

NEW NEF DIVISORS ON $\overline{M}_{0,n}$

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ABSTRACT. We give a direct proof, valid in arbitrary characteristic, of nefness for two families of F-nef divisors on $\overline{M}_{0,n}$. The divisors we consider include all type A level one conformal block divisors as well as divisors previously not known to be nef.

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1. INTRODUCTION

We prove nefness for two families of divisors on $\overline{M}_{0,n}$ by a new method. The first family \mathcal{D}_1 consists of all type A level one conformal block divisors and has many geometric incarnations; see Section 3. We give a new proof that every divisor in \mathcal{D}_1 defines a base-point-free linear system on $\overline{M}_{0,n}$, which is an isomorphism on $M_{0,n}$.

The conformal block divisors on $\overline{M}_{0,n}$ form an important family: They are nef by work of Fakhruddin [Fak12], often span extremal rays of the (symmetric) nef cone [AGSS12, AGS10], and are related to Veronese quotients [Gia13, GG12, GJM11, GJMS12]. In particular, CB divisors account for all known regular morphisms on $\overline{M}_{0,n}$ [BGM13, Section 19.5].

As an application of our method, we construct a family \mathcal{D}_2 of nef divisors on $\overline{M}_{0,n}$, which do not lie in the cone spanned by the conformal block divisors. The geometric meaning of this family is elusive, but we speculate

that it is connected to the morphisms defined by divisors in \mathcal{D}_1 ; see Section 4.¹

Our proof is elementary in that it relies only on Keel's relations in $\text{Pic}(\overline{M}_{0,n})$ and nothing else. One advantage of this approach is that it works in positive characteristic, where semiampleness of the conformal block divisors on $\overline{M}_{0,n}$ is not generally known.

The key observations that enable our proof are:

- (1) The family \mathcal{D}_i is functorial with respect to the boundary stratification, or, equivalently, satisfies factorization in the sense of [BG12].
- (2) Every divisor $D \in \mathcal{D}_i$ is linearly equivalent to an effective combination of the boundary divisors on $\overline{M}_{0,n}$.

A standard argument implies that all divisors in \mathcal{D}_i are nef. (To prove semiampleness of divisors in \mathcal{D}_1 , we show that every $D \in \mathcal{D}_1$ is linearly equivalent to an effective combination of boundary divisors in such a way as to avoid any given point of $\overline{M}_{0,n}$.)

The above argument is at the heart of the original inductive approach to the F-conjecture for $\overline{M}_{0,n}$; see [GKM02, Question (0.13)] and the discussion surrounding it. F-nef divisors obviously satisfy factorization and the strong F-conjecture says that every F-nef divisor can be written as an effective combination of boundary divisors. However, a recent result of Pixton shows that the strong F-conjecture is false: there exists a nef divisor on $\overline{M}_{0,12}$ which is not an effective combination of the boundary divisors [Pix13].

Nevertheless, one could still hope that the strong F-conjecture holds for a restricted class of F-nef divisors; restrictions of symmetric F-nef divisors to the boundary being one example. The divisors in \mathcal{D}_i are of this form. Thus the present paper can be regarded as further evidence for the symmetric F-conjecture.

It would be interesting to know whether all conformal block divisors on $\overline{M}_{0,n}$ are effective combinations of the boundary divisors (see the discussion after Theorem 3.4 below).

1.1. Notation. The i^{th} cotangent line bundle and its divisor class on $\overline{M}_{0,n}$ is denoted ψ_i . We use the notation $[n] = \{1, \dots, n\}$ and say that a partition $I \sqcup J = [n]$ is *proper* if $|I|, |J| \geq 2$. The boundary divisor on $\overline{M}_{0,n}$ corresponding

¹We do not know a way to prove that a given divisor is not an effective combination of CB divisors (but see [Swi11] for some results in this direction). Rather, there are explicit examples of divisors in \mathcal{D}_2 for which extensive numerical experimentation has failed to uncover CB divisors whose span could contain them.

to a proper partition $I \sqcup J = [n]$ is denoted $\Delta_{I,J}$. We write $\Delta = \sum_{I,J} \Delta_{I,J}$ for the total boundary divisor; here and elsewhere the summation is taken over all proper partitions of $[n]$, unless specified otherwise.

