

SEMIAMPLeness CRITERIA FOR DIVISORS ON $\overline{M}_{0,n}$

MAKSYM FEDORCHUK

ABSTRACT. We develop new characteristic-independent combinatorial criteria for semiample-ness of divisors on $\overline{M}_{0,n}$. As an application, we associate to a cyclic rational quadratic form satisfying a certain balancedness condition an infinite sequence of semiample line bundles. We also give several sufficient and effective conditions for an \mathfrak{S}_n -symmetric divisor on $\overline{M}_{0,n}$ to be semiample or nef.

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1. INTRODUCTION

Let $\overline{M}_{0,n}$ be the Deligne-Mumford moduli space. It is a smooth projective scheme over $\text{Spec } \mathbb{Z}$, which in addition to representing the functor of stable n -pointed rational curves, has misleadingly simple (but highly non-obvious) blow-up constructions out of \mathbb{P}^{n-3} and $(\mathbb{P}^1)^{n-3}$; see [Kee92, p.546, and Theorems 1, 2] and [Kap93, Chapter 4] for a precise functorial definition and two different blow-up constructions of $\overline{M}_{0,n}$.

The Picard group of $\overline{M}_{0,n}$ is well-understood (see §2.3). However, many finer questions about the birational geometry of $\overline{M}_{0,n}$ remain highly elusive. In particular, the following two open questions about divisor classes on $\overline{M}_{0,n}$ form the context for the present work:

Question 1.0.1. What are the nef divisors on $\overline{M}_{0,n}$?

Question 1.0.2. Is every nef divisor on $\overline{M}_{0,n}$ semiample?

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The famous *F-conjecture* purports to give an answer to Question 1.0.1 in terms of the F-curves, which are the 1-dimensional boundary strata in $\overline{M}_{0,n}$ (see §2.1). To state it, recall that a line bundle on $\overline{M}_{0,n}$ is *F-nef* if it intersects every F-curve non-negatively.

Conjecture 1.0.3 ([KM96, Question 1.1], [GKM02, Conjecture 0.2]). A line bundle on $\overline{M}_{0,n}$ is nef if and only if it is F-nef.

In what follows, the *F-cone* will be the set of F-nef divisor classes in $\text{NS}(\overline{M}_{0,n})$. Thus, the F-conjecture postulates that $\text{Nef}(\overline{M}_{0,n})$ equals to the finite polyhedral F-cone. An analogous conjecture has also been formulated for $\overline{M}_{g,n}$ (see [GKM02, Conjecture 0.2], [Gib09, p.1228]), but the positive genus case follows from the genus 0 case by a powerful result of Gibney, Keel, and Morrison [GKM02, Theorem 0.7] (which holds in every characteristic besides 2).

The F-conjecture for $\overline{M}_{0,n}$ has been verified only for $n \leq 7$ [Lar12], and not much is known for higher n . The situation for symmetric line bundles is slightly better. The *symmetric F-conjecture*, which states that an \mathfrak{S}_n -symmetric line bundle on $\overline{M}_{0,n}$ is nef if and only if it is F-nef, was proven by Gibney for all $n \leq 24$ [Gib09], greatly extending an earlier work of Keel and McKernan, who used Mori-theoretic techniques to prove the symmetric F-conjecture for $n \leq 11$ [KM96]. (Because of their reliance on the contraction theorem, the results of [KM96] and [Gib09] are restricted to characteristic 0.) As a corollary, Gibney establishes the F-conjecture for \overline{M}_{22} and \overline{M}_{24} , the first two moduli spaces of curves known to be of general type.

Since every nef line bundle on a Mori Dream Space is semiample by definition [HK00], Question 1.0.2 would have the answer “All” if the following question had an affirmative answer:

Question 1.0.4 ([HK00, Question 3.2]). Is $\overline{M}_{0,n}$ a Mori Dream Space?

The above question attracted a lot of attention and was sometimes formulated as a conjecture until Castravet and Tevelev proved that $\overline{M}_{0,n}$ is not a Mori Dream Space for $n \geq 134$ [CT13], a result later improved to $n \geq 13$ by González and Karu [GK14]. We note however that there is no known example of a nef, but not semiample divisor on $\overline{M}_{0,n}$; and that it is still an open problem whether $\overline{M}_{0,n}/\mathfrak{S}_n$ is a Mori Dream Space.

1.1. Approaches to semiampleness. Before we describe our results, we recall that there are roughly four methods to proving semiampleness of a divisor on $\overline{M}_{0,n}$ (the first three of which are applicable to almost any projective variety). The first method is by proving ampleness using intersection theory. The second method is by proving nefness and then applying Kawamata’s base point freeness theorem. Both of these methods are illustrated in the proof of ampleness of $\psi - \Delta$ in [KM96, Lemma 3.6]. The third method is by constructing regular morphisms from $\overline{M}_{0,n}$ to other varieties, which often have a modular interpretation (e.g., as GIT quotients), and by pulling back ample divisors from the target (see [AS12, Gia13, GJMS13]). Finally, Fakhruddin showed that in characteristic 0 there is an enormous collection of semiample divisors on $\overline{M}_{0,n}$ coming from the theory of conformal

blocks [Fak12]. In recent years, properties of conformal blocks divisors on $\overline{M}_{0,n}$ have been extensively studied in [Fak12, AGSS12, AGS14, GG12, BGM13]. It is an open question to what extent the conformal blocks divisors cover $\text{Nef}(\overline{M}_{0,n})$.

Even if we restrict ourselves to \mathfrak{S}_n -symmetric divisors on $\overline{M}_{0,n}$, of the four methods described above, only the second can be considered effective in the sense that given a symmetric divisor D , there is an algorithm for determining whether the method applies to prove that D is semiample. However, Kawamata's base point freeness theorem is restricted to characteristic 0, and still requires one to first prove that D is nef. Even then, the cone of log canonical F-nef divisors, that is, divisors that are semiample (assuming the F-conjecture) by Kawamata's base point freeness theorem, is well-known to be strictly contained inside the F-cone.

1.2. Our results. In this work, we do not take any position on the open questions discussed above. Instead, with the view toward these problems, we introduce a new combinatorial framework for proving semiamplessness (respectively, nefness) for a certain family of divisors on $\overline{M}_{0,n}$ over \mathbb{Z} (respectively, over fields of arbitrary characteristic).

Our method is elementary in that it relies only on Keel's relations in $\text{Pic}(\overline{M}_{0,n})$, but is also characteristic-independent, with most of the results holding over $\text{Spec}(\mathbb{Z})$. For \mathfrak{S}_n -symmetric divisors we give several effective numerical semiamplessness criteria. Namely, given a divisor $D = \sum_{r=2}^{\lfloor n/2 \rfloor} a_r \Delta_r$ on $\overline{M}_{0,n}$, our criteria verify semiamplessness of D by checking whether its coefficients satisfy finitely many linear inequalities (see Theorem 5.4.2) or finitely many quadratic inequalities (see Theorem 6.0.9). The running times of the algorithms implementing these criteria vary between $o(2^n)$ and $o(n^4)$.

Because of the effectivity of our criteria, and because semiample divisors that we produce often lie on the boundary of $\text{Nef}(\overline{M}_{0,n})$ and are generally not log canonical, our method is complementary to the four methods described in §1.1. There is also a non-trivial overlap. For example, the fact that the quadratic form

$$x_0^2 + x_1^2 + \cdots + x_{m-1}^2$$

is balanced in the sense of Definition 5.1.2 gives a new, characteristic-independent proof of semiamplessness of all \mathfrak{sl}_m level 1 conformal blocks divisors studied in [GG12] from the point of view of GIT and conformal blocks; see Example 5.3.10, and §8.1 for other connections to the conformal blocks divisors.

Questions about divisors on $\overline{M}_{0,n}$ tend to be much more difficult than questions about divisors on $\overline{M}_{0,n}/\mathfrak{S}_n$, or, equivalently, questions about symmetric divisors on $\overline{M}_{0,n}$. In particular, from the perspective of the F-conjecture, there are two downsides to working with the full Picard group of $\overline{M}_{0,n}$:

- (1) An absence of a natural basis for $\text{Pic}(\overline{M}_{0,n})$.
- (2) The combinatorial complexity of the F-cone inside $\text{Pic}(\overline{M}_{0,n})$.

The first issue will be discussed in §2.3. To illustrate the second issue, we recall that on $\overline{M}_{0,5}$ there are 5 extremal rays, and on $\overline{M}_{0,6}$ there are 3,190 extremal rays of the F-cone, and as far as we know the number of extremal rays of the F-cone on $\overline{M}_{0,7}$ is unknown.

In contrast, the subgroup of symmetric divisors in $\text{Pic}(\overline{M}_{0,n})$ has a natural basis given by the boundary divisors $\Delta_2, \Delta_3, \dots, \Delta_{\lfloor n/2 \rfloor}$, which incidentally generate the cone of effective divisors on $\overline{M}_{0,n}/\mathfrak{S}_n$ [KM96]. The combinatorial complexity of the symmetric F-cone also grows more slowly with n . For example, the symmetric F-cone has only 1, respectively 2, extremal rays for $n = 5$, respectively $n = 6$. For $n = 25$, the symmetric F-cone has 28,334 extremal rays, but can still be computed in under 3 minutes on a personal computer.

A major downside of working with symmetric divisors is that the restriction to a boundary of a symmetric divisor is no longer symmetric, whereas many arguments concerning nefness of divisors on $\overline{M}_{0,n}$ are inductive and make use of the restriction to the boundary. For example, one of the more involved parts of the proof of the symmetric F-conjecture for $n \leq 24$ involves verifying nefness of the *necessary boundary restrictions* [Gib09, Definition 4.6] of F-nef symmetric divisors.

A simple, but crucial idea behind this work is that we should consider families of divisors on $\overline{M}_{0,n}$ that are closed under the operation of restriction to the boundary while that are not too far from symmetric divisors. Plainly, we want to consider all the symmetric divisors and all the pullbacks of the symmetric divisors under various attaching morphisms. To formalize this idea, we find it convenient to use the language of symmetric functions on finite cyclic groups (or more generally on abelian groups) to describe the family of divisors we are working with. If G is an abelian group and $f: G \rightarrow \mathbb{Z}$ is a symmetric function, then for every n -tuple (d_1, \dots, d_n) of elements of G , we consider the divisor of the following form:

$$(1.2.1) \quad \mathfrak{D}(G, f; (d_1, \dots, d_n)) := \sum_{i=1}^n f(d_i) \psi_i - \sum_{I, J} f\left(\sum_{i \in I} d_i\right) \Delta_{I, J}.$$

To describe when the divisor (1.2.1) is F-nef, we introduce a notion of an F-nef function (Definition 3.1.11), defined by a numerical condition reminiscent of the F-nefness condition on a symmetric divisor. The correspondence between symmetric F-nef divisors and F-nef functions is not one-to-one. Instead, there are infinitely many F-nef divisors that are associated to a single F-nef function. Importantly, every symmetric F-nef divisor does arise from some F-nef function. The language of symmetric functions and accompanying notation is introduced in Section 3, where we also make explicit factorization properties of our divisors.

An important outcome of the new viewpoint is that every symmetric F-nef divisor can be put into an infinite family of (not necessarily symmetric) F-nef divisors associated to the same F-nef function. In Section 4, we introduce three effectivity notions for F-nef functions, namely effectivity, tree-effectivity, and cyclic effectivity. We show that we can prove semi-ampleness (respectively, nefness) of all the infinitely many divisors associated to a single

function by proving the function to be tree-effective (respectively, effective); see Theorem 4.3.1.

In Section 5, we reinterpret the condition for a function $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ to be cyclically effective in terms of a balancedness condition on a certain quadratic form associated to f . As a result, we obtain an “if and only if” criterion for cyclic effectivity of functions on \mathbb{Z}_m and an infinitude of semiample divisors. In Section 6, we prove a sufficient condition for a function to be tree-effective, thus obtaining another infinite collection of semiample divisors. In Section 7, we explain how to obtain new effective functions from certain cyclically effective functions. Thus we obtain new infinite families of nef divisors. Finally, in Section 8, we collect some examples and computational results, and obtain a proof of the symmetric F-conjecture in arbitrary characteristic for $n \leq 16$.

Notation and conventions: We use freely standard divisor theory of $\overline{M}_{0,n}$, as in [KM96] and [GKM02], where the reader can also find an extended discussion of F-nefness for divisors on $\overline{M}_{0,n}$. We let $[n] := \{1, \dots, n\}$ and write \mathfrak{S}_n for the symmetric group on $[n]$.

2. PRELIMINARIES

2.1. Boundary stratification of $\overline{M}_{0,n}$. In this subsection, we recall, in the language that will be used throughout this work, well-known facts about the stratification of $\overline{M}_{0,n}$ by topological type; see [AC98, Section 1], or more recent expository articles [Mor07, p.17] or [Vak03, p.650] for more details.

Dual graphs. Given $[C] \in \overline{M}_{0,n}$, the dual graph of C is the tree graph $G(C)$ such that

- Internal nodes of $G(C)$ correspond to the irreducible components of C .
- Internal edges of $G(C)$ correspond to the nodes of C .
- The leaves of $G(C)$ are labeled by $[n]$ and correspond to the marked points.
- Each internal node of $G(C)$ has valency at least 3.

We will refer to any tree on n leaves satisfying the above conditions as $[n]$ -labeled.

If T is an $[n]$ -labeled tree, then every edge e of T defines in a natural way the e -partition $[n] = I \sqcup J$ such that I and J are the sets of leaves of T lying on the two connected graph components resulting from deleting e from T . The e -partitions, where e ranges over the edges of a fixed T , will be called T -partitions.

With the above terminology, the locally closed boundary strata of $\overline{M}_{0,n}$ are in bijection with the set of all $[n]$ -labeled trees: The boundary stratum M_T associated to an $[n]$ -labeled tree T consists of all $[C] \in \overline{M}_{0,n}$ such that $G(C) \simeq T$. For example, if T_0 is the $[n]$ -labeled tree with one internal node, then $M_{T_0} = M_{0,n}$ is the interior of $\overline{M}_{0,n}$.

