Toward GIT stability of syzygies of canonical curves

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Abstract

We introduce the problem of GIT stability for syzygy points of canonical curves with a view toward a GIT construction of the canonical model of $\overline{M}_g$. As the first step in this direction, we prove semi-stability of the 1^{st} syzygy point for a general canonical curve of odd genus.

1. Introduction

By analogy with Hilbert points of Gieseker [Gie77, p.234], we introduce syzygy points of canonical curves and initiate the program of studying their GIT stability. The eventual goal of this program is a GIT construction of the canonical model of $\overline{M}_g$, a problem whose origins lie in the log minimal model program for the moduli space of stable curves. Introduced by Hassett and Keel, the log MMP for $\overline{M}_g$ aims to construct certain log canonical models of $\overline{M}_g$ in a way that allows modular interpretation [Has05]. The log canonical divisors on (the stack) $\overline{\mathcal{M}}_g$ considered in this program are

$$K_{\overline{\mathcal{M}}_g} + \alpha\delta = 13\lambda - (2-\alpha)\delta, \quad \text{for } \alpha \in [0,1] \cap \mathbb{Q}.$$

The work done so far suggests that we can construct some of these models as GIT quotients of spaces of Hilbert points of n-canonically embedded curves. This is already evidenced in the work of Gieseker [Gie82] and Schubert [Sch91], who analyzed the cases of $n \geqslant 5$ and $n = 3$, respectively. Recent work of Hassett and Hyeon [HH09, HH13] extends the GIT analysis to $n = 2$ and constructs the first two log canonical models of $\overline{M}_g$ corresponding to $\alpha > \frac{2}{3}$; see also [AFSvdW13]. Subsequent work along this direction suggests that the case of $n = 1$ and the use of finite Hilbert points would yield log canonical models corresponding to the values of α down to $\alpha = \frac{g+6}{7g+6}$ [AFS13, FJ12].

The ultimate goal of the Hassett–Keel program is to reach $\alpha = 0$, which corresponds to the canonical model of $\overline{M}_g$. To go beyond $\alpha = \frac{g+6}{7g+6}$ and indeed down to $\alpha = 0$, Farkas and Keel suggested that one should construct birational models of $\overline{M}_g$ as GIT quotients using syzygies of canonically embedded curves. In this paper, we make the first step toward this goal by proving a generic semi-stability result for the 1^{st} syzygies in odd genus.

Main Theorem. *A general canonical curve of odd genus $g \geqslant 5$ has a semi-stable 1^{st} syzygy point.*

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Our strategy for proving generic semi-stability of syzygy points follows that of [AFS13] for proving generic semi-stability of finite Hilbert points. Namely, we prove the semi-stability of the 1^{st} syzygy point of a singular curve with $\mathbb{G}_m$-action — the balanced ribbon — by a method of [MS11].

Ribbons and the problem of studying syzygies of their canonical embeddings were originally introduced by Bayer and Eisenbud in [BE95]. Their motivation for studying ribbons was in the context of Green's conjecture for smooth canonical curves. Namely, Bayer and Eisenbud asked whether rational ribbons satisfy an appropriate version of Green's conjecture [BE95, p.720]. Although this question remains open in its full generality, an affirmative answer to it implies the generic Green's conjecture, which is known thanks to the work of Voisin [Voi02, Voi05]. Green's conjecture makes an appearance in the study of GIT stability of syzygies of canonical curves by controlling which syzygy points are well-defined, see Remark 2.4.

Outline of the paper.

In Section 2, we define syzygy points of a canonically embedded Gorenstein curve and give a precise statement of our main result. In Section 3, we recall some preliminary results about balanced ribbons. In the most technical Section 4, we construct several monomial bases of cosyzygies for the balanced ribbon. Finally, in Section 5, we prove the main theorem by deducing semi-stability of the 1^{st} syzygy point of the balanced ribbon from the existence of the monomial bases constructed in Section 4.

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2. Syzygy points of canonical curves

In this section, we recall some basic notions of Koszul cohomology and set-up the GIT problems for the linear syzygies of a canonical curve. We refer to [Gre84] and [AF11b] for a complete treatment of Koszul cohomology and a detailed discussion of Green's conjecture.

We define a *canonical Gorenstein curve* to be a Gorenstein curve C with a very ample dualizing sheaf ω_C. The arithmetic genus of such C is at least 3. In this paper, we are exclusively concerned with Koszul cohomology of the pair (C, ω_C). Namely, associated to C and the line bundle ω_C is the Koszul complex

$$\bigwedge^{p+1} \mathrm{H}^0(\omega_C) \otimes \mathrm{H}^0(\omega_C^{q-1}) \xrightarrow{f_{p+1,q-1}} \bigwedge^{p} \mathrm{H}^0(\omega_C) \otimes \mathrm{H}^0(\omega_C^{q}) \xrightarrow{f_{p,q}} \bigwedge^{p-1} \mathrm{H}^0(\omega_C) \otimes \mathrm{H}^0(\omega_C^{q+1}) \quad (1)$$

where the differential $f_{p,q}$ is given by

$$f_{p,q}(x_0 \wedge x_1 \wedge \cdots \wedge x_{p-1} \otimes y) = \sum_{i=0}^{p-1} (-1)^i x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{p-1} \otimes x_i y.$$

The Koszul cohomology groups of (C,ω_C) are

$$K_{p,q}(C) := \ker f_{p,q} \big/ \operatorname{im} f_{p+1,q-1}. \tag{2}$$

We say that C satisfies property (N_p) if $K_{i,q}(C)=0$ for all (i,q) with $i \leqslant p$ and $q \geqslant 2$. In particular, property (N_0) means that the natural maps $\operatorname{Sym}^m \mathrm{H}^0(\omega_C) \to \mathrm{H}^0(\omega_C^m)$ are surjective for all m, i.e., that C is projectively normal in its canonical embedding. Property (N_p) for $p \geqslant 1$ means, in addition, that the ideal of C in the canonical embedding is generated by quadrics and the syzygies of order up to p are linear.

Remark 2.1 (On projective normality of canonical Gorenstein curves). A classical theorem of Max Noether says that a *smooth* curve C of genus $g \geqslant 3$ is non-hyperelliptic if and only if ω_C is very ample if and only if C satisfies property (N_0) [ACGH85, p.117]. A relatively recent result [KM09] extends these equivalences to the case when C is an *integral* Gorenstein curve. We are not aware of a general result concerning projective normality of non-integral Gorenstein curves. In particular, it appears that the case of reducible curves is open in general [BB11]. Because of this and because our primary object of study is a non-reduced curve (namely, a rational ribbon), we often specify projective normality as a hypothesis. We do note that projective normality of non-hyperelliptic ribbons is established in the original paper of Bayer and Eisenbud [BE95, Theorem 5.3]. An explicit proof for the case of the balanced ribbon appears in [AFS13, Proposition 3.5].

For $p \leqslant g-2$, set

$$\Gamma_p(C) := \left(\bigwedge^{p+1} \mathrm{H}^0(\omega_C) \otimes \mathrm{H}^0(\omega_C) \right) \Big/ \bigwedge^{p+2} \mathrm{H}^0(\omega_C). \tag{3}$$

The first four terms of the Koszul complex (1) in degree $p+2$ give the exact sequence

$$0 \to K_{p+1,1}(C) \to \Gamma_p(C) \to \ker f_{p,2} \to K_{p,2}(C) \to 0.$$

We can thus readily compute that

$$\begin{aligned} \dim \ker f_{p,2} &= (3g-2p-3)\binom{g-1}{p}, \text{ and} \\ \dim \Gamma_p(C) &= g\binom{g}{p+1} - \binom{g}{p+2}. \end{aligned} \tag{4}$$

Definition 2.2. We define the *space of p^{th} order linear syzygies* of C as the subspace of $\Gamma_p(C)$ given by

$$\operatorname{Syz}_p(C) := K_{p+1,1}(C).$$

Suppose C satisfies property (N_p) so that $K_{p,2}(C)=0$. We define the *space of p^{th} order linear cosyzygies* of C as the quotient space of $\Gamma_p(C)$ given by

$$\operatorname{CoSyz}_p(C) := \ker f_{p,2}.$$

The relation of the above definition to the definition of syzygies in terms of the homogeneous ideal of C is as follows. Let

$$I_m(C) = \ker\left(\operatorname{Sym}^m \mathrm{H}^0(\omega_C) \to \mathrm{H}^0(\omega_C^m)\right)$$

be the degree m graded piece of the homogeneous ideal of C. Then the space of p^{th} order linear syzygies among the defining quadrics of C is taken to be the kernel of the map

$$\bigwedge^{p} \mathrm{H}^0(\omega_C) \otimes I_2(C) \xrightarrow{\alpha} \bigwedge^{p-1} \mathrm{H}^0(\omega_C) \otimes I_3(C).$$

A simple diagram chase now gives a well-known isomorphism $\ker\alpha \simeq K_{p+1,1}(C)$.

Definition 2.3. Suppose C satisfies property (N_p). We define the p^{th} *syzygy point of* C to be the quotient of $\Gamma_p(C)$ given by

$$\left[\Gamma_p(C) \to \mathrm{CoSyz}_p(C) \to 0\right],$$

and interpreted as a point in the Grassmannian $\mathrm{Grass}\left((3g-2p-3)\binom{g-1}{p}, \Gamma_p(C)\right)$.

