TYPE A LEVEL ONE CONFORMAL BLOCKS DIVISORS REVISITED

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Abstract. We prove that all type A level one conformal blocks divisors on $\overline{M}_{0,n}$ are effective sums of boundary divisors. This leads to an elementary proof of their nefness.

Denote by $\langle a\rangle_m$ the representative in $\{0,1,\dots,m-1\}$ of the residue of a modulo m.

Definition 1.1. Consider n integer numbers $(d_1,\dots,d_n)$ and let m be an integer dividing $\sum_{i=1}^n d_i$. We define a divisor on $\overline{M}_{0,n}$ by the following formula

$$D((d_1,\dots,d_n),m)=\sum_{i=1}^n \langle d_i\rangle_m\langle m-d_i\rangle_m\psi_i-\sum_{I,J}\langle d(I)\rangle_m\langle d(J)\rangle_m\Delta_{I,J}, \tag{1}$$

where $d(I)=\sum_{i\in I}d_i$ for any $I\subset\{1,\dots,n\}$.

The motivation for this definition comes from the following result [Fed11, Proposition 4.8]:

Proposition 1.2. *For a weight vector $\vec{d}=(d_1,\dots,d_n)$, let m be an integer dividing $\sum_{i=1}^n d_i$. Let $\mathbb{E}$ be the pullback to $\overline{M}_{0,n}$ of the Hodge bundle over $\overline{\mathcal{M}}_g$ via the weighted cyclic m-covering morphism $f_{\vec{d},m}$ and let $\mathbb{E}_j$ be the eigenbundle of $\mathbb{E}$ associated to the character j of μ_m. Then*

$$\lambda_{\vec{d},m}(j):=\det\mathbb{E}_j=\frac{1}{2m^2}\left[\sum_{i=1}^n\langle jd_i\rangle_m\langle m-jd_i\rangle_m\psi_i-\sum_{I,J}\langle jd(I)\rangle_m\langle jd(J)\rangle_m\Delta_{I,J}\right].$$

Remark 1.3. To obtain a closed form formula for the $\mathfrak{sl}_m$ level 1 conformal blocks divisor $\mathbb{D}(\mathfrak{sl}_m,1,(d_1,\dots,d_n))$, set $j=1$ and multiply the class of $\lambda_{\vec{d},m}(1)$ by m.

The divisor $D((d_1,\dots,d_n),m)$ has at least three incarnations:

1. It is a determinant of a Hodge eigenbundle [Fed11].
2. It is a type A level one conformal blocks divisor [Fak09, AGSS10].
3. It is a pullback of a natural polarization on a GIT quotient of a parameter space of n-pointed rational normal curves [Gia10, GG11].

Each interpretation leads to an independent proof of nefness of $D=D((d_1,\dots,d_n),m)$: the first via the semipositivity of the Hodge bundle over $\overline{\mathcal{M}}_g$, which comes from Hodge theory [Kol90]; the second via the theory of conformal blocks which realizes D as a quotient of a trivial vector bundle over $\overline{M}_{0,n}$ [Fak09]; the third via GIT.

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We now propose a fourth proof of nefness of $D((d_1,\dots,d_n),m)$ which is independent of all of the above and is completely elementary.

Proposition 1.4. *$D((d_1,\dots,d_n),m)$ is a sum of effective boundary divisors on $\overline{M}_{0,n}$.*

Proof of Proposition 1.4. We use the following elementary relation in $\operatorname{Pic}(\overline{M}_{0,n})$:

$$\psi_i+\psi_j=\sum_{i\in I, j\in J}\Delta_{I,J}. \tag{2}$$

The claim of the proposition will follow from combining a number of Relations (2). To specify which relations we use, we introduce some notation. By replacing each d_i by $\langle d_i\rangle_m$ we can assume that $d_i\le m-1$. Let $d_1+\cdots+d_n=ms$. First, we let $S=\{p_1,\dots,p_{ms}\}$ be a collection of indices where each index $i\in\{1,\dots,n\}$ appears d_i times. We subdivide S into m subsets $S_1,\dots,S_m$ of cardinality s with the property that each S_k contains precisely one occurrence of each $i\in\{1,\dots,n\}$. We note that this can be done in many ways. One way is to arrange elements of S at the vertices of a regular ms-gon so that d_i occurrences of each i are adjacent and to take S_k's to be regular s-gons formed by the chords divisible by m. Since $d_i\le m-1$, this subdivision satisfies all desired properties. We also denote by $\Gamma(S)$ the set of all unordered pairs $(p_i,p_j)\in S$ such that $p_i\neq p_j$ as elements of $\{1,\dots,n\}$. We define $\Gamma(S_k)$ for each $k=1,\dots,m$ in a similar fashion. Note that by our construction, $\Gamma(S_k)$ is simply the set of all unordered pairs of distinct elements of S_k.