We note for the future that $[C] \in \Delta_{I,J}$ if and only if $I \sqcup J$ is a $G(C)$ -partition. Consequently, we have that

$$\overline{M}_T = \bigcap_{\substack{I \sqcup J = [n] \text{ is a proper} \\ T\text{-partition}}} \Delta_{I,J}.$$

Factorization of the restriction of a line bundle to the boundary. Suppose T is an $[n]$ -labeled tree. Let $V(T)$ be the set of internal nodes of T and, for every $x \in V(T)$, let $v(x)$ denote the valence of x . The closed boundary stratum \overline{M}_T has the following product description:

$$(2.1.1) \quad \overline{M}_T \simeq \prod_{x \in V(T)} \overline{M}_{0,v(x)}.$$

If L is a line bundle on $\overline{M}_{0,n}$, then the restriction of L to \overline{M}_T can be written as a product of pullbacks of line bundles on $\overline{M}_{0,v(x)}$. Namely,

$$L|_{\overline{M}_T} = \bigotimes_{x \in V(T)} \text{pr}_x^* L_x,$$

where $\text{pr}_x: \overline{M}_T \rightarrow \overline{M}_{0,v(x)}$ is the projection and $L_x \in \text{Pic}(\overline{M}_{0,v(x)})$; this follows, for example, from [GKM02, Lemma 1.1]. We will call L_x *the factors* of the restriction of L to \overline{M}_T . Importantly, each factor L_x can be realized in the following way. Let $j_x: \overline{M}_{0,v(x)} \rightarrow \overline{M}_{0,n}$ be (any) attaching morphism associated to the $v(x)$ edges emanating from x . Then $L_x = j_x^*(L)$.

Zero and one-dimensional boundary strata. Let c be the number of the internal edges of an $[n]$ -labeled tree T . Then the codimension of \overline{M}_T in $\overline{M}_{0,n}$ equals c . In particular, the 0-dimensional boundary strata of $\overline{M}_{0,n}$ are in bijection with the set of $[n]$ -labeled unrooted binary trees. We denote the set of 0-dimensional strata of $\overline{M}_{0,n}$ by $B(n)$. It is well-known (see e.g., [SS04, p.395]) that

$$\text{card}(B(n)) = (2n - 5)!!$$

The 1-dimensional closed boundary strata in $\overline{M}_{0,n}$ are called *F-curves*. The numerical classes of F-curves are in bijection with partitions

$$[n] = I \sqcup J \sqcup K \sqcup L$$

of the set of marked points into 4 non-empty subsets. The F-conjecture 1.0.3 is equivalent to the statement that the classes of F-curves generate the cone of curves of $\overline{M}_{0,n}$.

2.2. Weighted graphs. Our eventual goal is to arrive at combinatorial criteria for semi-ampleness of certain divisors on $\overline{M}_{0,n}$, which will be formally introduced in Section 3. These criteria will be stated Section 4 in the language of weightings on complete graphs, which we now introduce.

We denote by $\Gamma([n])$ the complete graph whose vertices are identified with $[n]$ and by $E([n])$ the set of all edges of $\Gamma([n])$. We write $(i \sim j)$ to denote the edge joining vertices i and j . A *weighting* or a *weight function* on $\Gamma([n])$ is a function $w: E([n]) \rightarrow \mathbb{Q}$. We write $w(i \sim j)$ to denote the weight of the edge $(i \sim j)$. Given a weight function w on $\Gamma([n])$, we make the following definitions:

- (1) The *w-flow through a vertex* $k \in [n]$ is defined to be

$$w(k) := \sum_{i \neq k} w(k \sim i).$$

(2) The w -flow across a partition $I \sqcup J = [n]$ is defined to be

$$w(I | J) := \sum_{i \in I, j \in J} w(i \sim j).$$

We will denote by P_n the regular n -gon and, by abuse of notation, the set of its vertices. With this notation, a *cyclic ordering* of $[n]$ will be a bijection $\sigma: [n] \rightarrow P_n$, modulo the rotations of P_n . We say that a partition $A \sqcup B = P_n$ is *contiguous*, if A (equivalently, B) are contiguous vertices in P_n . For a cyclic ordering σ of $[n]$, we say that a partition $I \sqcup J = [n]$ is σ -contiguous if $\sigma(I) \sqcup \sigma(J) = P_n$ is a contiguous partition.

2.3. Effective boundary criterion. The Picard group of $\overline{M}_{0,n}$ is well-understood (see [Kee92], [AC98, §2], or [GKM02, §1]). Namely, $\text{Pic}(\overline{M}_{0,n})$ is a torsion-free abelian group generated by the boundary divisors $\Delta_{I,J}$, where $I \sqcup J = [n]$ is a proper partition, and the cotangent line bundles (ψ -classes) ψ_k , where $k = 1, \dots, n$. From [AC98, Theorem 2.2(d)], which in turn follows from Keel's relations [Kee92], the group of relations among the generators is generated by the following elements, defined for every $i \neq j$:

$$(2.3.1) \quad \psi_i + \psi_j = \sum_{i \in I, j \in J} \Delta_{I,J}.$$

It follows that every line bundle on $\overline{M}_{0,n}$ can be written as

$$\sum_{k=1}^n a_k \psi_k - \sum_{I,J} b_{I,J} \Delta_{I,J},$$

but also that this representation is far from unique.

Remark 2.3.2 (Some bases of $\text{Pic}(\overline{M}_{0,n})$). It is possible to write down several bases of $\text{Pic}(\overline{M}_{0,n})$. For example,

$$\{\psi_1, \dots, \psi_n\} \cup \{\Delta_{I,J} \mid |I|, |J| \geq 3\},$$

is such a basis by [FG03, Lemma 2].

An important series of bases of $\text{Pic}(\overline{M}_{0,n})$ is given by *non-adjacent bases* discovered by Gibney and Keel. To describe a non-adjacent basis, fix a cyclic ordering $\sigma: [n] \rightarrow P_n$. Then the set

$$\mathcal{A}_\sigma := \{\Delta_{I,J} \mid I \sqcup J = [n] \text{ is not a } \sigma\text{-contiguous partition}\}$$

is a basis of $\text{Pic}(\overline{M}_{0,n})$ (see [Car09]). As we will see, the fact that \mathcal{A}_σ is a basis of $\text{Pic}(\overline{M}_{0,n})$ is equivalent to the existence of a certain weighting on $[n]$.

We now state a simple observation that we will use repeatedly in the sequel.

Lemma 2.3.3 (Effective Boundary Lemma). *Let $R = \mathbb{Z}$ or $R = \mathbb{Q}$. Consider*

$$D = \sum_{k=1}^n a_k \psi_k - \sum_{I,J} b_{I,J} \Delta_{I,J}.$$

Then $D = \sum_{I,J} c_{I,J} \Delta_{I,J}$ in $\text{Pic}(\overline{M}_{0,n}) \otimes R$ if and only if there is an R -valued weighting on $\Gamma([n])$ such that the flow through each vertex k is a_k and the flow across each proper partition $I \sqcup J = [n]$ is $b_{I,J} + c_{I,J}$.

In particular, D is represented by an effective R -linear combination of the boundary divisors on $\overline{M}_{0,n}$ if and only if there exists an R -valued weighting on $\Gamma([n])$ such that the flow through each vertex k is a_k and the flow across each proper partition $I \sqcup J = [n]$ is at least $b_{I,J}$.

Proof. Suppose that for each $i \neq j$ we use the relation (2.3.1) $w(i \sim j)$ times to rewrite D as $\sum_{I,J} c_{I,J} \Delta_{I,J}$. Then in the free R -module generated by $\{\psi_k\}_{k=1}^n$ and $\{\Delta_{I,J}\}$ we have

$$\begin{aligned} \sum_{I,J} c_{I,J} \Delta_{I,J} &= D - \sum_{i \neq j} w(i \sim j) \left(\psi_i + \psi_j - \sum_{i \in I, j \in J} \Delta_{I,J} \right) \\ &= \sum_{k=1}^n (a_k - w(k)) \psi_k - \sum_{I,J} (b_{I,J} - w(I | J)) \Delta_{I,J}. \end{aligned}$$

The claim follows. \square

3. DIVISORS ON $\overline{M}_{0,n}$ FROM SYMMETRIC FUNCTIONS ON ABELIAN GROUPS

3.1. Divisors on $\overline{M}_{0,n}$ from functions on abelian groups. Let G be an abelian group. A G - n -tuple $\vec{d} = (d_1, \dots, d_n)$ is an n -tuple of elements of G satisfying $\sum_{i=1}^n d_i = 0 \in G$. Given $I \subset [n]$, we set $\vec{d}(I) := \sum_{i \in I} d_i$.

A function $f: G \rightarrow \mathbb{Q}$ is *symmetric* if $f(a) = f(-a)$ for every $a \in G$. Given a symmetric function f on G and a G - n -tuple \vec{d} , we define the following line bundle on $\overline{M}_{0,n}$:

$$(3.1.1) \quad \mathfrak{D}(G, f; \vec{d}) = \sum_{i=1}^n f(d_i) \psi_i - \sum_{\substack{I \sqcup J = [n] \\ \text{is proper}}} f(\vec{d}(I)) \Delta_{I,J}.$$

The line bundles defined by the above formula will be called *parasyymmetric* (this name will be motivated in the following discussion that culminates in Proposition 3.1.10).

We begin by explaining why we are forced to consider the line bundles defined by (3.1.1) even if we are ultimately interested in symmetric line bundles on $\overline{M}_{0,n}$. To begin, note that (3.1.1) describes all symmetric divisors on $\overline{M}_{0,n}$. Indeed, the line bundles on $\overline{M}_{0,n}$ obtained by taking $G = \mathbb{Z}_m$, where $m \mid n$, and $d_1 = \dots = d_n = 1$, are symmetric. Conversely, any symmetric \mathbb{Q} -divisor $D = \sum_{r=2}^{\lfloor n/2 \rfloor} a_r \Delta_r$ on $\overline{M}_{0,n}$ can be written as $\mathfrak{D}(\mathbb{Z}_n, p_D; \underbrace{(1, \dots, 1)}_n)$, by

taking $p_D: \mathbb{Z}_n \rightarrow \mathbb{Q}$ defined by

$$(3.1.2) \quad \begin{aligned} p_D(0) &= p_D(1) = 0, \\ p_D(n-r) &= p_D(r) = -a_r, \text{ for } 2 \leq r \leq \lfloor n/2 \rfloor. \end{aligned}$$

Second, parasyymmetric line bundles satisfy the following property:

Lemma 3.1.3 (Functoriality with respect to the boundary stratification). *Fix a symmetric function $f: G \rightarrow \mathbb{Q}$. Then for every G - n -tuple $\vec{d} = (d_1, \dots, d_n)$ and every boundary divisor $\Delta_{I,J} \subset \overline{M}_{0,n}$, we have*

$$\mathfrak{D}(G, f; (d_1, \dots, d_n))|_{\Delta_{I,J}} = \mathfrak{D}(G, f; (\{d_i\}_{i \in I}, \sum_{j \in J} d_j)) \boxtimes \mathfrak{D}(G, f; (\sum_{i \in I} d_i, \{d_j\}_{j \in J})),$$

where we use the standard identification $\Delta_{I,J} \simeq \overline{M}_{0,I \cup p} \times \overline{M}_{0,q \cup J}$.

Consequently, for a fixed G and $f: G \rightarrow \mathbb{Q}$, every factor of $\mathfrak{D}(G, f; (d_1, \dots, d_n))$ on every closed boundary stratum of $\overline{M}_{0,n}$ is of the form $\mathfrak{D}(G, f; \vec{g})$, where \vec{g} is some G - k -tuple.

Proof. Follows from standard intersection theory on $\overline{M}_{0,n}$. Indeed, recall that for any $A \subsetneq I$ and $B = [n] - A$, we have

$$(3.1.4) \quad \mathcal{O}_{\overline{M}_{0,n}}(\Delta_{A,[n]-A})|_{\Delta_{I,J}} = \mathcal{O}_{\overline{M}_{0,I \cup p}}(\Delta_{A,I \cup p - A}) \boxtimes \mathcal{O}_{\overline{M}_{0,J \cup q}}$$

and for $B \subsetneq J$, we have

$$(3.1.5) \quad \mathcal{O}_{\overline{M}_{0,n}}(\Delta_{B,[n]-B})|_{\Delta_{I,J}} = \mathcal{O}_{\overline{M}_{0,I \cup p}} \boxtimes \mathcal{O}_{\overline{M}_{0,J \cup q}}(\Delta_{B,J \cup q - B}).$$

More importantly,

$$(3.1.6) \quad \mathcal{O}_{\overline{M}_{0,n}}(-\Delta_{I,J})|_{\Delta_{I,J}} = \psi_p \boxtimes \psi_q$$

on $\Delta_{I,J} \simeq \overline{M}_{0,I \cup p} \times \overline{M}_{0,J \cup q}$.