Abusing notation, we use $\mathrm{CoSyz}_p(C)$ to denote both the vector space itself and the point in $\mathrm{Grass}\left((3g-2p-3)\binom{g-1}{p}, \Gamma_p(C)\right)$ that it represents. Observe that the 0^{th} syzygy point is simply the 2^{nd} Hilbert point.

Remark 2.4. For which curves is the p^{th} syzygy point defined? According to a celebrated Green's conjecture, a smooth canonical curve C satisfies (N_p) if and only if p is less than the Clifford index of C. Formulated by Green in [Gre84], this conjecture remains open in its full generality. It is known to be true, however, for a large class of curves. Voisin proved that general canonical curves on $K3$ surfaces satisfy Green's conjecture [Voi02, Voi05]. More recently, Aprodu and Farkas proved the conjecture for all smooth curves on $K3$ surfaces [AF11a]. In particular, the p^{th} syzygy point of a generic curve of genus g is defined for all $p < \lfloor g/2 \rfloor$.

Definition 2.5. We define $\overline{\mathrm{Syz}}_p$ to be the closure in $\mathrm{Grass}\left((3g-2p-3)\binom{g-1}{p}, \Gamma_p(C)\right)$ of the locus of p^{th} syzygy points of canonical curves satisfying property (N_p).

Consider the group $\mathrm{SL}_g \simeq \mathrm{SL}(\mathrm{H}^0(\omega_C))$. Its natural action on $\mathrm{H}^0(\omega_C)$ induces actions on the vector space $\Gamma_p(C)$, the Grassmannian $\mathrm{Grass}\left((3g-2p-3)\binom{g-1}{p}, \Gamma_p(C)\right)$, and finally on the subvariety $\overline{\mathrm{Syz}}_p$. The Plücker line bundle on the Grassmannian comes with a natural SL_g linearization, and so does its restriction to $\overline{\mathrm{Syz}}_p$. A candidate for the p^{th} *syzygy model* of $\overline{M}_g$ is thus the GIT quotient

$$\overline{\mathrm{Syz}}_p /\!\!/ \mathrm{SL}_g\,.$$

Our main theorem shows that this quotient is non-empty for $p = 1$ and odd $g \geqslant 5$.

Theorem 2.6. *A general canonical curve of odd genus $g \geqslant 5$ has a semi-stable 1^{st} syzygy point.*

We prove this theorem in Section 5; see Corollary 5.5.

3. The balanced canonical ribbon

We prove Theorem 2.6 by explicitly writing down a semi-stable point in $\overline{\mathrm{Syz}}_1$. This point corresponds to the syzygies of the balanced ribbon. Our exposition of its properties closely follows [AFS13] where the semi-stability of Hilbert points of this ribbon was established. Nevertheless, we recall the necessary details for the reader's convenience.

Let $g = 2k+1$. The *balanced ribbon* of genus g is the scheme R obtained by identifying $U := \operatorname{Spec}\mathbb{C}[u,\epsilon]/(\epsilon^2)$ and $V := \operatorname{Spec}\mathbb{C}[v,\eta]/(\eta^2)$ along $U \setminus \{0\}$ and $V \setminus \{0\}$ via the isomorphism

$$\begin{aligned} u &\mapsto v^{-1} - v^{-k-2}\eta, \\ \epsilon &\mapsto v^{-g-1}\eta. \end{aligned} \tag{5}$$

The scheme R is an example of a rational ribbon. While our proofs use only the balanced ribbon, we refer the reader to [BE95] for a more extensive study of ribbons in general.

Being a Gorenstein curve, R has a dualizing line bundle ω, generated by $\frac{du\wedge d\epsilon}{\epsilon^2}$ on U, and by $\frac{dv\wedge d\eta}{\eta^2}$ on V. Since ω is very ample by [AFS13, Lemma 3.2], the global sections of ω embed R as an arithmetically Gorenstein curve in $\mathbb{P}^{g-1}$ by [BE95, Theorem 5.3]. As a result, we have $K_{p,q}(R) = 0$ for all $q \geqslant 3$ and $p \leqslant g-3$. In particular, for all $p < \lfloor g/2 \rfloor$, property (N_p) is equivalent to $K_{p,2}(R) = 0$ [Ein87].

The balanced ribbon R admits a $\mathbb{G}_m$-action, given by

$$\begin{aligned} t\cdot u &\mapsto tu, & t\cdot \epsilon &\mapsto t^{k+1}\epsilon\,, \\ t\cdot v &\mapsto t^{-1}v, & t\cdot\eta &\mapsto t^{-k-1}\eta. \end{aligned} \tag{6}$$

This action induces $\mathbb{G}_m$-actions on $\mathrm{H}^0(R,\omega^m)$ for all m. The next two propositions describe these spaces along with their decomposition into weight spaces.

Proposition 3.1. *A basis for $\mathrm{H}^0(R,\omega)$ is given by $x_0,\dots,x_{2k}$, where the x_i's restricted to U are given by*

$$x_i = \begin{cases} u^i\ \frac{du\wedge d\epsilon}{\epsilon^2} & \text{if } 0\leqslant i\leqslant k \\ \left(u^i + (i-k)u^{i-k-1}\epsilon\right)\ \frac{du\wedge d\epsilon}{\epsilon^2} & \text{if } k< i\leqslant 2k, \end{cases}$$

and where x_i is a $\mathbb{G}_m$-semi-invariant of weight $i-k$. In particular, $\mathrm{H}^0(R,\omega)$ splits as a direct sum of g distinct $\mathbb{G}_m$ weight-spaces of weights $-k,\dots,k$.

Proof. That x_i's form a basis follows from [BE95, Theorem 5.1]. The statement about the weights is obvious. □

Remark 3.2 ($\mathbb{Z}_2$-symmetry). Observe that R has a $\mathbb{Z}_2$-symmetry given by the isomorphism $V\simeq U$ defined by $u\leftrightarrow v$ and $\epsilon\leftrightarrow\eta$ and commuting with the gluing isomorphism (5). The $\mathbb{Z}_2$-symmetry exchanges x_i and x_{2k-i}.

The following observation from [AFS13, Lemma 3.4] deals with higher powers of ω:

Lemma 3.3 (Ribbon Product Lemma). *Let $0\leqslant i_1,\dots,i_m\leqslant 2k$ be such that $i_1,\dots,i_\ell\leqslant k$ and $i_{\ell+1},\dots,i_m > k$. On U, we have*

$$x_{i_1}\cdots x_{i_m} = \left(u^a + (a-b)u^{a-k-1}\epsilon\right)\left(\frac{du\wedge d\epsilon}{\epsilon^2}\right)^m,$$

where

$$\begin{aligned} a &= i_1+\cdots+i_m\ , \\ b &= i_1+\cdots+i_\ell + k(m-\ell). \end{aligned}$$

Definition 3.4. The *u-weight* (or *u-degree*) of a monomial $x_{i_1}\cdots x_{i_m}$ is the sum $i_1+\cdots+i_m$. Note that the u-weight of $x_{i_1}\cdots x_{i_m}$ equals to the $\mathbb{G}_m$-weight of $x_{i_1}\cdots x_{i_m}$ plus km.

Proposition 3.5. *Let $m\geqslant 2$. Let $\mathrm{H}^0(R,\omega^m)_d$ be the weight-space of $\mathrm{H}^0(R,\omega^m)$ of u-weight d. Then*

$$\dim \mathrm{H}^0(R,\omega^m)_d = \begin{cases} 1 & \text{if } \ 0\leqslant d\leqslant k, \\ 2 & \text{if } \ k<d<2km-k, \\ 1 & \text{if } \ 2km-k\leqslant d\leqslant 2km. \end{cases}$$

Moreover, the map $\mathrm{Sym}^m\,\mathrm{H}^0(R,\omega)\to \mathrm{H}^0(R,\omega^m)$ is surjective.

Proof. Using the generator $(\frac{du\wedge d\epsilon}{\epsilon^2})^m$ of ω^m on U, let us identify the sections of ω^m on U with the elements of $\mathbb{C}[x,\epsilon]/(\epsilon^2)$. Consider the following $(2m-1)(g-1)$ sections of ω^m on U:

$$\{u^i\}_{i=0}^{2mk-k-1}, \quad \{u^i + (i-mk)u^{i-k-1}\epsilon\}_{i=k+1}^{2mk}. \tag{7}$$

We claim that these sections are in the image of $\operatorname{Sym}^m \mathrm{H}^0(R,\omega)$. Indeed, for $0 \leqslant i \leqslant k$, the monomial $x_0^{m-1}x_i$ restricts to u^i. For $2mk-k \leqslant i \leqslant 2mk$, the monomial $x_{2k}^{m-1}x_{i+2k-2mk}$ restricts to $u^i + (i-mk)u^{i-k-1}$. For $k < i < 2mk-k$, it suffices to exhibit two monomials $x_{i_1}\cdots x_{i_m}$ with $i_1+\cdots+i_m = i$ whose restrictions to U are linearly independent. This is easy to do using Lemma 3.3; we leave this to the reader.

We conclude that the sections listed in (7) extend to global sections of ω^m. By construction, these global sections are in the image of $\operatorname{Sym}^m \mathrm{H}^0(R,\omega)$. Since these sections are linearly independent and their number equals $h^0(\omega^m)$, they form a basis of $\mathrm{H}^0(\omega^m)$. We conclude that $\operatorname{Sym}^m \mathrm{H}^0(R,\omega) \to \mathrm{H}^0(R,\omega^m)$ is surjective. The sections $u^i(\frac{du\wedge d\epsilon}{\epsilon^2})^m$ are semi-invariants of $\mathbb{G}_m$ with weights $-km,\dots,km-k-1$. The sections $(u^i+(i-mk)u^{i-k-1}\epsilon)(\frac{du\wedge d\epsilon}{\epsilon^2})^m$ are semi-invariants of $\mathbb{G}_m$ with weights $-km+k+1,\dots,km$. Combining the two, we get the dimensions of the weight spaces. □

The following is immediate from Proposition 3.5.