Consider now the following sum

$$\Sigma=\sum_{(p_i,p_j)\in\Gamma(S)}(\psi_{p_i}+\psi_{p_j})-m\sum_{k=1}^{m}\sum_{(p_i,p_j)\in\Gamma(S_k)}(\psi_{p_i}+\psi_{p_j}) \tag{3}$$

Claim 1.5.

$$\Sigma=\sum_{i=1}^{n}\langle d_i\rangle_m\langle m-d_i\rangle_m\psi_i$$

Proof of Claim. We need to count the number of occurrences in Σ of each ψ_i, $i=1,\dots,n$. First, in the first sum of (3), each ψ_i occurs $d_i(ms-d_i)$ times. In the second sum, each ψ_i occurs $d_i(s-1)$ times. The claim follows. ☐

By rewriting each occurrence of $\psi_i+\psi_j$ in Equation (3) using Relation (2), we see that Σ is an effective combination of the boundary divisors. Namely, we have

$$\Sigma=\sum_{(p_i,p_j)\in\Gamma(S)}(\psi_{p_i}+\psi_{p_j})-m\sum_{k=1}^{m}\sum_{(p_i,p_j)\in\Gamma(S_k)}(\psi_{p_i}+\psi_{p_j}) \tag{4}$$

$$=\sum_{(p_i,p_j)\in\Gamma(S)}\left(\sum_{p_i\in I,p_j\in J}\Delta_{I,J}\right)-m\sum_{k=1}^{m}\sum_{(p_i,p_j)\in\Gamma(S_k)}\left(\sum_{p_i\in I,p_j\in J}\Delta_{I,J}\right) \tag{5}$$

To prove the proposition it remains to show that each boundary divisor $\Delta_{I,J}$ occurs with coefficient at least $\langle d(I)\rangle_m\langle d(J)\rangle_m$ in (5). Recall that $d(I)=\sum_{i\in I}d_i$. Write $d(I)=mq+r$. Let $x_1,\dots,x_m$ be the number of indices from I occurring in each of the sets $S_1, \dots, S_m$. Then $x_1 + \cdots + x_m = d(I) = mq + r$. Tracing through the construction we see that the coefficient with which $\Delta_{I,J}$ occurs in Σ is

$$d(I)(ms - d(I)) - \sum_{k=1}^{m} x_k(s - x_k). \tag{6}$$

Since $x(s-x)$ is a concave function, the minimum in (6) is achieved when all x_i differ by at most 1 from each other, i.e., when there are r x_i's equal to $q+1$ and $m-r$ equal to q. A straightforward computation now shows that for these x_i's Equation (6) evaluates to

$$r(m-r) = \langle d(I)\rangle_m \langle d(J)\rangle_m.$$

This finishes the proof. □

Corollary 1.6. *$D((d_1, \dots, d_n), m)$ is a nef divisor on $\overline{M}_{0,n}$.*

Proof. Since $D = D((d_1, \dots, d_n), m)$ is an effective combination of boundary divisors, it intersects any irreducible curve meeting the interior $M_{0,n} \subset \overline{M}_{0,n}$ non-negatively. Next, we use the fact that $D((d_1, \dots, d_n), m)$ is functorial with respect to the boundary stratification (i.e., it restricts to a tensor product (of pullbacks) of divisors of the same form on each boundary) to conclude inductively that $D((d_1, \dots, d_n), m)$ intersects all curves non-negatively. □

We would like to end with the following conjecture

Conjecture 1.7. Suppose $m \geq 3$ is prime and $m \mid \sum_{i=1}^{n} d_i$. Then the divisor class

$$\begin{aligned} & f^*_{\vec{d},m}(2m^2\lambda_{\vec{d},m}(j) - \delta_{\mathrm{irr}}) \\ &\quad = \sum_{i=1}^{n} \langle jd_i\rangle_m \langle m - jd_i\rangle_m \psi_i - \sum_{I,J} \langle jd(I)\rangle_m \langle jd(J)\rangle_m \Delta_{I,J} - m \sum_{I \,:\, m \mid d(I)} \Delta_{I,J} \end{aligned} \tag{7}$$

is nef on $\overline{M}_{0,n}$ and generates an extremal ray of $\mathrm{Nef}(\overline{M}_{0,n}/S_n)$.

Example 1.8. By taking $n = 9$, $m = 3$, and $j = 1$, we obtain the extremal ray $\Delta_2 + \Delta_3 + 2\Delta_4$ of $\mathrm{Nef}(\overline{M}_{0,9}/S_9)$ not accounted for by conformal blocks divisors.

We proved the conjecture for $m = 3, 5$ (see [Fed11]) but for $m \geq 7$, the question seems to be open.

References

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