

$$\delta(\mathcal{X}_0 \smallsetminus D) = \delta(\mathcal{X}_1) - (2h + b - 1) = \delta(\mathcal{X}_1) - (pq - p - q + 1).$$

Since $\mathcal{X}_0$ and $\mathcal{X}_1$ are pencils of plane curves of the same degree, we have $\kappa(\mathcal{X}_0) = \kappa(\mathcal{X}_1)$.

Next, to compare intersection numbers of F_0 and F_1, treated as curves in $\overline{\mathcal{M}}_{\binom{d-1}{2}}$, with λ_1 and δ, we need to write down a family of stable curves over each F_i. There is nothing to do in the case of F_1, since it is already a Lefschetz pencil of plane curves of degree d. In particular, we have $\lambda_1 \cdot F_1 = (\kappa(\mathcal{X}_1) + \delta(\mathcal{X}_1))/12$ by Mumford's formula. To write down a stable family over F_0, we perform the stable reduction of $\mathcal{X}_0 \to F_0$ in two steps:

(1) *Base change.* To begin, make a base change of order p_0q_0b to obtain the family $\mathcal{Y}$ with local equation $x^p - y^q = t^{p_0q_0b}$. The numerical invariants of $\mathcal{Y}$ are

$$\begin{aligned} \kappa(\mathcal{Y}) &= p_0q_0b\kappa(\mathcal{X}_0) = p_0q_0b\kappa(\mathcal{X}_1), \quad \text{and} \\ \delta(\mathcal{Y} \smallsetminus D) &= p_0q_0b\delta(\mathcal{X}_0 \smallsetminus D) = p_0q_0b\big(\delta(\mathcal{X}_1) - (pq - p - q + 1)\big). \end{aligned}$$

(2) *Weighted blow-up.* Let $\mathcal{Z}$ be the weighted blow-up of $\mathcal{Y}$, centered at $x = y = t = 0$, with *weight*$(x, y, t) = (q_0, p_0, 1)$. The central fiber of $\mathcal{Z}$ becomes $\widetilde{D} \cup T$ of the form described in [Has00, Theorem 6.3], with smooth T. The self-intersection of the tail T inside $\mathcal{Z}$ is $(-b)$. By intersecting both sides of $K_{\mathcal{Z}} = \pi^* K_{\mathcal{Y}} + aT$ with T, we find that $a = p_0 + q_0 - p_0q_0b$. It follows that

$$\kappa(\mathcal{Z}) = \kappa(\mathcal{Y}) - b(p_0q_0b - p_0 - q_0)^2 = p_0q_0b\kappa(\mathcal{X}_1) - b(p_0q_0b - p_0 - q_0)^2.$$

Since $\delta(\mathcal{Z} \smallsetminus (\widetilde{D} \cup T)) = \delta(\mathcal{Y} \smallsetminus D)$ and the central fiber $\widetilde{D} \cup T$ has b nodes, we conclude that $\delta(\mathcal{Z}) = p_0q_0b\delta(\mathcal{X}_1) - p_0q_0b(pq - p - q + 1) + b$. Summarizing, we have

$$\begin{aligned} \delta \cdot E &= p_0q_0b\delta(\mathcal{X}_1) - \delta(\mathcal{Z}) = p_0q_0b(pq - p - q + 1) - b, \\ \kappa \cdot E &= p_0q_0b\kappa(\mathcal{X}_1) - \kappa(\mathcal{Z}) = b(p_0q_0b - p_0 - q_0)^2. \end{aligned}$$

Using Mumford's formula $\lambda_1 = (\kappa + \delta)/12$, we finally obtain

$$\lambda_1 \cdot E = \frac{b}{12}\left((p_0q_0b - p_0 - q_0)^2 + p_0q_0(pq - p - q + 1) - 1\right).$$

□

Corollary 4.4. *Let $X = \operatorname{Spec} \Bbbk[x, y]/(x^p - y^q)$ and C be the corresponding $\widehat{\mathcal{O}}_{X,0}$-atom. Then under the $\mathbb{G}_m$-action $\eta:\ \lambda \cdot (x, y) = (\lambda^{-q_0}x, \lambda^{-p_0}y)$, we have*

$$\begin{aligned} \chi_1(C, \eta) &= \chi_1(\bar{X}, \eta) = \frac{1}{12b}\left((pq - p - q)^2 + pq(pq - p - q + 1) - b^2\right), \\ \chi_\delta(C, \eta) &= \frac{pq}{b}(pq - p - q + 1) - b, \quad \chi_\delta(\bar{X}, \eta) = \frac{pq}{b}(pq - p - q + 1). \end{aligned}$$

Proof. Let E be the family from Proposition 4.3. Then the characters of C are computed by Theorem 4.2. For the characters of $\bar{X}$, note that η acts with weight -1 on each of the b branches of the normalization of X. We are done by Corollaries 3.3 and 3.6. □

An amusing consequence of Theorem 4.2 is that it enables to compute slopes of special subvarieties of $\overline{\mathcal{M}}_g$ simply by using characters. For example, by considering singularities $y^3 =$

x^{3k+1} and $y^3 = x^{3k+2}$, we recover the following result of Stankova [SF00] (see also Remark 3.14).