COROLLARY 3.6. *Let $\mathcal{B}$ be a set of monomials of degree m in the variables $x_0,\dots,x_{2k}$. Its image in $\mathrm{H}^0(R,\omega^m)$ forms a basis if and only if*

(i) For $0 \leqslant d \leqslant k$ and $2km-k \leqslant d \leqslant 2km$, $\mathcal{B}$ contains one monomial of u-weight d.

(ii) For $k < d < 2km-k$, $\mathcal{B}$ contains two monomials of u-weight d and these two monomials are linearly independent in $\mathrm{H}^0(R,\omega^m)$.

We recall the following result:

PROPOSITION 3.7. *The following are bases of $\mathrm{H}^0(R,\omega^2)$:*

$$\mathcal{B}^+ := \{x_0x_i\}_{i=0}^{2k} \cup \{x_kx_i\}_{i=1}^{2k-1} \cup \{x_{2k}x_i\}_{i=1}^{2k} \tag{8}$$

$$\mathcal{B}^- := \{x_i^2\}_{i=0}^{2k} \cup \{x_ix_{i+1}\}_{i=0}^{2k-1} \cup \{x_ix_{i+k}\}_{i=1}^{k-1} \cup \{x_ix_{i+k+1}\}_{i=0}^{k-1}. \tag{9}$$

Both $\mathcal{B}^+$ and $\mathcal{B}^-$ are symmetric with respect to the $\mathbb{Z}_2$-symmetry of R and consist of $\mathbb{G}_m$-semi-invariant sections. The breakdown of $\mathcal{B}^+$ by u-weight in the range $0 \leqslant d \leqslant 2k$ is:

$$\begin{array}{ll} x_0x_d & \text{for } 0 \leqslant d \leqslant k \\ x_0x_d,\ x_kx_{d-k} & \text{for } k < d \leqslant 2k. \end{array}$$

The breakdown of $\mathcal{B}^-$ by u-weight in the range $0 \leqslant d \leqslant 2k$ is:

$$\begin{array}{ll} x_{\lfloor d/2\rfloor}x_{\lceil d/2\rceil} & \text{for } 0 \leqslant d \leqslant k \\ x_{\lfloor d/2\rfloor}x_{\lceil d/2\rceil},\ x_{\lfloor (d-k)/2\rfloor}x_{\lceil (d+k)/2\rceil} & \text{for } k < d \leqslant 2k. \end{array}$$

The breakdown in the range $2k \leqslant d \leqslant 4k$ is obtained by using the $\mathbb{Z}_2$-symmetry.

Proof. The fact that $\mathcal{B}^+$ and $\mathcal{B}^-$ are bases of $\mathrm{H}^0(R,\omega^2)$ is the content of [AFS13, Lemma 4.3]. The weight decomposition statement is obvious. □

We record a simple observation about expressing arbitrary quadratic monomials in $\mathrm{H}^0(R,\omega^2)$ in terms of the monomials of $\mathcal{B}^-$ (it will be used repeatedly in Section 4.2):

Lemma 3.8 (Quadratic equations). *Consider $0 \leqslant i \leqslant j \leqslant 2k$ and set $d = i+j$. Then in $\mathrm{H}^0(R,\omega^2)$ we have a relation*

$$x_i x_j = \lambda x_{\lfloor d/2 \rfloor} x_{\lceil d/2 \rceil} + \mu x_{\lfloor (d-k)/2 \rfloor} x_{\lceil (d+k)/2 \rceil},$$

where λ and μ are uniquely determined rational numbers. In addition, λ and μ satisfy:

(i) $\lambda + \mu = 1$,
(ii) *if $j \leqslant k$ or $i \geqslant k$, then $\mu = 0$,*
(iii) *if $j - i = k$ or $j - i = k+1$, then $\lambda = 0$;*
(iv) *if $j - i < k$, then λ, $\mu > 0$;*
(v) *if $j - i > k+1$, then $\lambda < 0$, $\mu > 0$.*

Proof. The existence and uniqueness of the relation follows from Proposition 3.7. We now establish the claims about the coefficients for $k < d < 3k$, the remaining cases being clear. By the $\mathbb{Z}_2$-symmetry, we may take $k < d \leqslant 2k$. If $j \leqslant k$, the statement is clear. If $j > k$, then

$$\begin{aligned} x_{\lfloor d/2 \rfloor} x_{\lceil d/2 \rceil} &= u^d, \\ x_{\lfloor (d-k)/2 \rfloor} x_{\lceil (d+k)/2 \rceil} &= u^d + \lceil (d-k)/2 \rceil u^{d-k} \epsilon, \\ x_i x_j &= u^d + (j-k) u^{d-k} \epsilon. \end{aligned}$$

Now, (i) follows from equating the coefficients of u^d. If $j - i = k$ or $j - i = k+1$, then $(i,j) = (\lfloor (d-k)/2 \rfloor, \lceil (d+k)/2 \rceil)$; so (iii) follows. Finally, (iv) and (v) follow from equating the coefficients of $u^{d-k}\epsilon$ and observing that if $j - i < k$, then $j - k < \lceil (d-k)/2 \rceil$, and if $j - i > k+1$, then $j - k > \lceil (d-k)/2 \rceil$. ☐

4. Monomial bases of cosyzygies

In this technical heart of the paper, we introduce *monomial bases of cosyzygies* for a canonical Gorenstein curve C. These correspond to Plücker coordinates of the 1^{st} syzygy point of C with respect to a fixed basis of $\mathrm{H}^0(C,\omega_C)$ and thus can be used in verifying semi-stability of the 1^{st} syzygy point of C via the Hilbert-Mumford criterion. We then construct three particular monomial bases of cosyzygies $\mathcal{C}^+, \mathcal{C}^-$, and $\mathcal{C}^\star$ for the balanced ribbon R that will be used in the proof of Theorem 5.4. This construction is done in Subsections 4.1, 4.2, and 4.3, respectively.

Let C be a canonical Gorenstein curve of genus C. Set $\mathrm{H}^0(\omega) := \mathrm{H}^0(C,\omega_C)$ and let $\Gamma := \Gamma_1(C)$ be defined as in (3). For the 1^{st} syzygy point, the relevant strand of the Koszul complex is

$$0 \to \Gamma \xrightarrow{f_{2,1}} \mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2) \xrightarrow{f_{1,2}} \mathrm{H}^0(\omega^3) \to 0. \tag{10}$$

Fix a basis $\{x_0, \dots, x_{g-1}\}$ of $\mathrm{H}^0(\omega)$. For $x, y, z \in \{x_0, \dots, x_{g-1}\}$, we call the image of $(x \wedge y) \otimes z$ in Γ a *cosyzygy*. With this convention, the only linear relations among cosyzygies in Γ are

$$(x \wedge y) \otimes z + (y \wedge z) \otimes x + (z \wedge x) \otimes y = 0.$$

By Definition 2.3, the 1^{st} syzygy point of C is well-defined if and only if $K_{1,2}(C) = 0$ if and only if the map $\Gamma \to \ker f_{1,2}$ induced by the complex (10) is surjective.

Definition 4.1. A set $\mathcal{C} = \{(x_a \wedge x_b) \otimes x_c\}_{(a,b,c) \in S} \subset \Gamma$ is called a *monomial basis of cosyzygies* if $\{f_{2,1}\big((x_a \wedge x_b) \otimes x_c\big)\}_{(a,b,c) \in S}$ form a basis of $\ker f_{1,2}$.

Suppose that $K_{1,2}(C) = 0$, so that the 1^{st} syzygy point $\mathrm{CoSyz}_1(C)$ is well-defined. Let $T \subset \mathrm{SL}(\mathrm{H}^0(\omega))$ be the maximal torus acting diagonally on the fixed basis $\{x_0, \dots, x_{g-1}\}$ of

$\mathrm{H}^0(\omega)$. We obtain a distinguished basis of Γ consisting of the T-eigenvectors $(x_a \wedge x_b) \otimes x_c$. Then the monomial bases of cosyzygies of C correspond precisely to the non-zero Plücker coordinates of $\mathrm{CoSyz}_1(C) \in \mathrm{Grass}\,((3g-5)(g-1), \Gamma)$ with respect to this basis of eigenvectors in Γ. To every such coordinate, and in turn, to every monomial basis $\mathcal{C}$, we can associate a T-character, called the *T-state of* $\mathcal{C}$. We may represent the T-state as a linear combination of $x_0, \dots, x_{g-1}$. Precisely, the T-state of $\mathcal{C} = \{(x_a \wedge x_b) \otimes x_c\}_{(a,b,c)\in S}$ is given by

$$w_T(\mathcal{C}) := \sum_{(a,b,c)\in S} w_T\big((x_a \wedge x_b) \otimes x_c\big) = \sum_{(a,b,c)\in S} (x_a + x_b + x_c) = n_0 x_0 + \cdots + n_{g-1} x_{g-1},$$

where n_i is the number of occurrences of x_i among the cosyzygies in $\mathcal{C}$. Note that we always have

$$\sum_{i=0}^{g-1} n_i = 3(3g-5)(g-1).$$

Remark 4.2. Recall from Equation (4) that

$$\dim \ker f_{1,2} = (3g-5)(g-1).$$

Therefore, a set $\mathcal{C} = \{(x_a \wedge x_b) \otimes x_c\}_{(a,b,c)\in S} \subset \Gamma$ is a monomial basis of cosyzygies if and only if the following two conditions are satisfied

(i) $\mathcal{C}$ has $(3g-5)(g-1)$ elements,

(ii) $\big\{f_{2,1}\big((x_a \wedge x_b) \otimes x_c\big) = x_b \otimes x_a x_c - x_a \otimes x_b x_c\big\}_{(a,b,c)\in S}$ span $\ker f_{1,2}$.