Given a set S , we denote by $\Gamma(S)$ the complete graph on S and by $E(S)$ the set of all edges of $\Gamma(S)$. A *weighting* or a *weight function* on $\Gamma(S)$ is a function $w: E(S) \rightarrow \mathbb{Q}$. For the complete graph $\Gamma([n])$ on the set $[n]$, we write $(i - j)$ to denote the edge joining vertices i and j . Given a weight function w on $\Gamma([n])$, we make the following definitions:

- (1) The *w-flow through a vertex* $k \in [n]$ is defined to be

$$w(k) := \sum_{i \neq k} (w(k - i)).$$

- (2) The *w-flow across a partition* $I \sqcup J = [n]$ is defined to be

$$w(I | J) = \sum_{i \in I, j \in J} w(i - j).$$

A *degree function* on $\Gamma(S)$ is a function $S \rightarrow \mathbb{Z}$. Given a degree function $i \mapsto d_i$ on $\Gamma([n])$, we set $d(I) := \sum_{i \in I} d_i$ for any $I \subset [n]$. Given an integer $m \geq 2$, we say that $I \subset [n]$ is *m-divisible* if $m \mid d(I)$. We call $I \sqcup J = [n]$ an *m-partition* if $m \mid d(I)$ and $m \mid d(J)$.

We denote by $\langle a \rangle_m$ the representative in $\{0, 1, \dots, m - 1\}$ of a modulo m .

We work over an algebraically closed field of arbitrary characteristic.

2. WEIGHTED GRAPHS AND EFFECTIVE COMBINATIONS OF BOUNDARY

Every divisor on $\overline{M}_{0,n}$ can be written as

$$\sum_{i=1}^n a_i \psi_i - \sum_{I,J} b_{I,J} \Delta_{I,J}.$$

This representation is far from unique because we have the following relation in $\text{Pic}(\overline{M}_{0,n})$ for every $i \neq j$:

$$(2.1) \quad \psi_i + \psi_j = \sum_{i \in I, j \in J} \Delta_{I,J}.$$

Relations (2.1) generate the module of all relations among $\{\psi_i\}_{i=1}^n$ and $\{\Delta_{I,J}\}$; this follows, for example, from [AC98, Theorem 2.2(d)], which in turn follows from Keel's relations [Kee92]. (We note that the above representation is unique if we impose an additional condition $|I|, |J| \geq 3$; see [FG03, Lemma 2].)

We now state a simple observation that we will use repeatedly in the sequel.

Lemma 2.1 (Effectivity criterion). *Let $R = \mathbb{Z}$ or $R = \mathbb{Q}$. A divisor $D = \sum_{i=1}^n a_i \psi_i - \sum_{I,J} b_{I,J} \Delta_{I,J}$ is R -linearly equivalent to $\sum_{I,J} c_{I,J} \Delta_{I,J}$ if and only if there is an R -valued weighting of $\Gamma([n])$ such that the flow through each vertex i is a_i and the flow across each proper partition $I \sqcup J = [n]$ is $b_{I,J} + c_{I,J}$.*

In particular, D is an effective R -linear combination of the boundary divisors on $\overline{M}_{0,n}$ if and only if there exists an R -valued weighting of $\Gamma([n])$ such that the flow through each vertex i is a_i and the flow across each proper partition $I \sqcup J = [n]$ is at least $b_{I,J}$.

Proof. Suppose that for each $i \neq j$ we use the relation (2.1) $w(i-j)$ times to rewrite D as $\sum_{I,J} c_{I,J} \Delta_{I,J}$. Then in the free R -module generated by $\{\psi_i\}_{i=1}^n$ and $\{\Delta_{I,J}\}$ we have

$$\begin{aligned} \sum_{I,J} c_{I,J} \Delta_{I,J} &= D - \sum_{i \neq j} w(i-j) \left(\psi_i + \psi_j - \sum_{i \in I, j \in J} \Delta_{I,J} \right) \\ &= \sum_{i=1}^n (a_i - w(i)) \psi_i - \sum_{I,J} (b_{I,J} - w(I|J)) \Delta_{I,J}. \end{aligned}$$

The claim follows. \square

3. TYPE A LEVEL ONE CONFORMAL BLOCK DIVISORS REVISITED

In this section, we study the family \mathcal{D}_1 of type A level one conformal block divisors.

Definition 3.1. Consider n integers (d_1, \dots, d_n) and let $m \geq 2$ be an integer dividing $\sum_{i=1}^n d_i$. We define a divisor on $\overline{M}_{0,n}$ by the following formula

$$(3.1) \quad D((d_1, \dots, d_n), m) = \sum_{i=1}^n \langle d_i \rangle_m \langle m - d_i \rangle_m \psi_i - \sum_{I,J} \langle d(I) \rangle_m \langle d(J) \rangle_m \Delta_{I,J},$$

where $d(I) := \sum_{i \in I} d_i$.