All other boundary divisors restrict to 0 on $\Delta_{I,J}$, and the ψ -classes on $\overline{M}_{0,n}$ restrict to the corresponding ψ -classes on $\Delta_{I,J} \simeq \overline{M}_{0,I \cup p} \times \overline{M}_{0,J \cup q}$. We conclude that by summing (3.1.4) with the coefficient $-f(\sum_{i \in A} d_i)$, (3.1.5) with the coefficient $-f(\sum_{j \in B} d_j)$, (3.1.6) with the coefficient $f(\sum_{i \in I} d_i) = f(\sum_{j \in J} d_j)$, and each ψ_i with the coefficient $f(d_i)$, we obtain precisely

$$\mathfrak{D}(G, f; (d_1, \dots, d_n))|_{\Delta_{I,J}} = \mathfrak{D}(G, f; (\{d_i\}_{i \in I}, \sum_{j \in J} d_j)) \boxtimes \mathfrak{D}(G, f; (\sum_{i \in I} d_i, \{d_j\}_{j \in J})).$$

The second part follows from the first using the discussion of §2.1. \square

Third, if G is a finite cyclic group, then every parasymmetric divisor associated to G is a pullback of some symmetric divisor on a higher-dimensional moduli space under some attaching morphism:

Lemma 3.1.7. *Suppose $G = \mathbb{Z}_m$ and $\{d_i\}_{i=1}^n$ are positive integers such that $m \mid N := \sum_{i=1}^n d_i$. Consider an attaching morphism $j: \overline{M}_{0,n} \rightarrow \overline{M}_{0,N}$ defined by attaching to the i^{th} marked point a fixed element of \overline{M}_{0,d_i+1} . Then*

$$\mathfrak{D}(G, f; (d_1, \dots, d_n)) = j^* \left(\mathfrak{D}(G, f; \underbrace{(1, \dots, 1)}_N) \right).$$

Proof. This follows immediately from Lemma 3.1.3 and the discussion of §2.1. \square

Motivated by the above two lemmas, we consider the set of all parasymmetric line bundles in $\text{Pic}(\overline{M}_{0,n})$ associated to an abelian group G , a symmetric function $f: G \rightarrow \mathbb{Z}$, and an integer $n \geq 3$:

$$(3.1.8) \quad \text{PS}(G, f, n) := \left\{ \mathcal{D}(G, f; \vec{d}) \mid \vec{d} \text{ is a } G\text{-}n\text{-tuple} \right\}.$$

We also set

$$(3.1.9) \quad \text{PS}(G, n) := \bigcup_{\substack{f: G \rightarrow \mathbb{Z} \\ \text{is symmetric}}} \text{PS}(G, f, n).$$

Since every symmetric line bundle on $\overline{M}_{0,n}$ lies in $\text{PS}(\mathbb{Z}_n, f, n)$ for some symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Z}$ and since every line bundle in $\text{PS}(\mathbb{Z}_m, f, n)$ is a pullback of a symmetric line bundle on some $\overline{M}_{0,N}$ via an attaching morphism by Lemma 3.1.7, we obtain the following result:

Proposition 3.1.10. $\bigcup_{m \geq 2, n \geq 3} \text{PS}(\mathbb{Z}_m, n)$ is the smallest collection of line bundles on $\{\overline{M}_{0,n}\}_{n \geq 3}$ that include all symmetric line bundles and is closed under pullbacks via attaching morphisms.

From now on, we will be interested exclusively in nefness and semiampleness of parasymmetric divisors. In order to recast the F-conjecture for divisors in $\text{PS}(G, n)$ in terms of symmetric functions on the abelian group G , we make the following definition:

Definition 3.1.11. Let G be an abelian group. We say that a symmetric function $f: G \rightarrow \mathbb{Q}$ is *F-nef* if for any $a, b, c \in G$ we have

$$(3.1.12) \quad f(a) + f(b) + f(c) + f(a + b + c) \geq f(a + b) + f(a + c) + f(b + c).$$

First properties. For an arbitrary $f: G \rightarrow \mathbb{Q}$, we define $d_f: G \times G \rightarrow \mathbb{Q}$ by $d_f(a, b) = f(a) + f(b) - f(a + b)$. Then (3.1.12) is equivalent to subadditivity of d_f in each variable:

$$(3.1.13) \quad d_f(a, b) + d_f(a, c) \geq d_f(a, b + c).$$

Lemma 3.1.14. Suppose $f: G \rightarrow \mathbb{Q}$ is an F-nef function on a finite abelian group. Then d_f is non-negative, and hence f is subadditive. Namely, for any $a, b \in G$, we have

$$f(a) + f(b) \geq f(a + b).$$

Proof. Take a positive integer N so that $Nb = 0$. Applying (3.1.13), we obtain

$$(N + 1)d_f(a, b) \geq d_f(a, (N + 1)b) = d_f(a, b).$$

□

Some examples of *F*-nef functions. In the following examples, we take $G = \mathbb{Z}_m$. For $k \in \mathbb{Z}$, we denote by $\langle k \rangle_m$ the residue of k in $\{0, 1, \dots, m-1\}$.

Example 3.1.15. We define the *standard function* $A_m: \mathbb{Z}_m \rightarrow \mathbb{Z}$ by

$$A_m(i) = \langle i \rangle_m \langle m - i \rangle_m.$$

Given $a, b, c \in \mathbb{Z}_m$, set $d = \langle m - a - b - c \rangle_m$. It is easy to check that

$$(3.1.16) \quad \begin{aligned} & A_m(a) + A_m(b) + A_m(c) + A_m(a+b+c) - A_m(a+b) - A_m(a+c) - A_m(b+c) \\ &= \begin{cases} 0 & \text{if } a+b+c+d = m \text{ or } a+b+c+d = 3m, \\ 2m \cdot \min\{a, b, c, d, m-a, m-b, m-c, m-d\}, & \text{otherwise.} \end{cases} \end{aligned}$$

We conclude that A_m is *F*-nef.

Example 3.1.17. The 2^{nd} *standard function* $B_m: \mathbb{Z}_m \rightarrow \mathbb{Z}$ is defined by

$$B_m(i) = A_m(i) \text{ if } i \neq 1, m-1, \text{ and } B_m(1) = B_m(m-1) = 3m-1.$$

It is easy to verify using (3.1.16) that B_m is *F*-nef for all $m \geq 8$ and for $m = 4, 6$, but is not *F*-nef for $m = 5, 7$.

Example 3.1.18. The 3^{rd} *standard function* $E_m: \mathbb{Z}_m \rightarrow \mathbb{Z}$ is defined by

$$E_m(i) = A_m(i) \text{ if } i \neq 0, \text{ and } E_m(0) = m.$$

It is easy to verify using (3.1.16) that E_m is *F*-nef for all $m \geq 3$, but is not *F*-nef for $m = 2$.

3.2. *F*-nef divisors from *F*-nef functions. Define the following subset of $\text{PS}(G, n)$ from (3.1.9):

$$(3.2.1) \quad \text{PSFN}(G, n) := \left\{ \mathfrak{D}(G, f; \vec{d}) \mid f: G \rightarrow \mathbb{Z} \text{ is an } \text{F-nef function, } \vec{d} \text{ is a } G\text{-}n\text{-tuple} \right\}.$$

We begin by observing that every divisor in $\text{PSFN}(G, n)$ is *F*-nef on $\overline{M}_{0,n}$:

Lemma 3.2.2. *Suppose $f: G \rightarrow \mathbb{Q}$ is *F*-nef. Then the divisor $\mathfrak{D}(G, f; (d_1, \dots, d_n))$ is *F*-nef on $\overline{M}_{0,n}$.*

Proof. Consider a partition $I \sqcup J \sqcup K \sqcup L = [n]$ and the corresponding *F*-curve $F_{I,J,K,L}$. Set $A = \sum_{i \in I} d_i$, $B = \sum_{j \in J} d_j$, $C = \sum_{k \in K} d_k$, $D = \sum_{\ell \in L} d_\ell$. Note that $A + B + C + D = 0 \in G$ and so $f(D) = f(A + B + C)$. Then using Lemma 3.1.3, we obtain

$$\begin{aligned} & \mathfrak{D}(G, f; (d_1, \dots, d_n)) \cdot F_{I,J,K,L} \\ &= f(A) + f(B) + f(C) + f(D) - f(A+B) - f(A+C) - f(B+C) \geq 0, \end{aligned}$$

by *F*-nefness of f . \square

As already observed, every symmetric divisor D on $\overline{M}_{0,n}$ comes from a symmetric function on \mathbb{Z}_n . More importantly, every symmetric *F*-nef divisor on $\overline{M}_{0,n}$ comes from some *F*-nef function on \mathbb{Z}_n :

Lemma 3.2.3. *Suppose $D = \sum_{r=2}^{\lfloor n/2 \rfloor} a_r \Delta_r$ is a symmetric F-nef divisor on $\overline{M}_{0,n}$. Then there exists an F-nef function $f_D: \mathbb{Z}_n \rightarrow \mathbb{Q}$ such that*

$$(3.2.4) \quad D = \mathfrak{D}(\mathbb{Z}_n, f_D; (1)_n).$$

Proof. Let $p_D: \mathbb{Z}_n \rightarrow \mathbb{Z}$ be as defined in (3.1.2). Consider

$$\lambda_{\text{F-nef}}(D) := \max \left\{ \frac{p_D(a+b) + p_D(a+c) + p_D(b+c) - p_D(a) - p_D(b) - p_D(c) - p_D(d)}{2n \cdot \min\{a, b, c, d, n-a, n-b, n-c, n-d\}} \right\},$$

where the maximum is taken over $a, b, c, d \in \{1, \dots, n-1\}$ with $a+b+c+d = 2n$. Let A_n be the standard function on \mathbb{Z}_n as defined in Example 3.1.15. Then using (3.1.16), we see that $\lambda A_n + p_D$ is F-nef if and only if $\lambda \geq \lambda_{\text{F-nef}}(D)$. We define

$$(3.2.5) \quad f_D := \lambda_{\text{F-nef}}(D) A_n + p_D.$$

Observing that $\mathfrak{D}(\mathbb{Z}_n, A_n; (1)_n) = 0$ (again from (3.1.16)), we see that

$$\mathfrak{D}(\mathbb{Z}_n, f_D; (1)_n) = \mathfrak{D}(\mathbb{Z}_n, p_D; (1)_n) = D.$$

We are done. □

Corollary 3.2.6. *$\{\text{PSFN}(\mathbb{Z}_m, n)\}_{m \geq 2, n \geq 3}$ is the smallest collection of line bundles on $\{\overline{M}_{0,n}\}_{n \geq 3}$ that include all symmetric F-nef line bundles and is closed under pullbacks via attaching morphisms.*

Definition 3.2.7. The function f_D constructed in the proof of Lemma 3.2.3 (see (3.2.5)) will be called the *associated F-nef function* of the divisor D .

4. NEFNESS AND BASE POINT FREENESS

4.1. Effective boundary divisors of three kinds. A divisor on $\overline{M}_{0,n}$ of the form $\sum b_{I,J} \Delta_{I,J}$, where $b_{I,J} \geq 0$, is called an *effective boundary*. We say that a line bundle $L \in \text{Pic}(\overline{M}_{0,n})$ is an effective boundary if the linear system $|L|$ contains an effective boundary divisor.

Definition 4.1.1. We say that $L \in \text{Pic}(\overline{M}_{0,n})$ is a *stratally effective boundary* if for every closed boundary stratum $\overline{M}_T \subset \overline{M}_{0,n}$, every factor of the restriction of L to \overline{M}_T (as defined in §2.1) is an effective boundary.

Recall that, by a standard argument called *Effective Dichotomy* [Mor07, p.39], a stratally effective boundary divisor is nef. Indeed, suppose L is a stratally effective boundary on $\overline{M}_{0,n}$. Then L intersects non-negatively all irreducible curves not lying entirely in $\Delta = \overline{M}_{0,n} \setminus M_{0,n}$, simply by virtue of being linearly equivalent to an effective divisor with the support in Δ . Furthermore, the restriction of L to every irreducible boundary divisor in Δ is nef by the same argument using the assumption that L is stratally effective boundary. Hence L is nef on $\overline{M}_{0,n}$.

Historically, proving that every F-nef line bundle on $\overline{M}_{0,n}$ is stratally effective boundary was one of the original approaches to the F-conjecture, see for example [Mor07, Question 3.31, p.100] or [GKM02, Question 0.13]. Pixton's example [Pix13] of a semiample line bundle

on $\overline{M}_{0,12}$ that is not effective boundary shows that this approach was too optimistic. One motivation behind this work is a (perhaps just as overly optimistic) hope that every F-nef parasymmetric line bundle might still be stratally effective boundary.

Definition 4.1.2. We say that a line bundle $L \in \text{Pic}(\overline{M}_{0,n})$ is *boundary semiample* if for every $[C] \in \overline{M}_{0,n}$, there exists an effective boundary \mathbb{Q} -divisor D_C linearly equivalent to L such that $[C] \notin \text{Supp}(D_C)$.

Clearly, a boundary semiample line bundle is both semiample and stratally effective boundary. With this in mind, we make the following simple observation:

Lemma 4.1.3. *Suppose L is a line bundle on $\overline{M}_{0,n}$ such that for every 0-dimensional boundary stratum $B \in B(n)$, there exists an effective boundary $D_B \in |L|$ such that $B \notin \text{Supp}(D_B)$. Then $\langle D_B \mid B \in B(n) \rangle$ is a base point free linear subsystem of $|L|$ and L is a boundary semiample line bundle.*

Proof. This is obvious once we recall that every boundary stratum in $\overline{M}_{0,n}$ contains some 0-dimensional boundary stratum in its closure: Namely, for every $[C] \in \overline{M}_{0,n}$, consider a locally closed boundary stratum M_T such that $[C] \in M_T$. Let $B \in \overline{M}_T$ be a 0-dimensional boundary stratum and $D_B \in |L|$ be an effective boundary such that $B \notin \text{Supp}(D_B)$, then $[C] \notin \text{Supp}(D_B)$ as well. The claim follows. \square

4.2. Effective functions of three kinds. The goal of this section is to use the Effective Boundary Lemma 2.3.3 to reinterpret in terms of weightings on $\Gamma([n])$ (see §2.2) what it means for parasymmetric divisors $\mathfrak{D}(G, f; \vec{d})$ defined by (3.1.1) to be

- (1) Effective boundary.
- (2) Stratally effective boundary.
- (3) Boundary semiample.

We keep the terminology and notation of Section 2.

Definition 4.2.1. Let $f: G \rightarrow \mathbb{Q}$ be a symmetric function and $\vec{d} = (d_1, \dots, d_n)$ be a G - n -tuple. We introduce the following notions of *effectivity of f with respect to \vec{d}* , listed in the order of increasing strength:

- (1) We say that f is *effective with respect to \vec{d}* if there exists a weighting $w: E([n]) \rightarrow \mathbb{Q}$ such that
 - (a) $w(i) = f(d_i)$ for all $i \in [n]$.
 - (b) $w(I \sqcup J) \geq f(\vec{d}(I))$ for all proper partitions $I \sqcup J = [n]$.
- (2) Let T be an $[n]$ -labeled tree. We say that f is *T -effective with respect to \vec{d}* if there exists a weighting $w: E([n]) \rightarrow \mathbb{Q}$ such that
 - (a) $w(i) = f(d_i)$ for all $i \in [n]$.
 - (b) $w(I \sqcup J) \geq f(\vec{d}(I))$ for all proper partitions $I \sqcup J = [n]$.
 - (c) $w(I \sqcup J) = f(\vec{d}(I))$ for all T -partitions $I \sqcup J = [n]$.