From now on, we assume that $C = R$ is the balanced ribbon of genus $g = 2k+1$ and $\{x_0, \dots, x_{2k}\}$ is the basis of $\mathrm{H}^0(\omega)$ described in Proposition 3.1. Our goal for the rest of this section is to construct three monomial bases of cosyzygies for R that will be used in Section 5 to establish semi-stability of $\mathrm{CoSyz}_1(R)$.

Notation: The following terminology will be in force throughout the rest of the paper. We define the *u-degree* of a cosyzygy $(x_a \wedge x_b) \otimes x_c \in \Gamma$ to be $a+b+c$ and define the *level* of a tensor $x_a \otimes x_b x_c \in \mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$ to be a. To lighten notation, we often use $(x_a \wedge x_b) \otimes x_c$ to denote $f_{2,1}\big((x_a \wedge x_b) \otimes x_c\big) = x_b \otimes x_a x_c - x_a \otimes x_b x_c$ in $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$.

For $\alpha \in \mathbb{Q}$, set $\{\alpha\} = \lfloor \alpha + \frac{1}{2} \rfloor$. In other words, $\{\alpha\}$ is the integer closest to α. Observe that for $n \in \mathbb{Z}$, we have

$$n = \lfloor n/3 \rfloor + \{n/3\} + \lceil n/3 \rceil.$$

We use $\langle S \rangle$ to denote the linear span of elements in a subset S of a vector space.

Outline of the construction: We first describe our strategy for constructing monomial bases of cosyzygies for the balanced ribbon R. From Definition 4.1, a set $\mathcal{C} = \big\{(x_a \wedge x_b) \otimes x_c\big\}_{(a,b,c)\in S} \subset \Gamma$ of $(3g-5)(g-1)$ cosyzygies is a monomial basis of cosyzygies if and only if the images $f_{2,1}\big((x_a \wedge x_b) \otimes x_c\big)$, for $(a,b,c) \in S$, span $\ker f_{1,2}$. The first step in our construction is to write down a set $\mathcal{C}$ of $(3g-5)(g-1)$ cosyzygies. We do this heuristically.

Next, we make the following observation. Since $\operatorname{im} f_{2,1} \subseteq \ker f_{1,2}$ and $f_{1,2}$ is surjective onto $\mathrm{H}^0(\omega^3)$, to prove that the images of the cosyzygies in $\mathcal{C}$ span $\ker f_{1,2}$, it suffices to show that

$$\dim \big(\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)\big) / \langle f_{2,1}((x_a \wedge x_b) \otimes x_c) \rangle_{(a,b,c)\in S} \leqslant \dim \mathrm{H}^0(\omega^3) = 5(g-1).$$

In order to do this, we treat

$$f_{2,1}\big((x_a \wedge x_b) \otimes x_c\big) = x_b \otimes x_a x_c - x_a \otimes x_b x_c$$

as a relation among the elements of $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$. We therefore reduce to showing that the relations imposed by $\mathcal{C}$ reduce the dimension of $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$ to at most $5(g-1)$.

The final observation is that all of our results and constructions respect the $\mathbb{G}_m$-action on R described in Equation (6). In particular, we can run our argument by u-degree. This observation greatly simplifies our task because the relevant weight spaces have small dimensions. In particular, by Proposition 3.5 we have

$$\dim \mathrm{H}^0(\omega^3)_d = \begin{cases} 1 & \text{if } 0 \leqslant d \leqslant k \text{ or } 5k \leqslant d \leqslant 6k, \\ 2 & \text{if } k < d < 5k. \end{cases} \tag{11}$$

4.1 A construction of the first monomial basis

We define $\mathcal{C}^+$ to be the union of the following sets of cosyzygies:

(T1) $(x_0 \wedge x_i) \otimes x_j$, where $i \neq 0, 2k$ and $j \neq 2k$.

(T2) $(x_0 \wedge x_i) \otimes x_{2k}$, where $1 \leqslant i \leqslant k-1$.

(T3) $(x_0 \wedge x_{2k}) \otimes x_i$, where $i \leqslant k-1$.

(T4) $(x_{2k} \wedge x_i) \otimes x_j$, where $i \neq 0, 2k$ and $j \neq 0$.

(T5) $(x_{2k} \wedge x_0) \otimes x_i$, $i \geqslant k+1$.

(T6) $(x_{2k} \wedge x_i) \otimes x_0$, $k+1 \leqslant i \leqslant 2k-1$.

(T7) $(x_k \wedge x_i) \otimes x_j$, where $i \neq 0, k, 2k$ and $j \neq 0, 2k$.

(T8) $(x_k \wedge x_0) \otimes x_{2k}$ and $(x_k \wedge x_{2k}) \otimes x_0$.

(T9) $(x_i \wedge x_{k+i}) \otimes x_{k-i}$ where $1 \leqslant i \leqslant k-1$.

(T10) $(x_{2k-i} \wedge x_{k-i}) \otimes x_{k+i}$, $1 \leqslant i \leqslant k-1$

Proposition 4.3. *$\mathcal{C}^+$ is a monomial basis of cosyzygies for R with T-state*

$$w_T(\mathcal{C}^+) = (g^2-1)(x_0 + x_k + x_{2k}) + (6g-6) \sum_{i \neq 0,k,2k} x_i.$$

Proof. Notice that $\mathcal{C}^+$ contains precisely $(3g-5)(g-1)$ cosyzygies and that it is invariant under the $\mathbb{Z}_2$-involution of the ribbon described in Remark 3.2.

To calculate the T-state of $\mathcal{C}^+$, observe that x_0, x_k, x_{2k} each appear g^2-1 times, and x_i, for every $i \neq 0, k, 2k$, appears $6g-6$ times. It follows that

$$w_T(\mathcal{C}^+) = (g^2-1)(x_0 + x_k + x_{2k}) + (6g-6) \sum_{i \neq 0,k,2k} x_i.$$

We now verify that $\mathcal{C}^+$ is a monomial basis of cosyzygies. In view of the $\mathbb{Z}_2$-symmetry and the dimensions of $\mathrm{H}^0(\omega^3)_d$ from (11), we only need to verify that the quotient space $\big(\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)\big)_d / \langle \mathcal{C}^+ \rangle_d$ is at most one-dimensional in u-degrees $0 \leqslant d \leqslant k-1$, and at most two-dimensional in u-degrees $k \leqslant d \leqslant 3k$.

The key player in our argument is the monomial basis $\mathcal{B}^+$ from Proposition 3.7:

$$\mathcal{B}^+ = \Big\{ \{x_0 x_i\}_{i=0}^{2k},\ \{x_k x_i\}_{i=1}^{2k-1},\ \{x_{2k} x_i\}_{i=1}^{2k} \Big\}. \tag{12}$$

Tensoring $\mathcal{B}^+$ with the standard basis $\{x_0,\dots,x_{2k}\}$ of $\mathrm{H}^0(\omega)$, we obtain the following basis of $\mathrm{H}^0(\omega)\otimes\mathrm{H}^0(\omega^2)$:

$$\mathcal{B} := \{x_a\otimes\mathfrak{m}\colon\ 0\leqslant a\leqslant 2k, \mathfrak{m}\in\mathcal{B}^+\}$$

Our argument now proceeds by u-degree:

Degree $0\leqslant d\leqslant k$. We have $\langle\mathcal{B}\rangle_d = \langle x_a\otimes x_0x_{d-a}\colon$ where $0\leqslant a\leqslant d\rangle$. Evidently, we have $x_ax_{d-a}=x_0x_d$ in $\mathrm{H}^0(\omega^2)$. It follows that

$$x_a\otimes x_0x_{d-a} = x_0\otimes x_ax_{d-a} + (x_0\wedge x_a)\otimes x_{d-a} = x_0\otimes x_0x_d + (x_0\wedge x_a)\otimes x_{d-a},$$

where $(x_0\wedge x_a)\otimes x_{d-a}$ is a cosyzygy **(T1)**. We conclude that $\langle\mathcal{B}\rangle_d/\langle\mathcal{C}^+\rangle_d$ is spanned by $x_0\otimes x_0x_d$, hence is at most one-dimensional.