The motivation for this definition comes from the following observation [Fed11, Proposition 4.8]:

Proposition 3.2. *For a weight vector $\vec{d} = (d_1, \dots, d_n)$, let m be an integer dividing $\sum_{i=1}^n d_i$. Let \mathbb{E} be the pullback to $\overline{M}_{0,n}$ of the Hodge bundle over $\overline{\mathcal{M}}_g$ via the weighted cyclic m -covering morphism $f_{\vec{d},m}$ and let \mathbb{E}_j be the eigenbundle of \mathbb{E} associated to the character j of μ_m . Then*

$$\det \mathbb{E}_j = \frac{1}{2m^2} \left[\sum_{i=1}^n \langle jd_i \rangle_m \langle m - jd_i \rangle_m \psi_i - \sum_{I,J} \langle jd(I) \rangle_m \langle jd(J) \rangle_m \Delta_{I,J} \right].$$

The divisor $D((d_1, \dots, d_n), m)$ has at least three incarnations:

- (1) It is a type A level 1 conformal block divisor; see [Fak12] for the definition and [AGSS12] for a detailed study of these divisors. More precisely, the \mathfrak{sl}_m level 1 conformal block divisor $\mathbb{D}(\mathfrak{sl}_m, 1, (d_1, \dots, d_n))$ equals to $2m^2 D((d_1, \dots, d_n), m)$ [Fed11].
- (2) It is a pullback of a natural polarization on a GIT quotient of a parameter space of n -pointed rational normal curves [Gia13, GG12].
- (3) It is a determinant of a Hodge eigenbundle as in Proposition 3.2.

Each interpretation of $D((d_1, \dots, d_n), m)$ leads to an independent proof of its nefness. The first via the theory of conformal blocks which realizes $D((d_1, \dots, d_n), m)$ as a quotient of a trivial vector bundle over $\overline{M}_{0,n}$; the second via GIT, and the third via the semipositivity of the Hodge bundle over \overline{M}_g , which in turn comes from the Hodge theory [Kol90].

We now propose a fourth proof of nefness of $D((d_1, \dots, d_n), m)$ which is independent of all of the above and is completely elementary.

Since the definition of $D((d_1, \dots, d_n), m)$ depends only on $\langle d_i \rangle_m$, replacing each d_i by $\langle d_i \rangle_m$ we can assume that $d_i \leq m - 1$. We proceed with a construction of a certain weighting on the complete graph $\Gamma([n])$.

Proposition 3.3 (Standard construction). *Suppose $1 \leq d_i \leq m - 1$ and $m \mid \sum_{i=1}^n d_i$. There exists a weighting w of $\Gamma([n])$ such that*

- (1) *For every vertex $i \in \Gamma([n])$, we have $w(i) = d_i(m - d_i)$.*
- (2) *For every proper partition $I \sqcup J = [n]$, we have $w(I \mid J) \geq \langle d(I) \rangle_m \langle d(J) \rangle_m$.*

In addition,

- (3) *Given a curve $[C] \in \overline{M}_{0,n}$, we can choose w so that $w(I \mid J) = \langle d(I) \rangle_m \langle d(J) \rangle_m$ for any $I \sqcup J = [n]$ satisfying $[C] \in \Delta_{I,J}$.*
- (4) *For any fixed proper partition $I \sqcup J = [n]$, the weighting w can be chosen so that $w(I \mid J) \geq 2m + \langle d(I) \rangle_m \langle d(J) \rangle_m$.*

Proof. Let $\sum_{i=1}^n d_i = ms$ and let $S = \{p_1, \dots, p_{ms}\}$ be a multiset of indices where each index $i \in \{1, \dots, n\}$ appears d_i times.

Choose a cyclic permutation $\sigma \in \mathfrak{S}_n$ of $[n] = \{1, \dots, n\}$ and arrange the elements of S along a circle at the vertices of a regular ms -gon so that the d_i occurrences of each i are adjacent, and the order of $\{1, \dots, n\}$ along the circle is given by σ .

Define $\{S_k\}_{k=1}^m$ to be the regular s -gons formed by the chords divisible by m . Since $d_i \leq m - 1$, this subdivision satisfies the following property:

- Each S_k contains at most one occurrence of each $i \in \{1, \dots, n\}$.