- (3) We say that f is *cyclically effective with respect to \vec{d}* if for every cyclic ordering $\sigma: [n] \rightarrow P_n$ there exists a weighting $w: E([n]) \rightarrow \mathbb{Q}$ such that
- (a) $w(i) = f(d_i)$ for all $i \in [n]$.
 - (b) $w(I | J) \geq f(\vec{d}(I))$ for all proper partitions $I \sqcup J = [n]$.
 - (c) $w(I | J) = f(\vec{d}(I))$ for all σ -contiguous partitions $I \sqcup J = [n]$.

Remark 4.2.2 (Effectivity and F-nefness). Every symmetric function is cyclically effective with respect to every G -3-tuple $\vec{d} = (d_1, d_2, d_3)$. This is easily seen by taking

$$w(i \sim j) = \frac{f(d_i) + f(d_j) - f(d_i + d_j)}{2}.$$

It is also easy to see that effectivity with respect to a G -4-tuple \vec{d} is equivalent to cyclic effectivity with respect to \vec{d} . In addition, f is effective with respect to every G -4-tuple if and only if f is F-nef (cf. Definition 3.1.11).

Lemma 4.2.3. *We have the following implications for a symmetric function $f: G \rightarrow \mathbb{Q}$ with respect to a G - n -tuple \vec{d} :*

$$\text{cyclically effective} \implies T\text{-effective for all } [n]\text{-labeled trees } T \implies \text{effective}.$$

Proof. The second implication is obvious. For the first implication, observe that any $[n]$ -labeled tree T can be embedded into the plane so that the leaves are identified with the vertices of P_n . The resulting cyclic ordering $\sigma: [n] \rightarrow P_n$ satisfies the property that every T -partition $I \sqcup J$ of $[n]$ is σ -contiguous. \square

Lemma 4.2.4. *Let $f: G \rightarrow \mathbb{Q}$ be a symmetric function and $\vec{d} = (d_1, \dots, d_n)$ be a G - n -tuple. Consider the divisor $\mathfrak{D}(G, f; \vec{d})$ from (3.1.1).*

- (1) *If f is effective with respect to \vec{d} , then $\mathfrak{D}(G, f; \vec{d})$ is an effective boundary.*
- (2) *Let T be an $[n]$ -labeled tree. Suppose f is T -effective with respect to \vec{d} . Then there is an effective boundary \mathbb{Q} -divisor D on $\overline{M}_{0,n}$ such that $D \sim_{\mathbb{Q}} \mathfrak{D}(G, f; \vec{d})$ and $\text{Supp}(D)$ does not contain the generic point of $M_T \subset \overline{M}_{0,n}$.*

Proof. This follows immediately from the Effective Boundary Lemma 2.3.3 and the description of M_T from §2.1. \square

The previous lemma motivates the following definitions:

Definition 4.2.5. We say that a symmetric function $f: G \rightarrow \mathbb{Q}$ is *effective* if, for all $n \geq 3$, f is effective with respect to every G - n -tuple \vec{d} .

Definition 4.2.6. We say that a symmetric function $f: G \rightarrow \mathbb{Q}$ is *tree-effective* if, for all $n \geq 3$, and for every $[n]$ -labeled tree T , the function f is T -effective with respect to every G - n -tuple \vec{d} .

Remark 4.2.7. For tree-effectivity, it suffices to consider only $[n]$ -labeled unrooted binary trees, because every other $[n]$ -labeled tree is obtained from some unrooted binary tree by a series of edge contractions.

Definition 4.2.8. We say that a symmetric function $f: G \rightarrow \mathbb{Q}$ is *cyclically effective* if, for all $n \geq 3$, f is cyclically effective with respect to every G - n -tuple \vec{d} .

Remark 4.2.9. It follows from Remark 4.2.2 and Lemma 4.2.3 that for a symmetric function $f: G \rightarrow \mathbb{Q}$, we have the following implications

$$\text{cyclically effective} \implies \text{tree-effective} \implies \text{effective} \implies \text{F-nef}.$$

4.3. Semiampleness and nefness of divisors from effectivity of functions. A single tree-effective (resp., effective) function gives rise to an infinitude of semiample (resp., nef) divisors, obtained by increasing n and varying the choice of a G - n -tuple:

Theorem 4.3.1. *Suppose $f: G \rightarrow \mathbb{Q}$ is tree-effective (respectively, effective). Then for any G - n -tuple $\vec{d} = (d_1, \dots, d_n)$, the line bundle $\mathfrak{D}(G, f; (d_1, \dots, d_n))$ is boundary semiample (respectively, stratally effective boundary, hence nef) on $\overline{M}_{0,n}$.*

Proof. Suppose $f: G \rightarrow \mathbb{Q}$ is tree-effective. Let $\vec{d} = (d_1, \dots, d_n)$ be a G - n -tuple. Consider a 0-dimensional boundary stratum B of $\overline{M}_{0,n}$ and let T be the associated $[n]$ -labeled unrooted binary tree. By assumption, f is T -effective with respect to \vec{d} . It follows by Lemma 4.2.4(2) that there is a \mathbb{Q} -divisor D on $\overline{M}_{0,n}$ such that $D \sim_{\mathbb{Q}} \mathfrak{D}(G, f; \vec{d})$ and $\text{Supp}(D)$ does not contain B . We conclude that $\mathfrak{D}(G, f; \vec{d})$ is boundary semiample by Lemma 4.1.3.

Suppose $f: G \rightarrow \mathbb{Q}$ is effective. Let $\vec{d} = (d_1, \dots, d_n)$ be some G - n -tuple and set $L := \mathfrak{D}(G, f; (d_1, \dots, d_n))$. Then by Lemma 3.1.3, every factor of L on every closed boundary stratum of $\overline{M}_{0,n}$ is of the form $\mathfrak{D}(G, f; \vec{g})$ for some G - k -tuple \vec{g} . Since f is effective with respect to every such \vec{g} by assumption, we conclude that $\mathfrak{D}(G, f; \vec{g})$ is effective boundary by Lemma 4.2.4(1). \square

While Theorem 4.3.1 allows one to prove semiampness (resp., nefness) of an infinite collection of line bundles at once by proving tree-effectivity (resp., effectivity) of a single symmetric function, we are often interested in proving semiampness (resp., nefness) of a particular divisor on $\overline{M}_{0,n}$ without having to prove a more general result about the associated F-nef function. In what follows, we introduce weakened versions of effectivity for functions on finite cyclic groups that allow us to do precisely that.

Definition 4.3.2. We say that a symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Q}$ is *weakly effective* if f is effective with respect to every \mathbb{Z}_n - k -tuple \vec{d} , where d_1, \dots, d_k are non-negative integers satisfying $d_1 + \dots + d_k = n$.

Definition 4.3.3. We say that a symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Q}$ is *weakly tree-effective* if f is T -effective with respect to every \mathbb{Z}_n - k -tuple \vec{d} for every $[k]$ -labeled tree T , where d_1, \dots, d_k are non-negative integers satisfying $d_1 + \dots + d_k = n$.

Definition 4.3.4. We say that a symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Q}$ is *weakly cyclically effective* if f is cyclically effective with respect to every \mathbb{Z}_n - k -tuple \vec{d} , where d_1, \dots, d_k are non-negative integers satisfying $d_1 + \dots + d_k = n$.

Remark 4.3.5. For a symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Q}$, we have

weakly cyclically effective \implies weakly tree-effective \implies weakly effective.

Repeating the proof of Theorem 4.3.1 with minor modifications we obtain the following result:

Theorem 4.3.6. *Suppose $f: \mathbb{Z}_n \rightarrow \mathbb{Q}$ is weakly tree-effective (respectively, weakly effective). Then for any k -tuple $\vec{d} = (d_1, \dots, d_k)$ of non-negative integers satisfying $\sum_{i=1}^k d_i = n$, the line bundle $\mathcal{D}(\mathbb{Z}_n, f; (d_1, \dots, d_k))$ is boundary semiample (respectively, strattally effective boundary, hence nef) on $\overline{M}_{0,k}$.*

5. CYCLIC SEMIAMPLE DIVISORS

In this section, we reinterpret the (weak) cyclic effectivity of F-nef functions on \mathbb{Z}_m in terms of certain properties for cyclic quadratic forms in m variables, namely *balancedness* and *weak balancedness* (see Definition 5.1.2 and Proposition 5.3.1).

For a positive definite quadratic form, the balancedness conditions can be verified algorithmically. To a balanced quadratic form, we can associate an infinite series of semiample divisors on various $\overline{M}_{0,n}$. Conversely, to a symmetric divisor D on $\overline{M}_{0,n}$, we associate a cyclic quadratic form Q_D in n variables, whose coefficients depend linearly on the coefficients of D . We prove that D is base point free if Q_D is weakly balanced. Weak balancedness of Q_D is equivalent to finitely many linear inequalities on the coefficients of D and so can be verified by a direct computation. This gives an effective sufficient criterion for semiampleness (see Theorem 5.4.2).

Notation. Throughout this section, we always have $G = \mathbb{Z}_m$ for some positive integer m . As before, for $k \in \mathbb{Z}$, we denote by $\langle k \rangle_m$ the residue of k in $\{0, 1, \dots, m-1\}$. All quadratic forms have rational coefficients, unless specified otherwise. We use the shorthand $(a)_r$ for $\underbrace{a, \dots, a}_r$.

5.1. Balanced cyclic quadratic forms. A quadratic form $Q(x_0, \dots, x_{m-1}) = \sum_{0 \leq i, j \leq m-1} Q_{i,j} x_i x_j$ is *cyclic* if

$$Q(x_0, x_1, \dots, x_{m-1}) = Q(x_1, \dots, x_{m-1}, x_0).$$

Equivalently, Q is cyclic if and only if its symmetric matrix is circulant, that is

$$Q_{i,j} = q(i-j), \quad \text{where } q: \mathbb{Z}_m \rightarrow \mathbb{Q} \text{ is a symmetric function.}$$

We now introduce a series of conditions on cyclic quadratic forms in m variables:

Condition (n). *For an integer $n = mc + r$, where $r = \langle n \rangle_m$, the minimum value of Q at the integral points of the affine hyperplane*

$$\sum_{i=0}^{m-1} x_i = n$$

is achieved at the vector¹ $v_n := ((1+c)_r, (c)_{m-r})$.

Example 5.1.1. $x_0^2 + x_1^2 + \dots + x_{m-1}^2$ satisfies Condition (n) for all $n \in \mathbb{Z}$.

Definition 5.1.2 (Balanced quadratic forms). Let Q be a cyclic quadratic form in m variables.

- (a) We say that Q is *balanced* if Condition (r) holds for every $r \pmod{m}$.
- (b) We say that Q is *weakly balanced* if when restricted to $x_i \in \{0, 1\}$ Condition (r) holds for all $r = 1, \dots, m-1$.

Equivalently, Q is weakly balanced if and only if for $r = 1, \dots, m-1$, the leading principal $r \times r$ minor of Q has the minimal sum of entries among all principal $r \times r$ minors.

- (c) We say that Q is ℓ -*balanced* if when restricted to $0 \leq x_i \leq \ell$ Condition (r) holds for all $r \pmod{m}$. Thus, 1-balanced forms are weakly balanced and ∞ -balanced forms are balanced.

We make three simple observations:

Remark 5.1.3 (Finiteness of conditions). For $Q = \sum_{0 \leq i, j \leq m-1} q(i-j)x_i x_j$, we have

$$(5.1.4) \quad \begin{aligned} Q(x_0 + c, \dots, x_{m-1} + c) - Q(x_0, \dots, x_{m-1}) \\ = 2c \left(\sum_{k=0}^{m-1} q(k) \right) (x_0 + \dots + x_{m-1}) + Q(c, \dots, c). \end{aligned}$$

It follows that Condition (r) is satisfied if and only if Condition (n) is satisfied for all $n \equiv \pm r \pmod{m}$. Moreover, to show that Q is balanced, it suffices to check Condition (r) for every $r \gg 0$ but restricted to positive x_i 's only.

Remark 5.1.5 (Reduction to PSD). For each $k = 0, \dots, m-1$, Condition (0) is equivalent to positive semi-definiteness of the quadratic form

$$\tilde{Q}_k := Q(x_0, \dots, x_{k-1}, -\sum_{i \neq k} x_i, x_{k+1}, \dots, x_{m-1}).$$

Furthermore, note that $\tilde{Q} := \sum_{k=0}^{m-1} \tilde{Q}_k$ satisfies

$$(5.1.6) \quad \tilde{Q} = mQ + (mq(0) - 2 \sum_{k=0}^{m-1} q(k)) (x_0 + \dots + x_{m-1})^2.$$

We conclude that Q satisfies Condition (0) if and only if \tilde{Q} is positive semi-definite, and Q satisfies Condition (n) if and only if \tilde{Q} does.

¹We allow the minimum value to be achieved at other points; in particular, Q attains the same value at all cyclic shifts of v_n .

Remark 5.1.7 (Testing for balancedness). Suppose Q is a positive definite cyclic quadratic form. Then balancedness of Q is algorithmically verified in two steps by:

- (1) Finding all vectors v such that $Q(v) \leq \max\{Q(v_r) \mid r = 0, \dots, m-1\}$.
- (2) Testing whether the vectors found in Step (1) violate Conditions (n).

For an arbitrary quadratic form Q in m variables, one can determine whether Q is ℓ -balanced by a direct evaluation of Q at all integer points of the hypercube $[0, \ell]^m$.

We give several examples of balanced quadratic forms.