Degree $k+1\leqslant d\leqslant 2k$. We have

$$\langle\mathcal{B}\rangle_d = \langle x_a\otimes x_0x_{d-a},\ x_b\otimes x_kx_{d-k-b}\colon\ 0\leqslant a\leqslant d,\ 0\leqslant b< d-k\rangle.$$

If $b\geqslant 1$, using the cosyzygies **(T7)** and **(T1)** and Lemma 3.8, we obtain

$$\begin{aligned}
x_b\otimes x_kx_{d-k-b} &= x_k\otimes x_bx_{d-k-b} + (x_k\wedge x_b)\otimes x_{d-k-b}\\
&= x_k\otimes x_0x_{d-k} + (x_k\wedge x_b)\otimes x_{d-k-b}\\
&= x_0\otimes x_kx_{d-k} + (x_0\wedge x_k)\otimes x_{d-k} + (x_k\wedge x_b)\otimes x_{d-k-b}.
\end{aligned}$$

Using **(T1)**, we also have

$$x_a\otimes x_0x_{d-a} = x_0\otimes x_ax_{d-a} + (x_0\wedge x_a)\otimes x_{d-a},$$

It follows that $\langle\mathcal{B}\rangle_d/\langle\mathcal{C}^+\rangle_d = \langle x_0\otimes x_ax_{d-a}\colon\ 0\leqslant a\leqslant d\rangle/\langle\mathcal{C}^+\rangle_d$. In other words, every tensor of u-degree d is reduced to a tensor of level 0. Since $\dim\langle x_0\otimes x_ax_{d-a}\colon\ 0\leqslant a\leqslant d\rangle = \dim\mathrm{H}^0(\omega^2)_d=2$, we are done.

Degree $2k+1\leqslant d\leqslant 3k-1$. Write $d=2k+i$, $1\leqslant i\leqslant k-1$. It is easy to see that modulo $\mathcal{C}^+$, every tensor in $\left(\mathrm{H}^0(\omega)\otimes\mathrm{H}^0(\omega^2)\right)_d$ can be reduced to a tensor of level 0, k, or $2k$, by using cosyzygies **(T1)**–**(T4)** or **(T7)**. In other words,

$$\langle\mathcal{B}\rangle_d/\langle\mathcal{C}^+\rangle_d = \langle x_0\otimes x_kx_{k+i},\ x_0\otimes x_{2k}x_i,\ x_k\otimes x_0x_{k+i},\ x_k\otimes x_kx_i,\ x_{2k}\otimes x_0x_i\rangle/\langle\mathcal{C}^+\rangle_d.$$

Since $\dim\langle x_i\otimes x_ax_{2k-a}\colon\ 0\leqslant a\leqslant 2k\rangle = \dim\mathrm{H}^0(\omega^2)_{2k}=2$, it suffices to show that every tensor in the above display can be rewritten modulo $\mathcal{C}^+$ as a tensor of level i. First, we observe that

$$\begin{aligned}
x_{2k}\otimes x_0x_i &= x_0\otimes x_{2k}x_i + (x_0\wedge x_{2k})\otimes x_i &&\text{(using } \textbf{(T5)} \text{ cosyzygy)},\\
x_0\otimes x_ix_{2k} &= x_i\otimes x_0x_{2k} - (x_0\wedge x_i)\otimes x_{2k} &&\text{(using } \textbf{(T2)} \text{ cosyzygy)},\\
x_k\otimes x_ix_k &= x_i\otimes x_k^2 - (x_k\wedge x_i)\otimes x_k &&\text{(using } \textbf{(T7)} \text{ cosyzygy)}.
\end{aligned}$$

Since $x_k\otimes x_0x_{k+i} = x_0\otimes x_kx_{k+i} + (x_0\wedge x_k)\otimes x_{k+i}$, it remains to show that $x_0\otimes x_kx_{k+i}$ can be rewritten as a tensor of level i. To this end, we compute

$$\begin{aligned}
x_0\otimes x_kx_{k+i} &= x_{k+i}\otimes x_0x_k - (x_0\wedge x_{k+i})\otimes x_k = x_{k+i}\otimes x_ix_{k-i} - (x_0\wedge x_{k+i})\otimes x_k\\
&= x_i\otimes x_{k+i}x_{k-i} + (x_i\wedge x_{k+i})\otimes x_{k-i} - (x_0\wedge x_{k+i})\otimes x_k,
\end{aligned}$$

where we have used a cosyzygy **(T9)** in the second line.

Degree $d = 3k$. Using cosyzygies **(T1)**, **(T4)**, and **(T7)**, every tensor in $\langle \mathcal{B} \rangle_{3k}$ reduces to a tensor of level 0, k, or $2k$. It follows that

$$\langle \mathcal{B} \rangle_{3k} / \langle \mathcal{C}^+ \rangle_{3k} = \langle x_0 \otimes x_k x_{2k},\ x_k \otimes x_0 x_{2k},\ x_k \otimes x_k^2,\ x_{2k} \otimes x_0 x_k \rangle / \langle \mathcal{C}^+ \rangle_{3k}.$$

Using cosyzygies **(T8)**, we see that $x_0 \otimes x_k x_{2k} = x_k \otimes x_0 x_{2k}$ and $x_{2k} \otimes x_0 x_k = x_k \otimes x_0 x_{2k}$ modulo $\mathcal{C}^+$. It follows that $\langle \mathcal{B} \rangle_{3k} / \langle \mathcal{C}^+ \rangle_{3k}$ is spanned by tensors of level k, hence is at most two-dimensional. □

4.2 A construction of the second monomial basis

We define $\mathcal{C}^-$ to be the union of the following sets of cosyzygies:

(T1) $(x_i \wedge x_j) \otimes x_j$, where $i \notin \{j-k-1, j-k, j, j+k, j+k+1\}$.

(T2) $(x_i \wedge x_{j+1}) \otimes x_j$, where $i > j+1$ or $i = j-k+1$, but $i \neq j+k$ and $i \neq j+k+1$.

(T3) $(x_i \wedge x_{j-1}) \otimes x_j$, where $i < j-1$ or $i = j+k-1$, but $i \neq j-k$ and $i \neq j-k-1$.

(T4) $(x_i \wedge x_j) \otimes x_{j+k}$, where $0 < j < k$ and $i \geqslant k$.

(T5) $(x_i \wedge x_j) \otimes x_{j+k+1}$, where $0 \leqslant j < k$ and $i \geqslant k$.

(T6) $(x_i \wedge x_{j+k}) \otimes x_j$, where $0 < j < k$ and $i < k$.

(T7) $(x_i \wedge x_{j+k+1}) \otimes x_j$, where $0 \leqslant j < k$ and $i < k$.

(T8) $(x_k \wedge x_0) \otimes x_0$

(T9) $(x_k \wedge x_{2k}) \otimes x_{2k}$

(T10) $(x_{\lfloor (d-2k)/3 \rfloor} \wedge x_{\lceil (d+2k)/3 \rceil}) \otimes x_{d - \lfloor (d-2k)/3 \rfloor - \lceil (d+2k)/3 \rceil}$, where $2k \leqslant d \leqslant 4k$ with the following exception: If $k \equiv 1 \pmod 3$ and $d = 2k$, then take instead $(x_0 \wedge x_{\lfloor 4k/3 \rfloor}) \otimes x_{\lceil 2k/3 \rceil}$.

The construction of $\mathcal{C}^-$ is motivated by the following basis of $\mathrm{H}^0(\omega^2)$ from Proposition 3.7:

$$\mathcal{B}^- = \left\{ \{x_j^2\}_{j=0}^{2k}, \{x_j x_{j+1}\}_{j=0}^{2k-1}, \{x_j x_{j+k}\}_{j=1}^{k-1}, \{x_j x_{j+k+1}\}_{j=0}^{k-1} \right\}.$$

After tensoring with $\{x_0, \dots, x_{2k}\}$, the basis above yields the basis of $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$ given by

$$\mathcal{B} := \{x_i \otimes \mathfrak{m} \mid 0 \leqslant i \leqslant 2k,\ \mathfrak{m} \in \mathcal{B}^-\}.$$

Proposition 4.4. *Suppose $k \geqslant 5$, so that $g \geqslant 11$. Then $\mathcal{C}^-$ is a monomial basis of cosyzygies for R with T-state*

$$w_T(\mathcal{C}^-) = (7g-12)(x_0 + x_{2k}) + (7g-15)x_k + (9g-18) \sum_{i \neq 0,k,2k} x_i.$$

Remark 4.5. The exception for $k \equiv 1 \pmod 3$ and $d = 2k$ in **(T10)** is only to get the correct T-state for $\mathcal{C}^-$. One obtains a monomial basis regardless.

Proof. Counting cosyzygies of each type in $\mathcal{C}^-$, we get $12k^2 - 4k = (3g-5)(g-1)$ cosyzygies. The state calculation is also straightforward.

Let Λ be the span in $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$ of all cosyzygies in $\mathcal{C}^-$ and let Λ' be the span in $\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)$ of the cosyzygies **(T1)**–**(T7)**. The relations given by Λ reduce a tensor in $\mathcal{B}$ to a different tensor. For example, modulo **(T1)** we have

$$x_i \otimes x_j^2 = x_j \otimes x_i x_j.$$

Our goal is to show that the quotient $\left(\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)\right)/\Lambda$ is generated by at most one element in degrees $0 \leqslant d \leqslant k$ and $5k \leqslant d \leqslant 6k$, and by at most two elements in degrees $k < d < 5k$.

Proposition 4.6 does most of the heavy lifting towards this goal and, for the sake of the argument, we assume its statement for now.

By Proposition 4.6 and Remark 4.7, $\left(\mathrm{H}^0(\omega)\otimes \mathrm{H}^0(\omega^2)\right)/\Lambda'$ is generated by one element in degrees $0\leqslant d<k$ and $5k<d\leqslant 6k$, by two elements in degrees $k\leqslant d<2k$ and $4k<d\leqslant 5k$ and by three elements in degrees $2k\leqslant d\leqslant 4k$. Therefore to complete the argument, it suffices to prove that the cosyzygies **(T8)** and **(T9)** impose nontrivial linear relations on the two generators in degree k and $5k$, respectively, and that the cosyzygy **(T10)** imposes a nontrivial linear relation on the three generators in degrees $2k\leqslant d\leqslant 4k$.