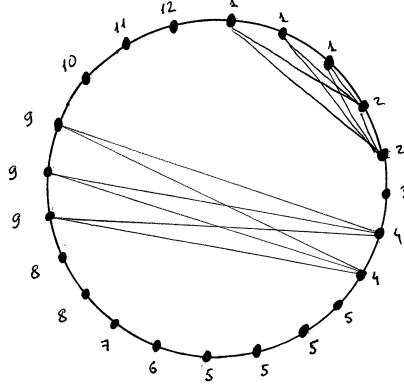


FIGURE 1. The standard construction of a weighting in the case $m = 11$ and $d_1 = 3, d_2 = 2, d_3 = 1, d_4 = 2, d_5 = 4, d_6 = d_7 = 1, d_8 = 2, d_9 = 3, d_{10} = d_{11} = d_{12} = 1$. The indices 1 and 2 are joined by 6 chords of w_1 -weight 1 and no chords of length 11. Thus $w(1 - 2) = 6$. The indices 4 and 9 are joined by 6 chords of w_1 -weight 1 and 2 chords of length 11 and w_2 -weight -11 . Thus $w(4 - 9) = -16$.

For every edge $e \in E(S)$, we define

$$w_1(e) = \begin{cases} 1 & \text{if } e \text{ joins distinct indices,} \\ 0 & \text{otherwise.} \end{cases}$$

$$w_2(e) = \begin{cases} -m & \text{if } e \in E(S_k) \text{ for some } k = 1, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

The weight function $w_1 + w_2$ on $\Gamma(S)$ induces in an obvious way a weight function w on $\Gamma([n])$. Namely, the w -weight of the edge $(i - j)$ in $\Gamma([n])$ is the sum of the $(w_1 + w_2)$ -weights of all edges in S joining the indices i and j . (Note that by construction, the $(w_1 + w_2)$ -weight of any edge in S joining two equal indices is 0).

We now compute the w -flow through each vertex $i \in [n]$. Clearly, the contribution of w_1 to $w(i)$ is $d_i(ms - d_i)$ and the contribution of w_2 to $w(i)$ is $-md_i(s - 1)$. Therefore,

$$w(i) = d_i(ms - d_i) - md_i(s - 1) = d_i(m - d_i).$$

This establishes (1). Next we show that the w -flow across each proper partition $I \sqcup J = [n]$ is at least $\langle d(I) \rangle_m \langle d(J) \rangle_m$.

Recall that $d(I) = \sum_{i \in I} d_i$. Write $d(I) = mq + r$. Let x_1, \dots, x_m be the number of indices from I occurring in each of the sets S_1, \dots, S_m . Then $x_1 + \dots + x_m = d(I) = mq + r$. Tracing through the construction we see that $w_1(I | J) = d(I)(ms - d(I))$ and $w_2(I | J) = -m \sum_{k=1}^m x_k(s - x_k)$. It

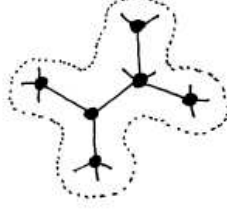


FIGURE 2. A planar realization of a dual graph of a 13-pointed rational curve C . The half-edges correspond to the marked points. To obtain a cyclic permutation σ with respect to which all partitions $I \sqcup J$ satisfying $[C] \in \Delta_{I,J}$ are balanced, choose a loop around the graph. The order in which the half-edges are encountered as one goes around the loop gives a requisite permutation.

follows that

$$(3.2) \quad w(I | J) = d(I)(ms - d(I)) - m \sum_{k=1}^m x_k(s - x_k).$$

Since $x(s - x)$ is a concave function of x , the minimum in (3.2) under the constraint $x_1 + \cdots + x_m = mq + r$ is achieved when the x_i 's differ by at most 1 from each other, i.e. when r of x_i 's are equal to $q + 1$ and $m - r$ of x_i 's are equal to q . (When this happens, we say that $I \sqcup J$ is balanced with respect to σ .) A straightforward computation now shows that for these values of x_i 's Equation (3.2) evaluates to

$$r(m - r) = \langle d(I) \rangle_m \langle d(J) \rangle_m.$$

This finishes the proof of (2).