Example 5.1.8. $A(x_0, \dots, x_{m-1}) := \sum_{i=0}^{m-1} x_i^2$.

Example 5.1.9. $B(x_0, \dots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2})^2$, where $m \geq 4$ and $m \not\equiv \pm 1 \pmod{6}$.

Example 5.1.10. $C(x_0, \dots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i + x_{i+k-1})^2$, where $m = 2k$ and k is odd.

Example 5.1.11. $D(x_0, \dots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i + x_{i+k})^2$, where $m = 2k + 1$.

Note that A is obviously balanced. We prove balancedness of B in the following lemma, and leave balancedness of C and D as an exercise to the reader.

Lemma 5.1.12. *The quadratic form*

$$(5.1.13) \quad B(x_0, \dots, x_{m-1}) = \sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2})^2$$

is balanced for all $m \geq 4$ such that $m \not\equiv \pm 1 \pmod{6}$.

Proof. Clearly, B is positive semi-definite and even positive definite if $m \not\equiv 0 \pmod{6}$. Thus B satisfies Condition (0). We begin by verifying Condition (r) for $2 \leq r \leq m-2$. For any $(a_0, \dots, a_{m-1}) \in \mathbb{Z}^m$ with $\sum_{i=0}^{m-1} a_i = r$, we have

$$\begin{aligned} B(a_0, \dots, a_{m-1}) &\geq \sum_{i=0}^{m-1} |a_i - a_{i+1} + a_{i+2}| \\ &\geq \left| \sum_{i=0}^{m-1} (a_i - a_{i+1} + a_{i+2}) \right| = \left| \sum_{i=0}^{m-1} a_i \right| = r = B(v_r). \end{aligned}$$

Hence Condition (r) is satisfied for $2 \leq r \leq m-2$.

When $r = 1$, we have $B(v_1) = 3$. By the above, for any $(a_i)_{i=0}^{m-1} \in \mathbb{Z}^m$ with $\sum_{i=0}^{m-1} a_i = 1$, we have the estimate $B(a_0, \dots, a_{m-1}) \geq 1$. Since $B(a_0, \dots, a_{m-1})$ is clearly odd, we conclude that B fails to satisfy Condition (1) if and only if there is $(a_i)_{i=0}^{m-1} \in \mathbb{Z}^m$ such that $a_0 - a_1 + a_2 = \pm 1$ and $a_i - a_{i+1} + a_{i+2} = 0$ for all $i = 1, \dots, m-1$. One checks that such a vector exists if and only if $m \equiv 1 \pmod{6}$ or $m \equiv -1 \pmod{6}$, in which case the vector is either $(-1, -1, 0, 1, 1, 0, \underbrace{-1, -1, 0, 1, 1, 0, \dots}_{\text{period}})$ or $(-1, -1, 0, 1, 0, \underbrace{-1, -1, 0, 1, 1, 0, \dots}_{\text{period}})$, up to a sign and cyclic shift. We are done by Remark 5.1.3. \square

We close this section with an open problem, whose solution will have an immediate bearing on the semiample cone of $\overline{M}_{0,n}$, as we shall see in the sequel.

Question 5.1.14. For an integer m , describe the convex cone of balanced cyclic quadratic forms in m variables. Given an integral linear form $L(x_0, x_1, \dots, x_{m-1})$, determine whether the cyclic quadratic form

$$Q(L)(x_0, \dots, x_{m-1}) := \sum_{k=0}^{m-1} L^2(x_k, x_{k+1}, \dots, x_{k+m-1})$$

is balanced. For example, for which m is the following form balanced:

$$\sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2} - x_{i+3} + x_{i+4})^2 ?$$

5.2. Quadratic forms of functions on cyclic groups. For a symmetric function $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$, we define $q_f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ by

$$(5.2.1) \quad q_f(a) := \frac{1}{2}(f(a+1) + f(a-1) - 2f(a)).$$

Remark 5.2.2. Given a function f as above, the number $q_f(a)$ can be interpreted as follows. Consider the divisor $D := \mathfrak{D}(\mathbb{Z}_m, f; (1)_m)$ on $\overline{M}_{0,m}$. Then for the F-curve $F_{1,1,a-1}$ (we follow the notation of [AGSS12, §2.2.2]), we have

$$D \cdot F_{1,1,a-1} = 2q_f(a) + 2f(1) - f(2).$$

In other words, q_f is determined (up to a constant) by the intersection numbers of D with the collection of F-curves $\{F_{1,1,a-1}\}_{a=2}^{\lfloor m/2 \rfloor}$, which form a basis for $N_1(\overline{M}_{0,m}/\mathfrak{S}_m)$ by [AGSS12, Proposition 4.1].

We associate to f a cyclic quadratic form defined by

$$(5.2.3) \quad Q_f(x_0, \dots, x_{m-1}) := \sum_{0 \leq i, j \leq m-1} q_f(i-j)x_i x_j,$$

and call Q_f the associated quadratic form of f . Note that $\sum_{k=0}^{m-1} q_f(k) = 0$ and consequently by (5.1.4), we have

$$(5.2.4) \quad Q_f(x_0 + c, \dots, x_{m-1} + c) = Q_f(x_0, \dots, x_{m-1}).$$

We are interested in exploring when Q_f is balanced or weakly balanced. We begin with a simple criterion for Q_f to be weakly balanced.

Lemma 5.2.5. Q_f is weakly balanced if and only if for every $S \subset \{0, \dots, m-1\}$, we have

$$(5.2.6) \quad f(|S|) - f(0) \geq \frac{1}{2} \sum_{i, j \in S} (f(i-j-1) + f(i-j+1) - 2f(i-j)).$$

Proof. This follows immediately from Definition 5.1.2 (b) by comparing the sum of entries in the principal minor of Q_f given by S to the sum of entries of the leading principal $|S| \times |S|$ minor of Q_f . \square

Remark 5.2.7. It follows from Remark 5.2.2 that (5.2.6) is equivalent to the condition that $\mathfrak{D}(f, \mathbb{Z}_m; (1)_m)$ intersects certain explicit curve classes on $\overline{M}_{0,m}$ non-negatively. It is not clear to us whether these curve classes have any geometric significance, or whether they are effective.

Corollary 5.2.8. *Suppose Q_f is weakly balanced and $f(0) = 0$. Then*

$$f(a) + f(b) + f(c) + f(a + b + c) \geq f(a + b) + f(a + c) + f(b + c)$$

for all $a, b, c \in \{0, \dots, m-1\}$ such that $a + b + c \leq m$.

Proof. Take $S = \{0, \dots, a-1\} \cup \{a+b, \dots, a+b+c-1\}$ in Lemma 5.2.5. □

Unfortunately, we do not have a criterion for balancedness of Q_f . We note however that, by Remark 5.1.5, Q_f is balanced if and only if

$$\tilde{Q}_f = Q_f + 2m(f(1) - f(0))(x_0 + \dots + x_{m-1})^2$$

is positive semi-definite and balanced.

5.3. Cyclic effectivity of F-nef functions and balancedness of quadratic forms.

Proposition 5.3.1. *Consider a symmetric function $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ with $f(0) = 0$.*

- (1) *The function f is cyclically effective if and only if the associated quadratic form Q_f is balanced.*
- (2) *The function f is weakly cyclically effective if and only if the associated quadratic form Q_f is weakly balanced.*

Our proof of the above proposition relies on a simple construction of a weighting on $\Gamma([n])$, called *the cyclic weighting*, which we proceed to explain.

Cyclic weighting. A cyclic weighting on $\Gamma([n])$ depends on a choice of a G - n -tuple \vec{d} and a cyclic ordering of $[n]$, which we think of as a bijection $\sigma: [n] \rightarrow P_n$, where P_n is a regular n -gon. Before we construct the cyclic weighting in Lemma 5.3.4, we need to introduce a bit of terminology and notation.

Definition 5.3.2. Given an n -tuple (d_1, \dots, d_n) of residues modulo m such that $\sum_{i=1}^n d_i \equiv 0 \pmod{m}$, a partition $I \sqcup J = [n]$, and a cyclic ordering σ of $[n]$, we define the vector $\sigma_I(\vec{d}) \in \mathbb{Z}^m$ as follows:

- (1) Replace each d_i by a positive integer representative of the residue. Set $N = \sum_{i=1}^n d_i$.
- (2) Consider a regular N -gon P_N and a correspondence between $[n]$ and the vertices of P_N such that
 - (a) Each i corresponds to some d_i contiguous vertices S_i of P_N .
 - (b) $\cup_{i=1}^n S_i = P_N$.
 - (c) The order in which the S_i 's occur in P_N as one goes in the clockwise direction along P_N is given by σ .

(3) For $j = 0, \dots, m-1$, we define z_j to be the number of vertices of P_N that belong to $\cup_{i \in I} S_i$ and whose index is congruent to j modulo m . At last, we set

$$\sigma_I(\vec{d}) := (z_0, \dots, z_{m-1}).$$

Remark 5.3.3. Of course, $\sigma_I(\vec{d}) = (z_0, \dots, z_{m-1})$ is only defined up to a multiple of $(1, 1, \dots, 1)$, and up to a cyclic permutation. Note that $\sum_{i=0}^{m-1} z_i = \sum_{i \in I} d_i = \vec{d}(I) \pmod{m}$. We also have $\sigma_I(\vec{d}) \equiv -\sigma_J(\vec{d}) \pmod{m}$ for a partition $I \sqcup J = [n]$. It is worth noting that if $I \sqcup J = [n]$ is σ -contiguous, then $\sigma_I(\vec{d}) = ((c+1)_r, (c)_{m-r})$, where $\vec{d}(I) = mc + r$.

Lemma 5.3.4 (Cyclic weighting). *Suppose $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ is a symmetric function and Q_f is the associated quadratic form. Then for any \mathbb{Z}_m - n -tuple $\vec{d} = (d_1, \dots, d_n)$ and any cyclic ordering $\sigma: [n] \rightarrow P_n$, there exists a unique weighting w_σ on $\Gamma([n])$ such that for every σ -contiguous partition $I \sqcup J = [n]$ we have*

$$(5.3.5) \quad w_\sigma(I | J) = Q_f(\sigma_I(\vec{d})) = f(\vec{d}(I)) - f(0).$$

Moreover, for such w_σ , it continues to hold that $w_\sigma(I | J) = Q_f(\sigma_I(\vec{d}))$ for every partition $I \sqcup J = [n]$.

Proof. It is clear that specifying the flow across all σ -contiguous partitions determines w_σ uniquely. For example, if i and j are adjacent in P_n , then we must have

$$w_\sigma(i \sim j) = \left(w_\sigma(i) + w_\sigma(j) - w_\sigma(\{i, j\} | [n] - \{i, j\}) \right) / 2,$$

and the weights of all the remaining edges are determined similarly.

It remains to prove the existence of the requisite w_σ . To begin, as in Definition 5.3.2 (1), replace each d_i by a positive integer representative of the residue and set $N = \sum_{i=1}^n d_i$. We have $m \mid N$. Consider the regular N -gon P_N and a correspondence between $[n]$ and the vertices of P_N as in Definition 5.3.2 (2): The vertex $i \in [n]$ corresponds to d_i adjacent vertices $S_i \subset P_N$ in such a way that $\cup_{i=1}^n S_i = P_N$ and S_i 's occur in the clockwise order in P_N given by σ . Let $q_f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ be as in (5.2.1). We define the weight function \tilde{w} on P_N by

$$\tilde{w}(i \sim j) := -q_f(i - j) = \frac{1}{2}(2f(i - j) - f(i - j - 1) - f(i - j + 1)).$$

Since f is a function on \mathbb{Z}_m , \tilde{w} is invariant under rotations of P_N by the angle $2\pi/m$. In particular,

- The weight $\tilde{w}(i \sim j)$ depends only on $i - j \pmod{m}$.
- All vertices have the same \tilde{w} -flow.
- The \tilde{w} -flow across a contiguous partition $A \sqcup B$ of P_N depends only on $|A|$.

We now compute the \tilde{w} -flow through the vertex $1 \in P_N$:

$$\begin{aligned} \tilde{w}(1) &= \sum_{j=2}^n \tilde{w}(1 \sim j) = -\sum_{j=2}^n q_f(j-1) = \frac{1}{2} \sum_{j=2}^n (2f(j-1) - f(j) - f(j-2)) \\ &= f(1) - f(0) = Q_f(1, 0, \dots, 0). \end{aligned}$$

Suppose that $A \sqcup B$ is an arbitrary partition of P_N . For $k = 0, \dots, m-1$, let z_j be the number of vertices in A whose index is congruent to $j \pmod{m}$. From the definition of \tilde{w} , we see that

$$(5.3.6) \quad \begin{aligned} \tilde{w}(A | B) &= - \sum_{0 \leq i, j \leq m-1} q_f(i-j) z_i (m - z_j) \\ &= -m \left(\sum_{i=0}^{m-1} z_i \right) \left(\sum_{k=0}^{m-1} q_f(k) \right) + \sum_{0 \leq i, j \leq m-1} q_f(i-j) z_i z_j = Q_f(z_0, \dots, z_{m-1}). \end{aligned}$$

It remains to observe that the weighting \tilde{w} on P_N induces a weighting on $\Gamma([n])$ that satisfies the conditions of the lemma. Namely, we define the w_σ -weight of $(i \sim j)$ in $E([n])$ to be the sum of \tilde{w} -weights of the edges in $E(P_N)$ joining S_i and S_j :

$$w_\sigma(i \sim j) := \sum_{a \in S_i, b \in S_j} \tilde{w}(a \sim b).$$

It follows from (5.3.6) and Definition 5.3.2 that for every partition $I \sqcup J = [n]$, we have

$$w_\sigma(I | J) = Q_f(\sigma_I(\vec{d})).$$

Since $Q_f(\sigma_I(\vec{d})) = f(\vec{d}(I)) - f(0)$ if $I \sqcup J$ is σ -contiguous, we are done. \square

Remark 5.3.7. That flows across all σ -contiguous partitions determines w_σ uniquely is equivalent to the fact that the collection \mathcal{A}_σ of divisors defined in Remark 2.3.2 forms a basis of $\text{Pic}(\overline{M}_{0,n})$. This follows immediately from the Effective Boundary Lemma 2.3.3. In view of this, the main contribution of Lemma 5.3.4 is the computation of w_σ -flows across *all* partitions of $[n]$ in terms of the values of the quadratic form Q_f .