Let $d=k$. The two generators of $\left(\mathrm{H}^0(\omega)\otimes \mathrm{H}^0(\omega^2)\right)/\Lambda'$ in this degree are

$$\begin{aligned}\sigma_1 &:= x_{\{k/3\}}\otimes x_{\lfloor k/3\rfloor}x_{\lceil k/3\rceil}, \text{ and}\\ \sigma_2 &:= x_k\otimes x_0^2.\end{aligned}$$

The relation imposed by **(T8)** is

$$x_k\otimes x_0^2 = x_0\otimes x_0x_k.$$

It is easy to see that modulo **(T1)**, **(T2)**, and **(T3)**, we have

$$x_0\otimes x_0x_k = x_0\otimes x_{\lfloor k/2\rfloor}x_{\lceil k/2\rceil}=\sigma_1.$$

Therefore, **(T8)** imposes the nontrivial relation

$$\sigma_2=\sigma_1.$$

The case of $d=5k$ follows symmetrically.

Let $2k\leqslant d\leqslant 4k$. The three generators of $\left(\mathrm{H}^0(\omega)\otimes \mathrm{H}^0(\omega^2)\right)/\Lambda'$ in degree d are

$$\begin{aligned}\sigma_1 &:= x_{\{d/3\}}\otimes x_{\lceil d/3\rceil}x_{\lfloor d/3\rfloor},\\ \sigma_2 &:= x_{\lceil (d+2k)/3\rceil}\otimes x_{\lfloor (d-k)/3\rfloor}x_{\{(d-k)/3\}}, \text{ and}\\ \sigma_3 &:= x_{\lfloor (d-2k)/3\rfloor}\otimes x_{\{(d+k)/3\}}x_{\lceil (d+k)/3\rceil}.\end{aligned}$$

For brevity, set $\ell=\lceil (d+2k)/3\rceil$ and $s=\lfloor (d-2k)/3\rfloor$. The relation imposed by **(T10)** is

$$x_\ell\otimes x_s x_{d-\ell-s}=x_s\otimes x_\ell x_{d-\ell-s}. \tag{13}$$

Assume that $d\leqslant 3k$; the case of $d\geqslant 3k$ follows symmetrically. Since $d\leqslant 3k$, we have

$$s<\lfloor (d-k)/3\rfloor \leqslant \{(d-k)/3\}<d-\ell-s\leqslant k.$$

On the left hand side of (13), we have by Lemma 3.8

$$\begin{aligned}x_\ell\otimes x_s x_{d-\ell-s} &= x_\ell\otimes x_{\lfloor (d-k)/3\rfloor}x_{\{(d-k)/3\}}\\ &=\sigma_2.\end{aligned}$$

On the right hand side of (13), working modulo **(T6)**–**(T7)**, and applying Lemma 3.8, we get

$$\begin{aligned}x_s\otimes x_\ell x_{d-\ell-s} &= \lambda x_s\otimes x_{\{(d+k)/3\}}x_{\lceil (d+k)/3\rceil}+\mu x_s\otimes x_{\lfloor (d-s-k)/2\rfloor}x_{\lceil (d-s+k)/2\rceil}\\ &=\lambda\sigma_3+\mu x_{\lceil (d-s+k)/2\rceil}\otimes m, \text{ where } m \text{ is balanced,}\\ &=\lambda\sigma_3+\mu(\alpha\sigma_1+\beta\sigma_2+\gamma\sigma_3),\end{aligned}$$

where the last step uses Proposition 4.6. Furthermore, since $k\geqslant 5$, we have $\lceil (d-s+k)/2\rceil > \lceil (d+2k)/3\rceil$. Hence Proposition 4.6 (Part 2(c)) implies that $\alpha<0$. Thus, **(T10)** imposes the relation

$$\sigma_2=\lambda\sigma_3+\mu(\alpha\sigma_1+\beta\sigma_2+\gamma\sigma_3).$$

If $\mu = 0$, then this relation is clearly nontrivial. If $\mu \neq 0$, the non-vanishing of the coefficient of σ_1 shows that the relation is nontrivial.

Finally, we verify that the exceptional cosyzygy in **(T10)** for $k \equiv 1 \pmod 3$ and $d = 2k$ imposes a nontrivial relation. The argument is almost the same. In this case, the cosyzygy gives

$$x_{\lfloor 4k/3\rfloor} \otimes x_0 x_{\lceil 2k/3\rceil} = x_0 \otimes x_{\lceil 2k/3\rceil} x_{\lfloor 4k/3\rfloor}. \tag{14}$$

Reducing the left hand side of (14) modulo **(T1)**–**(T7)**, we get

$$\begin{aligned} x_{\lfloor 4k/3\rfloor} \otimes x_0 x_{\lceil 2k/3\rceil} &= x_{\lfloor 4k/3\rfloor} \otimes m, \text{ where } m \text{ is balanced,} \\ &= \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3. \end{aligned}$$

Since $\lfloor (d-2k)/3\rfloor \leqslant \lfloor 4k/3\rfloor \leqslant \lceil (d+2k)/3\rceil$, Proposition 4.6 (Part 2(b)) implies that $\alpha > 0$.

Reducing the right hand side of (14), we get

$$\begin{aligned} x_0 \otimes x_{\lceil 2k/3\rceil} x_{\lfloor 4k/3\rfloor} &= \lambda x_0 \otimes x_k^2 + \mu x_0 \otimes x_{\lfloor k/2\rfloor} x_{\lceil 3k/2\rceil}, \text{ where } \lambda, \mu > 0 \\ &= \lambda\sigma_3 + \mu x_{\lceil 3k/2\rceil} \otimes m, \text{ where } m \text{ is balanced,} \\ &= \lambda\sigma_3 + \mu(\alpha'\sigma_1 + \beta'\sigma_2 + \gamma'\sigma_3). \end{aligned}$$

Since $\lceil 3k/2\rceil > \lceil (d+2k)/3\rceil$, Proposition 4.6 (Part 2(c)) implies that $\alpha' < 0$. Thus, **(T10)** imposes

$$\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 = \lambda\sigma_3 + \mu(\alpha'\sigma_1 + \beta'\sigma_2 + \gamma'\sigma_3).$$

Since $\alpha > 0$ whereas $\mu\alpha' < 0$, the relation is nontrivial. □

Before moving onto the key technical results needed in the proof of Proposition 4.4, we introduce some additional terminology. We call the forms x_j^2 and $x_j x_{j+1}$ *balanced* and the forms $x_j x_{j+k}$ and $x_j x_{j+k+1}$ *k-balanced.* Likewise, we call a tensor $x_i \otimes \mathfrak{m}$ *balanced* (resp. *k-balanced*) if $\mathfrak{m}$ is *balanced* (resp. *k-balanced*). Finally, we call a balanced tensor $x_i \otimes \mathfrak{m}$ of degree d *well-balanced* if $\lfloor (d-2k)/3\rfloor \leqslant i \leqslant \lceil (d+2k)/3\rceil$. Equivalently, a balanced tensor $x_i \otimes x_s x_\ell$ is well-balanced if $\max(|i-s|, |i-\ell|) \leqslant k+1$.

Proposition 4.6. *(Part 1) Every element of $\left(\mathrm{H}^0(\omega) \otimes \mathrm{H}^0(\omega^2)\right)/\Lambda'$ can be uniquely expressed as a linear combination of the following tensors:*

$$\text{Type 1}\begin{cases} x_i \otimes x_i^2, & \text{where } 0 \leqslant i \leqslant 2k \\ x_i \otimes x_i x_{i+1}, & \text{where } 0 \leqslant i \leqslant 2k-1 \\ x_i \otimes x_{i-1} x_i, & \text{where } 1 \leqslant i \leqslant 2k \end{cases}$$

$$\text{Type 2}\begin{cases} x_{i+k} \otimes x_i^2, & \text{where } 0 \leqslant i \leqslant k \\ x_{i+k+1} \otimes x_i^2, & \text{where } 0 \leqslant i \leqslant k-1 \\ x_{i+k+1} \otimes x_i x_{i+1}, & \text{where } 0 \leqslant i \leqslant k-1 \end{cases}$$

$$\text{Type 3}\begin{cases} x_{i-k} \otimes x_i^2, & \text{where } k \leqslant i \leqslant 2k \\ x_{i-k-1} \otimes x_i^2, & \text{where } k+1 \leqslant i \leqslant 2k \\ x_{i-k-1} \otimes x_i x_{i-1}, & \text{where } k+1 \leqslant i \leqslant 2k \end{cases}$$

(Part 2) Furthermore, let $2k \leqslant d \leqslant 4k$. Then there is precisely one tensor of degree d of each Type 1–3. Suppose the balanced tensor $\tau = x_i \otimes x_{\lfloor (d-i)/2\rfloor} x_{\lceil (d-i)/2\rceil}$ is expressed as

$$\tau = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3,$$

where σ_t is of Type t. Then,

(a) $\alpha + \beta + \gamma = 1$;

(b) if $\lfloor (d-2k)/3 \rfloor < i < \lceil (d+2k)/3 \rceil$*, then* $\alpha > 0$*,* $\beta \geqslant 0$*,* $\gamma \geqslant 0$ *(well-balanced case);*

(c) if $i > \lceil (d+2k)/3 \rceil$*, then* $\alpha < 0$*,* $\beta > 1$*,* $\gamma \leqslant 0$*;*

(d) if $i < \lfloor (d-2k)/3 \rfloor$*, then* $\alpha < 0$*,* $\beta \leqslant 0$*,* $\gamma > 1$*.*

Remark 4.7. In terms of the u-degree, the list of tensors in Proposition 4.6 can be written more compactly as follows:

(Type 1) $x_{\{d/3\}} \otimes x_{\lfloor d/3 \rfloor} x_{\lceil d/3 \rceil}$ where $0 \leqslant d \leqslant 6k$.
(Type 2) $x_{\lceil (d+2k)/3 \rceil} \otimes x_{\lfloor (d-k)/3 \rfloor} x_{\{(d-k)/3\}}$ where $k \leqslant d \leqslant 4k$.
(Type 3) $x_{\lfloor (d-2k)/3 \rfloor} \otimes x_{\lceil (d+k)/3 \rceil} x_{\{(d+k)/3\}}$ where $2k \leqslant d \leqslant 5k$.