Next, we note that if all indices from I occur contiguously in σ , then $I \sqcup J$ is balanced with respect to σ . Observe that for any $[C] \in \overline{M}_{0,n}$, there exists a cyclic permutation $\sigma \in \mathfrak{S}_n$ such that all marked points lying on one side of any node of C occur contiguously in σ . (This can be seen either by induction or by examining a planar representation of the dual graph of C as in Figure 2.) This finishes the proof of (3). By the above, the w -flow across $I \sqcup J$ is $\langle d(I) \rangle_m \langle d(J) \rangle_m$ if and only if $I \sqcup J$ is balanced with respect to σ . Otherwise, the w -flow across $I \sqcup J$ is at least $2m + \langle d(I) \rangle_m \langle d(J) \rangle_m$ (indeed, the values of $\sum_{k=1}^m x_k(s - x_k)$ have constant parity). To prove (4), it remains to observe that for any partition $I \sqcup J$, there exists σ such that $I \sqcup J$ is not balanced with respect to σ . For example, if $I = \{1, \dots, k\}$ and $J = \{k + 1, \dots, n\}$, then $\sigma = (k(k + 1))(12 \cdots n)(k(k + 1))$ works. \square

Theorem 3.4. $D((d_1, \dots, d_n), m)$ is an effective sum of boundary divisors on $\overline{M}_{0,n}$ and $|D((d_1, \dots, d_n), m)|$ is a base-point-free linear system on $\overline{M}_{0,n}$. Moreover, if $m \nmid d_1 \cdots d_n$, then $D((d_1, \dots, d_n), m)$ separates points of $M_{0,n}$.

Proof. If $m \mid d_i$, then $D((d_1, \dots, d_n), m) = f^*(D(d_1, \dots, \widehat{d}_i, \dots, d_n), m)$, where $f: \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ is the morphism forgetting the i^{th} marked point. We immediately reduce to the case when $m \nmid d_i$. In this case, the first claim follows immediately from Lemma 2.1 by applying Proposition 3.3(1) and (2). The second claim follows from Proposition 3.3(3), which says that for any $[C] \in \overline{M}_{0,n}$, we can find an effective combination of the boundary which is linearly equivalent to $D((d_1, \dots, d_n), m)$ and whose support does not contain $[C]$.

Finally, to prove that $D((d_1, \dots, d_n), m)$ separates points of $M_{0,n}$ when d_i 's are not divisible by m , it suffices to show that $D((d_1, \dots, d_n), m)$ has a positive degree on any complete irreducible curve $T \subset \overline{M}_{0,n}$ meeting the interior $M_{0,n}$. Let T be such a curve. Since $M_{0,n}$ is affine, there exists a boundary divisor $\Delta_{I,J}$ which meets T . By Proposition 3.3(4), we can rewrite $D((d_1, \dots, d_n), m)$ as an effective linear combination of boundary in such a way that the coefficient of $\Delta_{I,J}$ is positive. The claim follows. \square

It would be interesting to know if all conformal block divisors on $\overline{M}_{0,n}$ are effective combination of boundary. In view of Theorem 3.4, one possible strategy for proving this is to apply the technique of this paper to an explicit formula for the divisor classes of the conformal block divisors given by [Muk13, Proposition 4.3]. (Note that Mukhopadhyay's formula is a direct consequence of [Fak12] but presents the divisor class in a form most amenable to applying Lemma 2.1.)

4. DIVISOR FAMILY \mathcal{D}_2

In this section, we define and prove nefness for a new family of F-nef divisors on $\overline{M}_{0,n}$.

Definition 4.1. Suppose that $m \geq 3$ is an integer and $m \mid \sum_{i=1}^n d_i$. We define

$$\begin{aligned} E((d_1, \dots, d_n), m) &:= D((d_1, \dots, d_n), m) + m \left(\sum_{i:m \mid d_i} \psi_i - \sum_{I:m \mid d(I)} \Delta_{I,J} \right) \\ &= \sum_{i=1}^n \langle d_i \rangle_m \langle m - d_i \rangle_m \psi_i - \sum_{I,J} \langle d(I) \rangle_m \langle d(J) \rangle_m \Delta_{I,J} + m \left(\sum_{i:m \mid d_i} \psi_i - \sum_{I,J:m \mid d(I)} \Delta_{I,J} \right). \end{aligned}$$

The motivation for considering these divisors comes from the following observation. Suppose that $1 \leq d_i \leq m - 1$. Then $D((d_1, \dots, d_n), m)$ is a base-point-free divisor on $\overline{M}_{0,n}$. It is easy to see that the associated

morphism $f: \overline{M}_{0,n} \rightarrow X$ contracts the boundary divisor $\Delta_{I,J}$ whenever $m \mid d(I)$.² It follows that $E((d_1, \dots, d_n), m)$ is of the form $f^*A - E$, where A is a very ample divisor on X and E is an effective combination of f -exceptional divisors.