We are now ready to establish the equivalence between cyclic effectivity of $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ and the balancedness of Q_f .

Proof of Proposition 5.3.1. Suppose $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ is a symmetric function with $f(0) = 0$. Let $\vec{d} = (d_1, \dots, d_n)$ be a \mathbb{Z}_m - n -tuple, where we assume each d_i to be a positive integer. By definition, f is cyclically effective with respect to \vec{d} if and only if for every cyclic ordering $\sigma: [n] \rightarrow P_n$, the weighting w_σ constructed in Lemma 5.3.4 satisfies

$$(5.3.8) \quad w_\sigma(I | J) \geq f(\vec{d}(I))$$

for every partition $I \sqcup J = [n]$. (Note that the equality is already satisfied for all σ -contiguous partitions by the construction of w_σ .) But by Lemma 5.3.4, we have

$$w_\sigma(I | J) = Q_f(\sigma_I(\vec{d})),$$

where $\sigma_I(\vec{d}) = (z_0, \dots, z_{m-1}) \in \mathbb{Z}^m$ is a vector such that

$$z_0 + \dots + z_{m-1} = \vec{d}(I).$$

Suppose $\vec{d}(I) = mc + r$. Then

$$Q_f((1+c)_r, (c)_{m-r}) = f(r) = f(\vec{d}(I)).$$

We conclude that (5.3.8) holds for all \vec{d} if and only if

$$Q_f(z_0, \dots, z_{m-1}) \geq Q_f((1+c)_r, (c)_{m-r})$$

for all $(z_0, \dots, z_{m-1}) \in \mathbb{Z}_{>0}^m$ satisfying $\sum_{i=0}^{m-1} z_i = \vec{d}(I)$. However, this is precisely the condition for Q_f to be balanced.² This finishes the proof of the first part.

The second part follows by the same argument after minor modifications. Namely, we observe that the \mathbb{Z}_m - n -tuples \vec{d} that need to be considered when verifying weak cyclic effectivity give rise to the vectors $\sigma_I(\vec{d})$ whose every coordinate is either 0 or 1. Hence weak balancedness of Q_f is sufficient to conclude. \square

Theorem 5.3.9. *Suppose $f: \mathbb{Z}_m \rightarrow \mathbb{Z}$ is a symmetric function with $f(0) = 0$ such that the associated quadratic form Q_f is balanced. Then*

$$\mathfrak{D}(\mathbb{Z}_m, f; (d_1, \dots, d_n))$$

is boundary semiample on $\overline{M}_{0,n}$ for all \mathbb{Z}_m - n -tuples (d_1, \dots, d_n) .

Proof. We have that f is cyclically effective by Proposition 5.3.1(1). The result follows by Theorem 4.3.1. \square

Example 5.3.10 (Type A level 1 conformal blocks divisors). The quadratic form associated to the standard function A_m (see Example 3.1.15) is

$$Q_{A_m}(x_0, \dots, x_{m-1}) = m(x_0^2 + x_1^2 + \dots + x_{m-1}^2) - \left(\sum_{i=0}^{m-1} x_i \right)^2.$$

Note that Q_{A_m} is balanced because $x_0^2 + x_1^2 + \dots + x_{m-1}^2$ is balanced. Every \mathfrak{sl}_m level 1 conformal blocks divisor (see [Fak12] definition, and [Gia13, GG12] for GIT interpretation and other wonderful properties of these divisors) is a multiple of $\mathfrak{D}(\mathbb{Z}_m, A_m; (d_1, \dots, d_n))$ for some $d_1, \dots, d_n \in \mathbb{Z}_m$ such that $m \mid \sum_{i=1}^n d_i$ by [Fed11, Theorem A and Proposition 4.8]. Thus Theorem 5.3.9 gives a new proof of semiamplicity for these divisors, valid over $\text{Spec } \mathbb{Z}$.

5.4. The cyclic semiample cone of $\overline{M}_{0,n}$. The condition for a quadratic form to be balanced is quite stringent and checking whether a given quadratic form is balanced can be quite difficult. On the other hand, weak balancedness of a quadratic form can be easily checked. Thus we single out those symmetric divisors on $\overline{M}_{0,n}$ whose associated quadratic form is weakly balanced, in the process obtaining a simple sufficient criterion for a symmetric divisor on $\overline{M}_{0,n}$ to be semiample.

Suppose $D = \sum_{i=2}^{\lfloor n/2 \rfloor} a_i \Delta_i$ is a symmetric divisor on $\overline{M}_{0,n}$. Set $a_0 = a_1 = 0$ and $a_i = a_{n-i}$ for $\lfloor n/2 \rfloor + 1 \leq i \leq n-1$. Define a function $q_D: \mathbb{Z}_n \rightarrow \mathbb{Q}$ by

$$q_D(i) := \frac{1}{2}(2a_i - a_{i-1} - a_{i+1}).$$

²We have used Remark 5.1.3 to restrict to positive z_i 's here.

and a quadratic form

$$(5.4.1) \quad Q_D(x_0, \dots, x_{n-1}) := \sum_{0 \leq i, j \leq n-1} q_D(i-j)x_i x_j.$$

We call Q_D the associated quadratic form of D .

Theorem 5.4.2 (Cyclic Semiample Criterion). *Let $D = \sum_{i=2}^{\lfloor n/2 \rfloor} a_i \Delta_i$ be a symmetric divisor on $\overline{M}_{0,n}$. Suppose that for every $S \subset \{0, \dots, n-1\}$, the following inequality holds:*

$$(5.4.3) \quad 2a_{|S|} \geq \sum_{i, j \in S} (2a_{i-j} - a_{i-j+1} - a_{i-j-1}),$$

where as before $a_0 = a_1 = 0$ and $a_i = a_{n-i}$. Then D is semiample. Moreover, there exists a base point free linear subsystem of $|2D|$ of dimension at most $(2n-5)!!$

Proof. By Lemma 5.2.5 applied to the symmetric function $p_D: \mathbb{Z}_n \rightarrow \mathbb{Z}$ defined in (3.1.2), the associated quadratic form $Q_D = Q_{p_D}$ is weakly balanced. Proposition 5.3.1(2) now shows that $f := 2p_D$ is a weakly cyclically effective function, with values in even integers. Since

$$\mathfrak{D}(\mathbb{Z}_n, f; (1, \dots, 1)) = 2D,$$

the semiampleness follows at once from Theorem 4.3.6. The claim about base point free linear system follows once we observe that because the cyclic weighting w_σ constructed in Lemma 5.3.4 is integer-valued when the function f takes values in even integers, we actually obtain \mathbb{Z} -linear equivalence in Lemma 4.2.4(2). \square

Definition 5.4.4. We say that the symmetric divisor D on $\overline{M}_{0,n}$ is *cyclic semiample* if D satisfies the criterion of Theorem 5.4.2. Clearly, cyclic semiample divisors form a finite polyhedral subcone in $\text{NS}(\overline{M}_{0,n})^{\mathfrak{S}_n}$, which we call *the cyclic semiample cone*.

Proposition 5.4.5. *The cyclic semiample cone is a full-dimensional subcone in $\text{NS}(\overline{M}_{0,n})^{\mathfrak{S}_n}$*

Proof. Consider the following expression for a well-known ample divisor $\psi - \Delta$ on $\overline{M}_{0,n}$:

$$\psi - \Delta = \sum_{i=2}^{\lfloor n/2 \rfloor} \frac{i(n-i) - (n-1)}{n-1} \Delta_i.$$

The coefficients of Δ_i 's above *strictly* satisfy Inequalities (5.4.3). It follows that Theorem 5.4.2 gives a full-dimensional subcone in $\text{NS}(\overline{M}_{0,n})^{\mathfrak{S}_n}$ consisting of semiample divisors. \square

Remark 5.4.6. By Example 5.3.10, all symmetric \mathfrak{S}_n level 1 conformal blocks divisors lie in the cyclic semiample cone of $\overline{M}_{0,n}$. These $\lfloor n/2 \rfloor$ divisors are known to be extremal rays of the nef cone by [AGSS12, Theorem 1.2]. It follows from [Kaz14, Corollary 1.3] that all \mathfrak{S}_n -invariant type A conformal blocks vector bundles of rank 1 on $\overline{M}_{0,n}$ lie in the cyclic semiample cone. Computer computations show that the cyclic semiample cone shares many more than $\lfloor n/2 \rfloor$ extremal rays with the symmetric F-cone of $\overline{M}_{0,n}$. For example, when $n = 20$, there are 739 extremal rays of the symmetric F-cone and 60 of them are semiample by Theorem 5.4.2.

5.5. Geometric interpretation of the cyclic semiample cone. Let R be \mathbb{Z} or \mathbb{Q} . Suppose X is a smooth projective variety with a finitely generated Picard group. We say that a collection of effective divisors $\mathcal{B} = \{D_1, \dots, D_\rho\}$ on X is an R -basis if the classes of D_1, \dots, D_ρ form a basis of $\text{Pic}(X) \otimes R$.

Suppose $\mathcal{B} = \{D_1, \dots, D_\rho\}$ is an R -basis. Define *the complement of \mathcal{B}* to be the following open subset of X :

$$U(\mathcal{B}) := X \setminus \bigcup_{i=1}^{\rho} D_i.$$

Every R -basis \mathcal{B} defines ρ R -linear functions $c_i^{\mathcal{B}}: \text{Pic}(X) \otimes R \rightarrow R$, where $c_i^{\mathcal{B}}(L)$ for $L \in \text{Pic}(X) \otimes R$ are uniquely defined by

$$L = \mathcal{O} \left(\sum_{i=1}^{\rho} c_i^{\mathcal{B}} D_i \right).$$

Let $E(\mathcal{B}) \subset \text{Pic}(X) \otimes R$ be the convex cone of those line bundles whose coefficients in the basis \mathcal{B} are all non-negative. Namely,

$$E(\mathcal{B}) := \left\{ L \in \text{Pic}(X) \otimes R \mid c_i^{\mathcal{B}}(L) \geq 0 \right\}.$$

Definition 5.5.1. We say that a collection of R -bases $\mathcal{B}_1, \dots, \mathcal{B}_N$ is *exhaustive* if

$$(5.5.2) \quad U(\mathcal{B}_1) \cup U(\mathcal{B}_2) \cup \dots \cup U(\mathcal{B}_N) = X.$$

Given an exhaustive collection $\mathcal{B}_1, \dots, \mathcal{B}_N$, we define

$$\text{Nef}(\mathcal{B}_1, \dots, \mathcal{B}_N) := E(\mathcal{B}_1) \cap E(\mathcal{B}_2) \cap \dots \cap E(\mathcal{B}_N).$$

Lemma 5.5.3. *Suppose $\mathcal{B}_1, \dots, \mathcal{B}_N$ is an exhaustive collection. Then $\text{Nef}(\mathcal{B}_1, \dots, \mathcal{B}_N) \subset \text{Nef}(X)$. Moreover, if $\text{Pic}(X)$ is in addition torsion-free, then $\text{Nef}(\mathcal{B}_1, \dots, \mathcal{B}_N) \subset \text{Semiample}(X)$.*

Proof. This is obvious. By definition, $L \in E(\mathcal{B}_i)$ if and only if L can be represented modulo torsion by an effective divisor with support in \mathcal{B}_i . Condition (5.5.2) says that the supports of \mathcal{B}_i 's do not intersect. \square

The following observation is known to experts (we learned about it from Jenia Tevelev):

Lemma 5.5.4. *The collection of non-adjacent bases on $\overline{M}_{0,n}$ (see Remark 2.3.2) taken over all cyclic orderings of $[n]$ is exhaustive.*

Proof. This is just the reformulation of the implication

$$\text{cyclically effective} \Rightarrow \text{tree-effective}$$

from Lemma 4.2.3. Namely, if T is an $[n]$ -labeled tree, consider its embedding into the plane so that the leaves are identified with the vertices of P_n . The resulting cyclic ordering $\sigma: [n] \rightarrow P_n$ satisfies the property that every T -partition of $[n]$ is σ -contiguous. In particular, the generic point of M_T is not in the support of the divisors in \mathcal{A}_σ . \square

Lemma 5.5.5. *The cyclic semiample cone of $\overline{M}_{0,n}$ is $\text{Easy}(\{\mathcal{A}_\sigma \mid \sigma \text{ is a cyclic ordering of } [n]\})$.*

Proof. This follows by unwinding Definitions 5.4.4 and 5.5.1. \square

6. TREE-EFFECTIVITY AND ANOTHER SEMIAMPLENESS CRITERION

In the previous section, we have explored the strongest of the three effectivity conditions on a symmetric function, namely cyclic effectivity. In particular, we obtained an “if and only if” characterization of cyclic effectivity of a function f in terms of the balancedness of the associated quadratic form Q_f . In this section, we turn our attention to the tree-effectivity, but with more modest results. Namely, we prove a sufficient criterion for a function to be tree-effective (resp., weakly tree-effective). This leads to another sufficient criterion for semiample divisors on $\overline{M}_{0,n}$. On one hand, this criterion allows to exhibit boundary semiample divisors on $\overline{M}_{0,n}$ that are not cyclic semiample; on the other hand, this criterion is too stringent because many cyclically effective functions do not satisfy it, whereas every cyclically effective function is tree-effective by Remark 4.2.9. Both of these phenomena are illustrated by Table 1 in Section 8.

Proposition 6.0.6.

(1) *Suppose $f: G \rightarrow \mathbb{Q}_{\geq 0}$ is a subadditive symmetric function satisfying*

$$(6.0.7) \quad \begin{aligned} & (f(b) - f(a) + f(a+b))f(A) \\ & + (f(a) - f(b) + f(a+b))f(B) \\ & + (f(a) + f(b) - f(a+b))f(a+b) \\ & \geq 2f(B+b)f(a+b). \end{aligned}$$

for all $a, b, A, B \in G$ such that $a + b + A + B = 0 \in G$. Then f is tree-effective.