Corollary 4.5. *For $g \equiv 0, 1 \pmod 3$, there is a complete family B of generically smooth genus g stable trigonal curves such that $(\delta \cdot B)/(\lambda_1 \cdot B) = 36(g+1)/(5g+1)$.*

Proof. Let C be the monomial unibranch singularity $y^3 = x^{3k+1}$. From Equation (18) the restriction of its miniversal deformation to $\operatorname{Def}^-(C)$ is

$$y^3 = x^{3k+1} + y(a_{2k}x^{2k} + \cdots + a_0) + (b_{3k-1}x^{3k-1} + \cdots + b_0). \tag{19}$$

It follows that $[(\operatorname{Def}^-(C) - 0)/\mathbb{G}_m] \simeq \mathcal{P}(2, 5, \dots, 6k+2, 6, 9, \dots, 9k+3)$ is the moduli space of curves defined by Equation (19) on $\mathbb{P}(3, 3k+1, 1)$. Evidently, there exists a complete family $B \to [(\operatorname{Def}^-(C) - 0)/\mathbb{G}_m]$ of at-worst nodal irreducible curves. Applying Theorem 4.2 to this family and using computations of Section 3.1.7, we find that $\lambda_1 \cdot B = \chi_1(C) = g(5g+1)/6$ and $\delta \cdot B = 13\chi_1(C) - \chi_2(C) = 6g(g+1)$. This gives slope $36(g+1)/(5g+1)$.

Considering $y^3 = x^{3k+2}$, we obtain in an analogous fashion a complete family of trigonal curves of genus $g = 3k+1$ with slope $36(g+1)/(5g+1)$. ☐

We note that in contrast to a simple construction above, an extremal family achieving slope $36(g+1)/(5g+1)$ is obtained by a laborious construction in [SF00]. However, our methods do not establish a stronger result, also due to Stankova, that $36(g+1)/(5g+1)$ is the *maximal possible* among slopes of trigonal families of genus g.

5. Connections to Geometric Invariant Theory

In this section, we show that the α-invariant can be reinterpreted in terms of the Hilbert-Mumford index in GIT, and explain how the character computations of Section 3 can be used to prove the instability of certain Hilbert points.

Recall that for every $g \geq 3$, $n \geq 1$, and $m \geq 2$, we have the Hilbert and Chow GIT quotients

$$\overline{\mathrm{Hilb}}_{g,n}^{m,ss} /\!\!/ \mathrm{SL}_N \quad \text{and} \quad \overline{\mathrm{Chow}}_{g,n}^{ss} /\!\!/ \mathrm{SL}_N$$

parameterizing, respectively, semistable m^{th} Hilbert points of n-canonically embedded genus g curves, and semistable Chow points of n-canonically embedded genus g curves, up to projectivities. Here, $N = g$ if $n = 1$, and $N = (2n-1)(g-1)$ if $n > 1$. We refer the reader to [Mor09] and [HH13, Sect. 2] for more details on these GIT quotients.

Proposition 5.1. *Let C be a Gorenstein n-canonically embedded genus g curve which admits a $\mathbb{G}_m$-action $\eta\colon \mathbb{G}_m \to \operatorname{Aut}(C)$. Consider the induced one-parameter subgroup $\tilde{\eta}\colon \mathbb{G}_m \to \mathrm{SL}_N$. Then the Hilbert-Mumford indices with respect to $\tilde{\eta}$ of the m^{th} Hilbert point of C and the Chow point of C are*

$$\mu^{\overline{\mathrm{Hilb}}_{g,n}^{m}}([C], \tilde{\eta}) = \begin{cases} \chi_1 + (m-1)\Big[\big((4g+2)m - g + 1\big)\chi_1 - \dfrac{gm}{2}\chi_\delta\Big], & \text{if } n = 1, \\ (m-1)(g-1)\Big[(6mn^2 - 2mn - 2n + 1)\chi_1 - \dfrac{mn^2}{2}\chi_\delta\Big], & \text{if } n > 1, \end{cases}$$

and, respectively,

$$\mu^{\overline{\mathrm{Chow}}_{g,n}}([C], \tilde{\eta}) = \begin{cases} (4g+2)\chi_1 - \dfrac{g}{2}\chi_\delta\ , & \text{if } n = 1, \\ (g-1)n\Big[(6n-2)\chi_1 - \dfrac{n}{2}\chi_\delta\Big], & \text{if } n > 1. \end{cases}$$