Theorem 4.2. *Suppose $m \geq 3$ and $\{d_i\}_{i=1}^n$ are such that $m \mid \sum_{i=1}^n d_i$. Then:*

- (a) $E((d_1, \dots, d_n), m)$ is an effective combination of boundary divisors on $\overline{M}_{0,n}$.
- (b) $E((d_1, \dots, d_n), m)$ is nef on $\overline{M}_{0,n}$.

Example 4.3. By taking $n = 9$, $d_1 = \dots = d_9 = 1$, and $m = 3$, we obtain the divisor $\Delta_2 + \Delta_3 + 2\Delta_4$ in $\text{Nef}(\overline{M}_{0,9})$. This divisor generates an extremal ray of the nef cone of $\overline{M}_{0,9}$ and is not known to come from the conformal block bundles; see [Swi11].

It is proved in [Fed11], that in the case $d_1 = \dots = d_n$ and $m = 3$, the divisor $E((d_1, \dots, d_n), m)$ generates an extremal ray of the symmetric nef cone of $\overline{M}_{0,n}$. We expect this to be true more generally whenever $m \geq 5$ is prime and $d_1 = \dots = d_n$.

Proof of Theorem 4.2. Replacing d_i by $\langle d_i \rangle_m$, we can assume that $0 \leq d_i \leq m - 1$. Next, we observe that if $d_i = 0$, then

$$E((d_1, \dots, d_n), m) = f^* \left(E((d_1, \dots, \widehat{d}_i, \dots, d_n), m) \right) + m\psi_i,$$

where $f: \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ is the morphism forgetting the i^{th} marked point. Since ψ_i is well-known to be an effective combination of boundary (see [FG03, Lemma 1]), we reduce to the case $1 \leq d_i \leq m - 1$. Here, Part (a) follows immediately from Lemma 2.1 once we establish the existence of a certain weighting on $\Gamma([n])$. This is achieved in Proposition 4.5 below.

We proceed to prove Part (b). Since $E((d_1, \dots, d_n), m)$ is an effective combination of the boundary divisors on $\overline{M}_{0,n}$ by Part (a), it has non-negative degree on any irreducible curve intersecting the interior $M_{0,n}$.

Next, observe that $E((d_1, \dots, d_n), m)$ satisfies factorization, that is for any boundary divisor $\Delta_{I,J} \subset \overline{M}_{0,n}$, we have

$$E((d_1, \dots, d_n), m)|_{\Delta_{I,J}} = E(\left(\{d_i\}_{i \in I}, \sum_{j \in J} d_j\right), m) \boxtimes E(\left(\{d_j\}_{j \in J}, \sum_{i \in I} d_i\right), m),$$

where we use the usual identification $\Delta_{I,J} \simeq \overline{M}_{0,I \cup p} \times \overline{M}_{0,J \cup q}$. It follows by a standard argument that $E((d_1, \dots, d_n), m)$ is nef on $\overline{M}_{0,n}$. \square

² A test family computation in [Fed11, Corollary 2.6(2)] establishing this for the case $d_1 = \dots = d_n$ applies verbatim in the more general case.

In the remainder of this section, we finish the proof of Part (a) of Theorem 4.2.

Definition 4.4. Suppose $1 \leq d_i \leq m - 1$ and $m \mid \sum_{i=1}^n d_i$. Let w be a weighting of a complete graph $\Gamma([n])$. We say that

- (P1) w satisfies (P1) with respect to a vertex $i \in \Gamma([n])$ if the w -flow through i is exactly $d_i(m - d_i)$.
- (P2) w satisfies (P2) with respect to a proper partition $I \sqcup J = [n]$ if the w -flow across $I \sqcup J$ is at least $\langle d(I) \rangle_m \langle d(J) \rangle_m$.
- (P3) w satisfies (P3) with respect to a proper m -partition $I \sqcup J = S$ if the w -flow across $I \sqcup J$ is at least m .

We say that w satisfies (P1), (P2), and (P3), if it satisfies (P1), (P2), and (P3) with respect to all vertices, all proper partitions, and all proper m -partitions, respectively.