(2) *Suppose $f: \mathbb{Z}_n \rightarrow \mathbb{Q}_{\geq 0}$ is a subadditive symmetric function satisfying (6.0.7) for all non-negative integers a, b, A, B such that $a + b + A + B = n$. Then f is weakly tree-effective.*

Proof. We begin by establishing the first part.

Let $\vec{d} = (d_1, \dots, d_n)$ be a G - n -tuple. We need to verify that f is T -effective with respect to \vec{d} for every $[n]$ -labeled unrooted binary tree T . We proceed by induction on n starting with the base case $n = 3$, which holds by Remark 4.2.2.

Suppose $n \geq 4$. Take an internal node p of T from which two leaves emanate. Without loss of generality, these leaves are $n-1$ and n . Let $v = \overline{pq}$ be the internal edge of T containing p . We define T_0 to be the $[n-1]$ -labeled tree obtained by contracting the leaves $n-1$ and n of T into p , so that p becomes the $(n-1)^{\text{st}}$ leaf of T_0 . By our inductive assumption, f is T_0 -effective with respect to $\vec{d}_0 := (d_1, \dots, d_{n-2}, d_{n-1} + d_n)$. Equivalently, there exists a weighting w_0 on $\Gamma([n-1])$ such that

- (1) $w_0(i) = f(d_i)$ for all $i = 1, \dots, n-2$.
- (2) $w_0(n-1) = f(d_{n-1} + d_n)$.
- (3) $w_0(I \sqcup J) \geq f(\vec{d}_0(I))$ for all partitions $I \sqcup J = [n-1]$.

(4) $w_0(I | J) = f(\vec{d}_0(I))$ for all T_0 -partitions $I \sqcup J = [n-1]$.

Set $a := d_{n-1}$ and $b := d_n$. We define a weighting w on $\Gamma([n])$ as follows:

$$(6.0.8) \quad \begin{aligned} w(i \sim j) &= w_0(i \sim j) \text{ if } i, j \in [n-2], \\ w((n-1) \sim n) &= \frac{f(a) + f(b) - f(a+b)}{2}, \\ w(n \sim i) &= \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} w_0((n-1) \sim i), \text{ for } i \in [n-2], \\ w((n-1) \sim i) &= \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} w_0((n-1) \sim i), \text{ for } i \in [n-2]. \end{aligned}$$

Clearly, the w -flow through the vertex n of T is

$$\begin{aligned} w(n) &= \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} w_0(n-1) + \frac{f(a) + f(b) - f(a+b)}{2} \\ &= \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} f(a+b) + \frac{f(a) + f(b) - f(a+b)}{2} = f(a) = f(d_n), \end{aligned}$$

and the w -flow through the vertex $n-1$ of T is

$$w(n-1) = \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} w_0(n-1) + \frac{f(a) + f(b) - f(a+b)}{2} = f(d_{n-1}).$$

We also have

$$w(i) = w_0(i) = f(d_i), \quad \text{for all } i \in [n-2].$$

Thus w satisfies Condition (a) of Definition 4.2.1 (2).

Suppose $I \sqcup J = [n]$ is a partition such that $n-1$ and n lie in the same subset, say, J . Then from (6.0.8), we obtain $w(I | J) = w_0(I | [n-1] - I)$. It follows from T_0 -effectivity of f that

$$w(I | J) = w_0(I | [n-1] - I) \geq f(\vec{d}_0(I)) = f(\vec{d}(I)),$$

with the equality achieved when $I \sqcup J = [n]$ is a T -partition. Thus w satisfies Condition (c) of Definition 4.2.1 (2).

Consider now a partition of $[n]$ given by $I = n \cup I'$ and $J = (n-1) \cup J'$, where $I' \sqcup J' = [n-2]$. We set

$$\begin{aligned} A &:= \vec{d}(I') = \sum_{i \in I'} d_i, \\ B &:= \vec{d}(J') = \sum_{j \in J'} d_j. \end{aligned}$$

Then $a + b + A + B = 0 \in G$. We proceed to estimate the w -flow across $I \sqcup J$, which we compute using (6.0.8):

$$\begin{aligned}
w(I | J) &= w((n-1) \sim n) + \sum_{i \in I', j \in J'} w(i \sim j) + \sum_{i \in I'} w(i \sim (n-1)) + \sum_{j \in J'} w(j \sim n) \\
&= w((n-1) \sim n) + \sum_{i \in I', j \in J'} w_0(i \sim j) \\
&\quad + \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} \sum_{i \in I'} w_0(i \sim (n-1)) + \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} \sum_{j \in J'} w_0(j \sim n) \\
&= \frac{f(a) + f(b) - f(a+b)}{2} \\
&\quad + \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} w_0(I' | J' \cup \{n-1\}) + \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} w_0(J' | I' \cup \{n-1\}) \\
&\geq \frac{f(a) + f(b) - f(a+b)}{2} + \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} f(\vec{d}_0(I')) + \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} f(\vec{d}_0(J')) \\
&= \frac{f(a) + f(b) - f(a+b)}{2} + \frac{f(b) - f(a) + f(a+b)}{2f(a+b)} f(A) + \frac{f(a) - f(b) + f(a+b)}{2f(a+b)} f(B) \\
&\geq f(B+b) = f(\vec{d}(I)).
\end{aligned}$$

Thus w satisfies Condition (c) of Definition 4.2.1 (2). This finishes the proof of the first part. The second part follows by exactly the same argument, noting that the induction step replaces a k -tuple $\vec{d} = (d_1, \dots, d_k)$ of non-negative integers satisfying $\sum_{i=1}^k d_i = n$ by a $(k-1)$ -tuple $\vec{d}_0 = (d_1, \dots, d_{k-2}, d_{k-1} + d_k)$ of non-negative integers satisfying the same condition. \square

Theorem 6.0.9. *Let $D = \sum_{r=2}^{\lfloor n/2 \rfloor} a_r \Delta_r$ be a symmetric divisor on $\overline{M}_{0,n}$ and $p_D: \mathbb{Z}_n \rightarrow \mathbb{Z}$ be the function defined by (3.1.2). Suppose that for some*

$$\lambda \geq \lambda_{F\text{-nef}}(D)$$

we have that the function $f_\lambda: \mathbb{Z}_n \rightarrow \mathbb{Q}$ defined by $f_\lambda = \lambda A_n + p_D$ (i.e., $f_\lambda(r) = \lambda r(n-r) - a_r$) satisfies (6.0.7) for all non-negative integers a, b, A, B such that $a + b + A + B = n$. Then D is boundary semiample.

Proof. By the assumption on λ , f_λ is F-nef (cf. the proof of Lemma 3.2.3). Hence f_λ takes only non-negative values and f_λ is subadditive by Lemma 3.1.14. By Proposition 6.0.6(2), we have that f_λ is weakly tree-effective. Noting that

$$D = \mathfrak{D}(\mathbb{Z}_n, f_\lambda; \underbrace{(1, \dots, 1)}_n),$$

we conclude that D is boundary semiample by Theorem 4.3.1. \square

7. NEW NEF DIVISORS ON $\overline{M}_{0,n}$

In Proposition 5.3.1, we showed that a balanced cyclic quadratic form $Q_f(x_0, \dots, x_{m-1})$ corresponds to a cyclically effective F-nef function $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ with $f(0) = 0$, and so gives

rise to an infinite family of boundary semiample divisors (Theorem 5.3.9). In this section, we prove that if Q_f (or, equivalently, f) satisfies some further positivity conditions, then we can construct a new F-nef function \tilde{f} out of f in such a way that \tilde{f} is effective. This procedure gives rise to new infinite families of nef divisors $\mathfrak{D}(\mathbb{Z}_m, \tilde{f}; \vec{d})$.

Definition 7.0.10. Suppose $f: G \rightarrow \mathbb{Q}$ is a symmetric function with $f(0) = 0$. Set

$$m(f) := \frac{1}{2} \min\{2f(a) + 2f(b) - f(a+b) - f(a-b) \mid a, b \in G\}.$$

We define $\tilde{f}: G \rightarrow \mathbb{Q}$ by

$$(7.0.11) \quad \tilde{f}(k) = \begin{cases} f(k) & \text{for } k \neq 0, \\ m(f) & \text{for } k = 0. \end{cases}$$

The point of this definition is that for an F-nef function $f: G \rightarrow \mathbb{Q}$ with $f(0) = 0$, we can increase the value of the function at 0 up to $m(f)$ without losing the F-nefness property:

Lemma 7.0.12. *If $f: G \rightarrow \mathbb{Q}$, with $f(0) = 0$, is F-nef, then $\tilde{f}: G \rightarrow \mathbb{Q}$ is also F-nef.*

Proof. This follows easily from Definition 3.1.11. \square

Proposition 7.0.13. *Let $m \geq 3$. Suppose $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ is a cyclically effective F-nef function such that the balanced associated quadratic form Q_f satisfies the following additional condition:*

(†) *The minimum value of Q_f on the set*

$$\{(x_0, \dots, x_{m-1}) \in \mathbb{Z}^m \setminus (0, \dots, 0) \mid \sum_{i=0}^{m-1} x_i = 0\}$$

is at least $m(f)$ from Definition 7.0.10.

Then $\tilde{f}: \mathbb{Z}_m \rightarrow \mathbb{Q}$ is effective. Moreover, for any \mathbb{Z}_m - n -tuple \vec{d} , we have that \tilde{f} is T-effective with respect to \vec{d} for every $[n]$ -labeled tree whose all T-partitions $I \sqcup J = [n]$ satisfy $\vec{d}(I) \neq 0 \in \mathbb{Z}_m$.

Before we give a proof of the above proposition, we formulate its implications for the nef cone of $\overline{M}_{0,n}$:

Theorem 7.0.14. *Let $f: \mathbb{Z}_m \rightarrow \mathbb{Q}$ be as in Proposition 7.0.13. Then for any \mathbb{Z}_m - n -tuple $\vec{d} = (d_1, \dots, d_n)$, the divisor*

$$(7.0.15) \quad \mathfrak{D}(\mathbb{Z}_m, \tilde{f}; \vec{d}) = \sum_{i=1}^n \tilde{f}(d_i) \psi_i - \sum_{I,J} \tilde{f}(\vec{d}(I)) \Delta_{I,J} = \mathfrak{D}(\mathbb{Z}_m, f; \vec{d}) - \tilde{f}(0) \sum_{\substack{I \sqcup J = [n] \\ m \mid \vec{d}(I)}} \Delta_{I,J}$$

is stratally effective boundary, hence nef, on $\overline{M}_{0,n}$. Moreover, $\mathfrak{D}(\mathbb{Z}_m, \tilde{f}; \vec{d})$ is base point free away from $\sum_{\substack{I \sqcup J = [n] \\ m \mid \vec{d}(I)}} \Delta_{I,J}$.

Proof. The first part follows from effectivity of \tilde{f} (by Proposition 7.0.13) and Theorem 4.3.1. For the second part, denote

$$\Delta_0 := \sum_{\substack{I \sqcup J = [n] \\ m \mid \vec{d}(I)}} \Delta_{I,J}.$$

Let T be the dual graph of a 0-dimensional boundary B on $\overline{M}_{0,n}$ that does not lie in Δ_0 . Then by Proposition 7.0.13, there is an effective linear combination of the boundary that avoids B and is linearly equivalent to $\mathfrak{D}(\mathbb{Z}_m, \tilde{f}; (d_1, \dots, d_n))$. The claim now follows from the observation that any boundary stratum not entirely lying in Δ_0 contains in its closure some 0-dimensional boundary stratum B that also does not lie in Δ_0 (this is where we use $m \geq 3$). \square

Proof of Proposition 7.0.13. Let $\vec{d} = (d_1, \dots, d_n)$ be a \mathbb{Z}_m - n -tuple. To prove effectivity of \tilde{f} , we first reduce to the case where no d_i is 0 in \mathbb{Z}_m . Indeed, if $d_n = 0$, then effectivity of \tilde{f} with respect to (d_1, \dots, d_{n-1}) implies the effectivity of \tilde{f} with respect to \vec{d} by considering any triangle in $\Gamma([n])$ with a vertex at n , and with edges emanating from \tilde{f} having weight $\tilde{f}(0)/2$ and the remaining edge having weight $-\tilde{f}(0)/2$.

Second, we observe that if \vec{d} contains no 0's, then there exists an $[n]$ -labeled unrooted binary tree whose all T -partitions $I \sqcup J = [n]$ satisfy $\vec{d}(I) \neq 0 \in \mathbb{Z}_m$. Thus to prove effectivity of \tilde{f} with respect to \vec{d} , it remains to prove T -effectivity of \tilde{f} with respect to \vec{d} for all such trees. This is achieved by constructing a T -cyclic weighting, which we now describe.

To begin, we recall that any planar embedding of an $[n]$ -labeled tree T defines a cyclic ordering $\sigma: [n] \rightarrow P_n$, obtained by going around the embedded graph in the clockwise direction. Because every T -partition of $[n]$ gives a σ -contiguous partition of $[n]$, it follows from cyclic effectivity of f that the cyclic weighting w_σ on $\Gamma([n])$ constructed in Lemma 5.3.4 satisfies

$$w_\sigma(I \mid J) \geq f(\vec{d}(I)),$$

with the equality holding for every T -partition $I \sqcup J = [n]$. There are 2^{n-3} distinct planar embeddings of T . We define the T -cyclic weighting w_T on $\Gamma([n])$ to be the average of all cyclic weightings w_σ on $\Gamma([n])$ taken over all planar embeddings of T , as described above. We proceed to describe the properties of w_T :

Claim (Property 1). *We have that $w_T(I \mid J) \geq f(\vec{d}(I))$ for every partition $I \sqcup J$. Thus $w_T(I \mid J) \geq \tilde{f}(\vec{d}(I))$ for every partition $I \sqcup J = [n]$ satisfying $\vec{d}(I) \neq 0 \in \mathbb{Z}_m$.*

Claim (Property 2). *If e is an edge of T and $I \sqcup J = [n]$ is the e -partition satisfying $\vec{d}(I) = 0 \in \mathbb{Z}_m$, then $w_T(I \mid J) = Q_f(0, \dots, 0) = 0$.*

Claim (Property 3). *Suppose all T -partitions $I \sqcup J = [n]$ satisfy $\vec{d}(I) \neq 0 \in \mathbb{Z}_m$. We have that $w_T(I \mid J) \geq \tilde{f}(0)$ for any partition $I \sqcup J = [n]$ such that $\vec{d}(I) = 0 \in \mathbb{Z}_m$.*

Proof. The first two claims follow from the properties of the cyclic weighting w_σ given in Lemma 5.3.4 and the cyclic effectivity of f . We proceed to prove the third claim.