Proof. Using the cosyzygies **(T1)–(T7)**, we reduce every element of the basis $\mathcal{B}$ to a linear combination of the tensors of Type 1, 2, and 3. Uniqueness then follows by counting the dimensions.

Step 1 (Reducing k-balanced tensors to balanced tensors): Consider a k-balanced tensor $x_i \otimes x_{\lfloor (d-i-k)/2 \rfloor} x_{\lceil (d-i+k)/2 \rceil}$, where $k \leqslant d-i \leqslant 3k$. Suppose $i \geqslant k$. Then modulo the cosyzygy **(T4)** or **(T5)**, we get

$$x_i \otimes x_{\lfloor (d-i-k)/2 \rfloor} x_{\lceil (d-i+k)/2 \rceil} = x_{\lfloor (d-i-k)/2 \rfloor} \otimes x_i x_{\lceil (d-i+k)/2 \rceil}.$$

Since $i \geqslant k$ and $\lceil (d-i+k)/2 \rceil \geqslant k$, the form $x_i x_{\lceil (d-i+k)/2 \rceil}$ equals a balanced form in $\mathrm{H}^0(\omega^2)$ by Lemma 3.8. The case of $i < k$ is analogous using cosyzygies **(T6)** or **(T7)**.

Step 2 (Reducing balanced tensors to well-balanced tensors): Consider a balanced tensor $x_i \otimes x_{\lfloor (d-i)/2 \rfloor} x_{\lceil (d-i)/2 \rceil}$ that is not well-balanced. For brevity, set

$$s = \lfloor (d-i)/2 \rfloor, \quad \ell = \lceil (d-i)/2 \rceil.$$

Assume that $i > \lceil (d+2k)/3 \rceil$ (the case of $i < \lfloor (d-2k)/3 \rfloor$ follows symmetrically). We then have $i - s > k+1$ and hence $i > \ell + k > \ell$. Modulo the cosyzygy **(T1)** or **(T2)**, we get

$$x_i \otimes x_s x_\ell = x_\ell \otimes x_s x_i.$$

By Lemma 3.8, we have

$$x_s x_i = \lambda m_1 + \mu m_1',$$

where m_1 is balanced, m_1' is k-balanced, and $\lambda + \mu = 1$. Since $i - s > k+1$, we also have $\lambda < 0$. Reducing the k-balanced tensor $x_\ell \otimes m_1'$ as in Step 1, we get

$$\begin{aligned} x_\ell \otimes m_1' &= x_\ell \otimes x_{\lfloor (d-\ell-k)/2 \rfloor} x_{\lceil (d-\ell+k)/2 \rceil} \\ &= x_{\lceil (d-\ell+k)/2 \rceil} \otimes m_2 \quad \text{modulo } \textbf{(T6)} \text{ or } \textbf{(T7)} \end{aligned}$$

where m_2 is balanced. We thus get an expression

$$x_i \otimes x_s x_\ell = \lambda x_\ell \otimes m_1 + \mu x_{\lceil (d-\ell+k)/2 \rceil} \otimes m_2, \tag{15}$$

where m_1 and m_2 are balanced, $s = \lfloor (d-i)/2 \rfloor$, $\ell = \lceil (d-i)/2 \rceil$, $\lambda + \mu = 1$, $\lambda < 0$.

Note that we have the inequalities

$$\begin{aligned} &\lfloor (d-2k)/3 \rfloor \leqslant \ell \leqslant \lceil (d+2k)/3 \rceil, \text{ and} \\ &\lceil (d+2k)/3 \rceil \leqslant \lceil (d-\ell+k)/2 \rceil < i. \end{aligned}$$

In other words, the first tensor on the right in (15) is already well-balanced and the second is strictly closer to being well-balanced than the original tensor. By repeated application of (15), we arrive at a linear combination of well-balanced tensors.

Step 3 (Reducing the well-balanced tensors): We now show that all well-balanced tensors reduce to linear combinations of tensors of Type 1, 2, and 3. We will make use of the following result.

Lemma 4.8. *Let $\tau = x_i \otimes x_{\lfloor (d-i)/2 \rfloor} x_{\lceil (d-i)/2 \rceil}$ be a well-balanced tensor of degree d. Modulo* **(T1)–(T7)***, we have a reduction*

$$\tau = \lambda \tau_1 + \mu \tau_2, \tag{16}$$

where τ_1 and τ_2 are well-balanced, $\lambda + \mu = 1$, and $\lambda,\ \mu \geqslant 0$. Moreover, if τ is not of Type 2 or 3, then $\lambda > 0$. And, if τ is not of Type 1, 2, or 3, then $\tau_1 = x_j \otimes x_{\lfloor (d-j)/2 \rfloor} x_{\lceil (d-j)/2 \rceil}$, where $|\{d/3\} - j| < |\{d/3\} - i|$, and τ_1 is not of Type 2 or 3.

Proof of the lemma. Let $\tau = x_i \otimes x_{\lfloor (d-i)/2 \rfloor} x_{\lceil (d-i)/2 \rceil}$. For brevity, set $s = \lfloor (d-i)/2 \rfloor$ and $\ell = \lceil (d-i)/2 \rceil$. If $i = \ell$ or $i = s$, then τ is of Type 1. In this case, we take $\tau_1 = \tau$ and $\lambda = 1,\ \mu = 0$. If both $x_i x_\ell$ and $x_i x_s$ are k-balanced, then τ is of Type 2 or 3. In this case, we take $\tau_2 = \tau$ and $\lambda = 0, \mu = 1$. Suppose neither of these is the case. We consider the case of $i > \ell$; the case of $i < s$ follows symmetrically. Note that ℓ satisfies

$$\{(d-k)/3\} \leqslant \ell \leqslant \{(d+k)/3\}. \tag{17}$$

We first treat the special case $i = s + k$. Since not both $x_i x_\ell$ and $x_i x_s$ are k-balanced, we must have $s = \ell - 1$. Therefore, we get

$$\begin{aligned}
\tau &= x_i \otimes x_{\ell-1} x_\ell \\
&= x_{\ell-1} \otimes x_i x_\ell \quad \text{modulo } \textbf{(T3)} \\
&= \lambda x_{\ell-1} \otimes m_1 + \mu x_{\ell-1} \otimes x_{\ell+k} x_{\ell-1}, \\
&\qquad \text{where } m_1 \text{ is balanced, } \lambda > 0,\ \mu \geqslant 0, \text{ and } \lambda + \mu = 1 \text{ (Lemma 3.8)}, \\
&= \lambda x_{\ell-1} \otimes m_1 + \mu x_{\ell+k} \otimes x_{\ell-1} x_{\ell-1} \quad \text{modulo } \textbf{(T6)} \\
&= \lambda \tau_1 + \mu \tau_2, \quad \text{as desired.}
\end{aligned}$$

Now assume that $i \neq s + k$. Then $0 < i - \ell \leqslant i - s < k$. In this case, we get

$$\begin{aligned}
\tau &= x_i \otimes x_s x_\ell \\
&= x_\ell \otimes x_s x_i \quad \text{modulo } \textbf{(T1)} \text{ or } \textbf{(T2)}.
\end{aligned}$$

We now write using Lemma 3.8

$$x_s x_i = \lambda m_1 + \mu m_1',$$

where m_1 is balanced, m_1' is k-balanced and $\lambda + \mu = 1$. Since $0 < i - s < k$, we have $\lambda > 0$ and $\mu \geqslant 0$. Reducing the k-balanced tensor $x_\ell \otimes m_1'$ as in Step 1, we get

$$x_\ell \otimes m_1' = x_p \otimes m_2,$$

where m_2 is balanced and

$$p = \begin{cases} \lfloor (d - \ell - k)/2 \rfloor & \text{if } \ell \geqslant k, \\ \lceil (d - \ell + k)/2 \rceil & \text{if } \ell < k. \end{cases}$$

In either case, (17) implies that

$$\lfloor (d-2k)/3 \rfloor \leqslant p \leqslant \lceil (d+2k)/3 \rceil.$$

Setting $\tau_1 = x_\ell \otimes m_1$ and $\tau_2 = x_p \otimes m_2$, we thus get

$$\tau = \lambda \tau_1 + \mu \tau_2,$$

$$\begin{tikzcd} & \boxed{\tau} \arrow[dl, bend right, "\lambda"'] \arrow[dr, bend left, "\mu"] & \\ \boxed{\tau_1} & & \boxed{\tau_2} \end{tikzcd}$$

Figure 1. The relations among well-balanced tensors as a Markov chain

as claimed.