Proposition 4.5. Let $m \geq 3$. Suppose $1 \leq d_i \leq m - 1$ and $m \mid \sum_{i=1}^n d_i$. Then there exists a weighting w of $\Gamma([n])$ satisfying (P1)-(P3).

Proof. In what follows we say that two partitions $A \sqcup B$ and $C \sqcup D$ are *transverse* if $\text{card}(A \cap C) \text{card}(A \cap D) \text{card}(B \cap C) \text{card}(B \cap D) > 0$.

Note that the Standard Construction of Proposition 3.3 produces a weighting satisfying (P1) and (P2). The most delicate part of the proof is ensuring that (P3) holds. We will construct the requisite weighting w by breaking S into smaller pieces using m -partitions of $[n]$ and averaging.

Construction 4.6. Suppose $[n] = S_1 \sqcup S_2$ is a proper m -partition. By the inductive hypothesis, there exist weightings w_1 and w_2 of $\Gamma(S_1)$ and $\Gamma(S_2)$, respectively, satisfying (P1)-(P3). These define a weighting $w_{S_1|S_2}$ of $\Gamma([n])$ in an obvious way:

$$w_{S_1|S_2}(e) = \begin{cases} w_1(e) & \text{if } e \in E(S_1), \\ w_2(e) & \text{if } e \in E(S_2), \\ 0 & \text{otherwise} \end{cases}$$

Claim 4.7.

- (1) $w_{S_1|S_2}$ satisfies (P1).
- (2) $w_{S_1|S_2}$ satisfies (P2).
- (3) The $w_{S_1|S_2}$ -flow across every m -partitions of $[n]$, with the exception of $S_1 \sqcup S_2$, is at least m . The $w_{S_1|S_2}$ -flow across $S_1 \sqcup S_2$ is 0. The $w_{S_1|S_2}$ -flow across every m -partition $I \sqcup J$ transverse to $S_1 \sqcup S_2$ is at least $2m - 2$, and is at least $2m$ if in addition $m \mid d(S_1 \cap I)$.

Proof. The first claim is clear. To prove the second claim, consider a partition $I \sqcup J = [n]$. Let $r_1 = \langle d(I \cap S_1) \rangle_m$ and $r_2 = \langle d(I \cap S_2) \rangle_m$. Without loss of generality, we can assume that $r_1 + r_2 \leq m$. Then the $w_{S_1|S_2}$ -flow between I and J is at least $r_1(m - r_1) + r_2(m - r_2) \geq (r_1 + r_2)(m - r_1 - r_2)$, as desired.

We proceed to prove the third claim. Clearly, $w_{S_1|S_2}$ -flow across $S_1 \sqcup S_2$ is 0. Let $I \sqcup J$ be another m -partition of $[n]$. Suppose first that $I \sqcup J$ is transverse to $S_1 \sqcup S_2$. The $w_{S_1|S_2}$ -flow across $I \sqcup J$ is the sum of the w_1 -flow across $(I \cap S_1) \sqcup (J \cap S_1)$ in S_1 and the w_2 -flow across $(I \cap S_2) \sqcup (J \cap S_2)$ in S_2 . Let $r_1 = \langle d(I \cap S_1) \rangle_p$. Applying the inductive hypothesis we see that if $r_1 = 0$, then the resulting $w_{S_1|S_2}$ -flow is at least $2m$. If $1 \leq r_1 \leq m - 1$, then the $w_{S_1|S_2}$ -flow across $I \sqcup J$ is at least $2r_1(m - r_1) \geq 2m - 2$.

Finally, suppose that $I \sqcup J$ is not transverse to $S_1 \sqcup S_2$. Without loss of generality, we can assume that $S_1 \subsetneq I$. Then the $w_{S_1|S_2}$ -flow across $I \sqcup J$ equals to the w_2 -flow across $(I \cap S_2) \sqcup (J \cap S_2)$, which is at least m by the inductive hypothesis because $(I \cap S_2) \sqcup (J \cap S_2)$ is a proper m -partition of S_2 . \square

Using Construction 4.6, we proceed to construct the weighting w using averaging and induction on $\sum_{i=1}^n d_i$. The case of $\sum_{i=1}^n d_i = m$ follows from Proposition 3.3 because there are no proper m -partitions. Suppose $\sum_{i=1}^n d_i = sm$, where $s \geq 2$.

Case 1: There is at most one proper m -partition $I \mid J$ of $[n]$. In this case, the claim follows by Proposition 3.3(4) because we can arrange the flow across the unique m -partition to be at least $2m$.