Fix $I \sqcup J = [n]$ with $\vec{d}(I) = 0 \in \mathbb{Z}_m$. By assumption on T , $I \sqcup J$ is not a T -partition. Let $e_0 = \overline{pq}$ be the internal edge of T satisfying the following property. Suppose e_1, e_2 are the other two edges from p . Let A be the set of leaves of T lying to the side of e_1 opposite to p and B be the set of leaves of T lying to the side of e_2 opposite to p . Then we require $A \subset I$ and $B \subset J$. It is easy to see that such e_0 exists because $I \sqcup J$ is not a T -partition. We set $a = \sum_{i \in A} d_i$ and $b = \sum_{j \in B} d_j$.

Let ι be the involution on the set of the planar embeddings of T given by the change of orientation of the edges e_0, e_1, e_2 at p . Suppose σ is a cyclic ordering corresponding to some planar embedding of T . We denote by $\iota(\sigma)$ the cyclic ordering corresponding to the embedding of T obtained by applying ι to the planar embedding of σ . Recall the definition of $\sigma_I(\vec{d})$ from Definition 5.3.2. The key observation now is that whenever $\sigma_I(\vec{d}) = (z_0, \dots, z_{m-1})$, we have

$$\iota(\sigma)_I(\vec{d}) = (z_0, \dots, z_{m-1}) + (-1)^\epsilon (\underbrace{1, \dots, 1}_a, \underbrace{0, \dots, 0}_b, 0, \dots) + (-1)^{\epsilon+1} (\underbrace{0, \dots, 0}_b, \underbrace{1, \dots, 1}_a, 0, \dots),$$

where $\epsilon \in \{0, 1\}$ depends on the original orientation of $\{e_0, e_1, e_2\}$ at p .

To prove the lemma, it remains to show that the average of $w(I | J)$ -flows for any pair $(\sigma, \iota(\sigma))$ of embeddings is at least $\tilde{f}(0)$.

Suppose that neither $\sigma_I(\vec{d})$ nor $\iota(\sigma)_I(\vec{d})$ is $(0, \dots, 0)$. Then because Q_f satisfies condition (†) of Proposition 7.0.13, we have both $w_\sigma(I | J) \geq \tilde{f}(0)$ and $w_{\iota(\sigma)}(I | J) \geq \tilde{f}(0)$ as desired.

Suppose now that $\sigma_I(\vec{d}) = (0, \dots, 0)$. Then $\iota(\sigma)_I(\vec{d}) = (\underbrace{1, \dots, 1}_b, \underbrace{0, \dots, 0}_{a-b}, \underbrace{-1, \dots, -1}_b, 0, \dots, 0)$.

It follows that

$$\begin{aligned} \frac{1}{2} \left(w_\sigma(I | J) + w_{\iota(\sigma)}(I | J) \right) &= \frac{1}{2} Q_f(\underbrace{1, \dots, 1}_b, \underbrace{0, \dots, 0}_{a-b}, \underbrace{-1, \dots, -1}_b, 0, \dots, 0) \\ &= 2f(a) + 2f(b) - f(a+b) - f(a-b) \geq \tilde{f}(0), \end{aligned}$$

as desired. \square

At last, the T -effectivity of \tilde{f} with respect to \vec{d} follows from Properties (1–3) of the T -cyclic weighting w_T . \square

Example 7.0.16. Suppose $m \geq 3$. The 1^{st} standard function $A_m: \mathbb{Z}_m \rightarrow \mathbb{Z}$ has the associated quadratic form $Q_{A_m} = m(x_0^2 + \dots + x_{m-1}^2) - (x_0 + \dots + x_{m-1})^2$ that satisfies the assumptions of Proposition 7.0.13. It is straightforward to compute that $m(A_m) = m$. As a result, Theorem 7.0.14 shows that the 3^{rd} standard function from Example 3.1.18 is effective. This

gives a streamlined proof of Theorem 4.2 from unpublished [Fed13] stating that the following divisor is nef on $\overline{M}_{0,n}$ for any \mathbb{Z}_m - n -tuple $\vec{d} = (d_1, \dots, d_n)$:

$$(7.0.17) \quad \begin{aligned} \mathfrak{D}(\mathbb{Z}_m, E_m; \vec{d}) &= \sum_{i=1}^n \langle d_i \rangle_m \langle m - d_i \rangle_m \psi_i - \sum_{I \sqcup J = [n]} \langle \vec{d}(I) \rangle_m \langle \vec{d}(J) \rangle_m \Delta_{I,J} \\ &+ m \left(\sum_{\substack{i: \\ m|d_i}} \psi_i - \sum_{\substack{I \sqcup J = [n] \\ m|\vec{d}(I)}} \Delta_{I,J} \right). \end{aligned}$$

By taking $n = 9$, $d_1 = \dots = d_9 = 1$, and $m = 3$, we obtain the divisor $\Delta_2 + \Delta_3 + 2\Delta_4$ in $\text{Nef}(\overline{M}_{0,9})$. This divisor generates an extremal ray of the symmetric nef cone of $\overline{M}_{0,9}$ and is not known to come from the conformal blocks bundles [Swi11], or any other geometric construction. This is also the only extremal ray of $\text{Nef}(\overline{M}_{0,9}/\mathfrak{S}_9)$ not known to be semiample, see Table 1 in §8.3.

It is proved in [Fed11], that in the case $d_1 = \dots = d_n$ and $m = 3$, the divisor (7.0.17) generates an extremal ray of the symmetric nef cone of $\overline{M}_{0,n}$. We believe the following:

Conjecture 7.0.18. Suppose $m \geq 5$ is prime, $m \mid n$ and $d_1 = \dots = d_n$. Then the divisor (7.0.17) generates an extremal ray of the symmetric nef cone of $\overline{M}_{0,n}$.

We do not know whether divisors given by Theorem 7.0.14 are in general semiample.

8. APPLICATIONS AND EXAMPLES

8.1. Applications to conformal blocks divisors. We have already seen in Example 5.3.10 that the semiampleness of all \mathfrak{sl}_m level 1 conformal blocks divisors studied in [GG12] follows from the balancedness of the quadratic form

$$x_0^2 + x_1^2 + \dots + x_{m-1}^2.$$

In fact, Theorem 4.3.1 implies that all such divisors are boundary semiample over $\text{Spec } \mathbb{Z}$. By [Kaz14], the above result implies that all symmetric type A conformal blocks vector bundles of rank 1 are also boundary semiample over $\text{Spec } \mathbb{Z}$.

There are several other notable connections of our results with the theory of conformal blocks divisors on $\overline{M}_{0,n}$. For example, by applying the Effective Boundary Lemma 2.3.3, Mukhopadhyay showed that all level 1 conformal blocks divisors of type B and D are effective boundary [Muk14]. It would be interesting to determine whether the associated quadratic forms of these divisors enjoy any balancedness properties.

In [GJMS13], Gibney, Jensen, Moon, and Swinarski introduce and study Veronese quotient divisors on $\overline{M}_{0,n}$ and relate them to higher level \mathfrak{sl}_2 conformal blocks divisors. For example, for every positive integer g and an integer $1 \leq \ell \leq g$, they consider the following Veronese quotient divisors on $\overline{M}_{0,2g+2}$ [GJMS13, Example 2.13]:

$$D(g, \ell) := \mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{g+1}}.$$

It follows from the formulae for the intersection numbers of this divisor with the F-curves $F_{1,1,a}$ given in [GJMS13, Corollary 2.2] and Remark 5.2.2 that the associated quadratic form $q_{D(g,\ell)}$ is weakly balanced if and only if ℓ is odd. Therefore, we conclude that $D(g,\ell)$ is cyclic semiample for all odd ℓ . Applying [GJMS13, Corollary 4.5], we deduce that the \mathfrak{sl}_2 level ℓ conformal blocks divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is cyclic semiample on $\overline{M}_{0,n}$ for all odd ℓ .

Computations show that the F-nef function $f_{D(g,\ell)}$ often satisfies tree-effectivity criterion of Proposition 6.0.6 for even values of ℓ . Determining whether $f_{D(g,\ell)}$ is tree-effective for these divisors would be another interesting problem.

8.2. Democratic weighting. While the major focus of this paper has been on proving semi-ampleness of divisors by proving tree-effectivity of the associated function, it is often easier to verify effectivity (or weak-effectivity) of a function. One obvious way to do this is as follows.

Given a function $f: G \rightarrow \mathbb{Q}$ and a G - n -tuple (d_1, \dots, d_n) , we define the *democratic weighting* w_{dem} on $\Gamma([n])$ by the following recipe. Set

$$\Sigma := \sum_{i=1}^n f(d_i),$$

and

$$c_i := \frac{1}{n-2} f(d_i) - \frac{1}{2(n-1)(n-2)} \Sigma.$$

We now set $w_{\text{dem}}(i \sim j) = c_i + c_j$. It is easy to check that the flow through each vertex i is $f(d_i)$. Next, consider a partition $I \sqcup J = [n]$ with $\text{card}(I) = k$ and $\text{card}(J) = n - k$. The w -flow across $I \sqcup J$ is then

$$\begin{aligned} & \frac{k}{n-2} \sum_{j \in J} f(d_j) + \frac{n-k}{n-2} \sum_{i \in I} f(d_i) - \frac{k(n-k)}{(n-1)(n-2)} \Sigma \\ &= \left(\frac{k}{n-2} - \frac{k(n-k)}{(n-1)(n-2)} \right) \sum_{j \in J} f(d_j) + \left(\frac{n-k}{n-2} - \frac{k(n-k)}{(n-1)(n-2)} \right) \sum_{i \in I} f(d_i) \\ &= \frac{k(k+1)}{(n-2)(n-1)} \sum_{j \in J} f(d_j) + \frac{(n-k)(n-k-1)}{(n-1)(n-2)} \sum_{i \in I} f(d_i). \end{aligned}$$

Proposition 8.2.1. *Suppose $f: G \rightarrow \mathbb{Q}$ is an F-nef function such that*

$$f(a) + f(b) \geq \frac{3}{2} f(a+b) \quad \text{for all } a, b \in G.$$

Then f is an effective function. In particular, $\mathfrak{D}(G, f; (d_1, \dots, d_n))$ is nef on $\overline{M}_{0,n}$ for all G - n -tuples (d_1, \dots, d_n) .

Proof. It is easy to see using the democratic weighting that f is effective with respect to any G - n -tuple. The claim follows by Theorem 4.3.1. \square

Definition 8.2.2. We say that a divisor $D = \sum_{r=2}^{\lfloor n/2 \rfloor} a_r \Delta_r$ on $\overline{M}_{0,n}$ is *democratic* if for some symmetric function $f: \mathbb{Z}_n \rightarrow \mathbb{Q}_{\geq 0}$ such that $D = \mathfrak{D}(\mathbb{Z}_n, f; (1)_n)$ and for all k -tuples of positive integers $\vec{d} = (d_1, \dots, d_k)$ satisfying $d_1 + \dots + d_k = n$, the democratic weighting proves that f is effective with respect to \vec{d} . Note that by Theorem 4.3.6, a democratic divisor is strally effective boundary, hence nef.

8.3. Experimental results. We wrote a simple worksheet in SAGE [S⁺14] (which in turn uses PARI/GP [The14] for certain computations with quadratic forms), available at <https://www2.bc.edu/maksym-fedorchuk/Semiamplessness-criterion.sws>, that can compute, among other things, whether:

- A positive definite cyclic quadratic form is good.³
- A given symmetric divisor on $\overline{M}_{0,n}$ satisfies semiamplessness criterion of Theorem 5.4.2.
- A given symmetric divisor on $\overline{M}_{0,n}$ satisfies semiamplessness criterion of Theorem 6.0.9.
- A given symmetric divisor on $\overline{M}_{0,n}$ is democratic, hence nef.

We compute the number of extremal rays of the symmetric F-cone of $\overline{M}_{0,n}$ that are semiample by Theorems 5.4.2 and 6.0.9 in Table 1. For the remaining extremal rays, we check whether they are democratic, hence nef. The output for a given n is obtained by running the command `SemiampleTest(n)` of the worksheet.

n	Extremal rays of the symmetric F-cone	Cyclic semiample, i.e., satisfy Theorem 5.4.2	Satisfy Theorem 6.0.9, but not cyclic semiample	Democratic divisors among the remaining extremal rays
8	4	3	1	0
9	4	3	0	1
10	7	6	1	0
11	10	6	0	4
12	10	6	1	3
13	18	9	0	9
14	27	13	1	13
15	26	11	0	15
16	74	19	7	48
17	113	22	0	84

TABLE 1. Experimental results

³Unfortunately, SAGE or PARI/GP quickly runs out of memory for more interesting quadratic forms.

Referring to the table above, we obtain the following result, which extends the validity of the symmetric F-conjecture to positive characteristic for $n \leq 16$ (recall that in characteristic 0, it is known for $n \leq 24$ by [Gib09]):

Theorem 8.3.1. *Over $\text{Spec } \mathbb{Z}$, we have that every symmetric F-nef divisor on $\overline{M}_{0,n}$ is stratally effective boundary (hence nef over any field) for $n \leq 16$. Moreover, all symmetric nef divisors on $\overline{M}_{0,8}$ and $\overline{M}_{0,10}$ are semiample.*

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