Proof. This result follows directly from the divisor class computation of the polarizations on the GIT quotients. The case of Hilbert quotients is worked out in [Mor09, Sect. 3.5] and [HH13, Sect. 3], following Mumford's computation for Chow quotients [Mum77, Theorem 5.15]. □

The above proposition implies that if one can compute the characters of λ_1 and δ (or equivalently λ_1 and λ_2) with respect to the one-parameter subgroups of the automorphism group of a curve C, then one immediately knows the Hilbert-Mumford indices of C in all Hilbert and Chow quotients with respect to such one-parameter subgroups. This allows us to use character computations to prove instability of certain Hilbert points of curves with $\mathbb{G}_m$-action (see [Hye10, Props. 2 and 3] for related results).

As an example, let us see how to use Proposition 5.1 and the computations of Section 3.3 to prove some new results on the instability of Hilbert points of canonically embedded rational ribbons. We keep the notation of Section 3.3.

Theorem 5.2 (Hilbert stability of ribbons)**.** *Let C_ℓ be a ribbon defined by $f(y) = y^{-\ell}$ for some $\ell \in \{1, \dots, g-2\}$. Then C_ℓ admits a $\mathbb{G}_m$-action $\eta\colon\ \mathbb{G}_m \to \mathrm{Aut}(C_\ell)$ and*

1. *If $\ell \neq (g-1)/2$, then the m^{th} Hilbert point of the n-canonical embedding of C_ℓ is unstable for all $m \geq 2$ and $n \geq 1$.*
2. *If $\ell = (g-1)/2$, then the m^{th} Hilbert point of the n-canonical embedding of C_ℓ is unstable for all $m \geq 2$ and $n \geq 2$.*
3. *If $\ell = (g-1)/2$, then the m^{th} Hilbert point of the* canonical embedding *of C_ℓ is strictly semistable* with respect to η *for all $m \geq 2$.*

Proof. The ribbon C_ℓ is obtained by gluing $\operatorname{Spec} \Bbbk[x, \epsilon]/(\epsilon^2)$ and $\operatorname{Spec} \Bbbk[y, \eta]/(\eta^2)$ along open affines $\operatorname{Spec} \Bbbk[x, x^{-1}, \varepsilon]/(\varepsilon^2)$ and $\operatorname{Spec} \Bbbk[y, y^{-1}, \eta]/(\eta^2)$ by

$$x \mapsto y^{-1} - y^{-\ell-2}\eta,$$
$$\varepsilon \mapsto y^{-g-1}\eta.$$

We consider the $\mathbb{G}_m$-action on C_ℓ given by $\lambda\cdot(x, y, \varepsilon, \eta) = (\lambda x, \lambda^{-1}y, \lambda^{g-\ell}\varepsilon, \lambda^{-\ell-1}\eta)$. It induces a one-parameter subgroup $\eta\colon\ \mathbb{G}_m \to \mathrm{Aut}(C_\ell)$. From Section 3.3, the characters of C_ℓ are

$$\chi_1(C_\ell, \eta) = g\left(\ell - \frac{g-1}{2}\right), \qquad \chi_\delta(C_\ell, \eta) = (8g+4)\left(\ell - \frac{g-1}{2}\right).$$

It follows by Proposition 5.1 that the Hilbert-Mumford index with respect to η of the m^{th} Hilbert point of the canonical embedding of C_ℓ is

$$\mu^{\overline{\mathrm{Hilb}}^m_{g,1}}([C], \eta) = g(g + m - gm)\left(\ell - \frac{g-1}{2}\right).$$

In particular, it is 0 if and only if $\ell = (g-1)/2$. Similarly, we verify that the Hilbert-Mumford index $\mu^{\overline{\mathrm{Hilb}}^m_{g,n}}([C], \eta)$ of the m^{th} Hilbert point of the n-canonical embedding of C_ℓ is non-zero for all $n, m \geq 2$. This finishes the proof. □

Corollary 5.3. *If g is even, then every canonically embedded genus g ribbon with $\mathbb{G}_m$-action has unstable m^{th} Hilbert point for all $m \geq 2$.*

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