Case 2: There are exactly two distinct m -partitions of $[n]$. Call them $A \sqcup B$ and $C \sqcup D$. By the assumption these must be transverse (otherwise, there would exist at least 3 distinct m -partitions). By Proposition 3.3(4), there exists a weighting w_1 of $\Gamma([n])$ satisfying (P1)-(P2) and $w_1(A \mid B) \geq 2m$. Similarly, there exists a weighting w_2 of $\Gamma([n])$ satisfying (P1)-(P2) and $w_2(C \mid D) \geq 2m$. It follows that $w := (w_1 + w_2)/2$ is a weighting of $\Gamma([n])$ satisfying (P1)-(P3).

Case 3: $[n]$ is not a disjoint union of three non-trivial m -divisible subsets and there are $k \geq 3$ distinct m -partitions of $[n]$. Let $\{A_i \sqcup B_i\}_{i=1}^k$ be all m -partitions of $[n]$. Let $w_i := w_{A_i|B_i}$ as constructed in Construction 4.6. Then $w := (\sum_{i=1}^k w_i)/k$ is a weighting of $\Gamma([n])$ satisfying (P1)-(P3). Indeed, (P1)-(P2) clearly hold. Furthermore, for any m -partition $A_i \sqcup B_i$ and $j \neq i$, we have $w_j(A_i \sqcup B_i) \geq 2m - 2$ by Claim 4.7(3). It follows that

$$w(I \mid J) \geq (k - 1)(2m - 2)/k \geq m,$$

if $m \geq 4$ or if $m = 3$ and $k \geq 4$. (If $m = 3$, it is easy to see that $k \geq 4$.)

Case 3: $[n]$ is a disjoint union of four non-trivial m -divisible subsets. Let $[n] = A \sqcup B \sqcup C \sqcup D$, where $m \mid d(A), d(B), d(C), d(D)$. Using Construction 4.6, we obtain the following three weightings of $[n]$:

$$w_1 := w_{(A \cup B)|(C \cup D)}, \quad w_2 := w_{(A \cup C)|(B \cup D)}, \quad w_3 := w_{(A \cup D)|(B \cup C)}.$$

Set $w := (w_1 + w_2 + w_3)/3$. Then

$$w((A \cup B) \sqcup (C \cup D)) \geq (0 + 2m + 2m)/3 > m.$$

Hence w satisfies (P3) with respect to $(A \cup B) \sqcup (C \cup D)$. Similarly, it is easy to see from the construction that w satisfies (P3) with respect to all proper m -partitions.

Case 4: $[n]$ is a disjoint union of three non-trivial m -divisible subset but not a disjoint union of four non-trivial m -divisible subsets. The case of $m = 3$ is straightforward and so we assume $m \geq 4$ in what follows. Let $[n] = S_1 \sqcup S_2 \sqcup S_3$, where $m \mid d(S_i)$.

Suppose first that $S_i \sqcup S_j$ is the unique m -partition of $(S_i \cup S_j)$ for all $i \neq j$. Let $w_{12} := w_{(S_1 \cup S_2)|S_3}$ be the weighting of $\Gamma([n])$ from Construction 4.6. Since $S_1 \cup S_2$ has a unique m -partition, we can arrange the w_{12} -flow across $S_1 \sqcup S_2$ in $S_1 \cup S_2$ to be at least $2m$ by Proposition 3.3(4). Define analogously w_{13} and w_{23} . Then

$$w := (w_{12} + w_{13} + w_{23})/3$$

is a weighting of $\Gamma([n])$ satisfying (P1)-(P3).

Finally, without loss of generality, suppose that $S_1 \cup S_2$ has an m -partition $A \sqcup B$ distinct from $S_1 \sqcup S_2$. Then $A \sqcup B$ must be transverse to $S_1 \sqcup S_2$. The average of the following weightings of $\Gamma([n])$ constructed using Construction 4.6:

$$w_{S_1|(S_2 \cup S_3)}, \quad w_{S_2|(S_1 \cup S_3)}, \quad w_{A|(B \cup S_3)}, \quad w_{B|(A \cup S_3)}$$

satisfies (P1)-(P3). Indeed, we must verify that the flow across $S_1 \sqcup (S_2 \cup S_3)$ is at least m , the other cases being analogous or easier. By construction, the flow across $S_1 \sqcup (S_2 \cup S_3)$ is at least $(0 + m + (2m - 2) + (2m - 2))/4 \geq m$. \square

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