Finally, we note that if $\tau = x_i \otimes x_s x_\ell$ was not of Type 1, 2, or 3, then by construction τ_1 has level j where either $j = s$ in the case of $i = s + k$, or $j = \ell$ in all other cases. In either case, it is clear that $|\{d/3\} - j| < |\{d/3\} - i|$. (Informally, this means that τ_1 is closer to being Type 1 than τ.) This finishes the proof of the lemma. $\square$

We continue the proof of Proposition 4.6. Let Ω be the set of well-balanced tensors. Define a linear operator $P\colon \mathbb{C}\langle\Omega\rangle \to \mathbb{C}\langle\Omega\rangle$ that encodes (16), namely

$$P\colon \tau \mapsto \lambda\tau_1 + \mu\tau_2.$$

By Lemma 4.8, we can interpret P as a Markov process on Ω (see Figure 1). Notice that the absorbing states of this Markov chain are precisely the tensors of Type 1, 2, and 3. Furthermore, from every other tensor, the path $\tau \to \tau_1 \to \dots$ eventually leads to a tensor of Type 1, again by Lemma 4.8. As a result, P is an absorbing Markov chain. By basic theory of Markov chains, for every $v \in \mathbb{C}\langle\Omega\rangle$, the limit $\lim_{n\to\infty} P^n v$ exists and is supported on the absorbing states. Taking $v = 1 \cdot \tau$, we conclude that τ reduces to a linear combination of the absorbing states. We thus get a linear relation

$$\tau = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3,$$

where σ_t is of Type t, as claimed.

The above analysis also lets us deduce the claims about the coefficients from Part 2 of the proposition. Let $2k \leqslant d \leqslant 4k$. Say $\tau = x_i \otimes x_s x_\ell$ reduces as

$$\tau = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3,$$

where σ_t is of Type t.

For Part 2(a), we note that $\alpha + \beta + \gamma = 1$ follows by passing to $\mathrm{H}^0(\omega^3)$ and comparing the coefficients of u^d.

For Part 2(b), assume that $\lfloor (d-2k)/3 \rfloor < i < \lceil (d+2k)/3 \rceil$. Then τ is well-balanced. The non-negativity of P implies the non-negativity of α, β, and γ. Furthermore, since there is a path of positive weight from τ to σ_1, we have $\alpha > 0$.

For Part 2(c), note that if $i = \lceil (d+2k)/3 \rceil$, then $\alpha = 0$, $\beta = 1$, and $\gamma = 0$. For $i > \lceil (d+2k)/3 \rceil$, we show by descending induction on i that $\alpha < 0$ and $\gamma \leqslant 0$. Then since $\alpha + \beta + \gamma = 1$, it follows that $\beta > 1$. For the induction, recall the reduction (15):

$$\tau = \lambda x_\ell \otimes m_1 + \mu x_{\lceil (d-\ell+k)/2 \rceil} \otimes m_2,$$

where the m_i are balanced, $\lambda < 0$, $\mu > 0$, and $\lambda + \mu = 1$. Recall also the inequalities

$$\lfloor (d-2k)/3 \rfloor \leqslant \ell \leqslant \lceil (d+2k)/3 \rceil \text{ and}$$
$$\lceil (d+2k)/3 \rceil \leqslant \lceil (d-\ell+k)/2 \rceil < i.$$

Except in the extreme case $(d,i)=(2k,2k)$, both inequalities in the first line are strict. Say we have the reductions

$$x_\ell \otimes m_1 = \alpha'\sigma_1 + \beta'\sigma_2 + \gamma'\sigma_3, \text{ and}$$
$$x_{\lceil (d-\ell+k)/2\rceil} \otimes m_2 = \alpha''\sigma_1 + \beta''\sigma_2 + \gamma''\sigma_3.$$

By Part 2(b), we have $\alpha' > 0$, and $\gamma' \geqslant 0$. By the inductive assumption, we have $\alpha'' \leqslant 0$, and $\gamma'' \leqslant 0$. Since $\lambda < 0$ and $\mu > 0$ in (4.2), we conclude the induction step. In the extreme case $(d,i)=(2k,2k)$, the reduction (4.2) becomes

$$\tau = \lambda\sigma_3 + \mu x_{\lceil 3k/2\rceil} \otimes m_2.$$

The assertion now follows from that for $x_{\lceil 3k/2\rceil} \otimes m_2$.

Finally, Part 2(d) follows symmetrically from Part 2(c).

□

4.3 A construction of the third (and final!) monomial basis

Let $\mathcal{C}^\star$ be the union of the following sets of cosyzygies:

(S1) The cosyzygies **(T1)**–**(T9)** in the description of $\mathcal{C}^-$.

(S2) $(x_{d-k} \wedge x_0) \otimes x_k$ for $2k \leqslant d < 3k$,

(S3) $(x_{2k} \wedge x_0) \otimes x_k$

(S4) $(x_{d-3k} \wedge x_{2k}) \otimes x_k$ for $3k < d \leqslant 4k$.

Proposition 4.9. *$\mathcal{C}^\star$ is a monomial basis of cosyzygies for R with T-state*

$$w_T(\mathcal{C}^\star) = \frac{15g-29}{2}(x_0 + x_{2k}) + (8g-16)x_k + (9g-20)\sum_{i\neq 0,k,2k} x_i$$

Proof. Let Λ' be the span in $\mathrm{H}^0(\omega)\otimes \mathrm{H}^0(\omega^2)$ of the cosyzygies in **(S1)**. Then by Proposition 4.6, the quotient $\left(\mathrm{H}^0(\omega)\otimes \mathrm{H}^0(\omega^2)\right)/\Lambda'$ is generated by one element in degrees $0 \leqslant d \leqslant k$ and $5k \leqslant d \leqslant 6k$, by two elements in degrees $k < d < 2k$ and $4k < d < 5k$, and by three elements in degrees $2k \leqslant d \leqslant 4k$. It suffices to prove that the cosyzygies **(S2)**–**(S4)** impose a nontrivial linear relation among the three generators in degrees $2k \leqslant d \leqslant 4k$.

Let $2k \leqslant d < 3k$. Recall that the three generators in this degree are

$$\begin{aligned}
\sigma_1 &:= x_{\{d/3\}} \otimes x_{\lceil d/3\rceil} x_{\lfloor d/3\rfloor},\\
\sigma_2 &:= x_{\lceil (d+2k)/3\rceil} \otimes x_{\lfloor (d-k)/3\rfloor} x_{\{(d-k)/3\}}, \text{ and}\\
\sigma_3 &:= x_{\lfloor (d-2k)/3\rfloor} \otimes x_{\{(d+k)/3\}} x_{\lceil (d+k)/3\rceil}.
\end{aligned}$$

The relation given by **(S2)** is

$$x_0 \otimes x_{d-k}x_k = x_{d-k} \otimes x_0 x_k.$$

We reduce both sides modulo Λ'. Note that $x_0 \otimes x_{d-k}x_k = x_0 \otimes m_1$ and $x_{d-k}\otimes x_0 x_k = x_{d-k}\otimes m_2$ where the m_i are balanced. Modulo Λ', we have by Proposition 4.6

$$x_0 \otimes m_1 = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\ , \quad \text{and}$$
$$x_{d-k} \otimes m_2 = \alpha'\sigma_1 + \beta'\sigma_2 + \gamma'\sigma_3.$$

The relation imposed by **(S2)** is therefore

$$\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 = \alpha'\sigma_1 + \beta'\sigma_2 + \gamma'\sigma_3. \tag{18}$$

On one hand, since $0 \leqslant \lfloor (d-2k)/3 \rfloor$, Proposition 4.6 implies that $\gamma > 0$, $\alpha \leqslant 0$, and $\beta \leqslant 0$. On the other hand, since $\lfloor (d-2k)/3 \rfloor < d-k$, we either have $\alpha > 0$ (if $d - k < \lceil (d+2k)/3 \rceil$) or $\beta > 0$ (if $\lceil (d+2k)/3 \rceil \leqslant d-k$). In either case, the relation (18) is nontrivial.

The same argument goes through for $d = 3k$.

The case of $3k < d \leqslant 4k$ follows symmetrically. □

We finish this section with an existence result for monomial bases of cosyzygies in low genus.

Proposition 4.10. *As before, R is the balanced ribbon of genus g.*

(i) Suppose $g = 7$. There exists a monomial basis $\mathcal{C}^-$ of cosyzygies for R with T-state

$$w_T(\mathcal{C}^-) = 40x_0 + 42x_1 + 42x_2 + 40x_3 + 42x_4 + 42x_5 + 40x_6.$$

(ii) Suppose $g = 9$. There exists a monomial basis $\mathcal{C}^-$ of cosyzygies for R with T-state

$$w_T(\mathcal{C}^-) = 56x_0 + 60x_1 + 60x_2 + 60x_3 + 56x_4 + 60x_5 + 60x_6 + 60x_7 + 56x_8.$$

Proof. Both statements are proved by explicitly writing down a monomial basis of cosyzygies of R. We provide details only in the case $g = 7$, the case of $g = 9$ being similar. Let $\mathcal{C}^-$ be the following set of cosyzygies:

(i) u-degree 1: $x_0 \wedge x_1 \otimes x_0$.

(ii) u-degree 2: $x_0 \wedge x_2 \otimes x_0,\ x_0 \wedge x_1 \otimes x_1$.

(iii) u-degree 3: $x_0 \wedge x_3 \otimes x_0,\ x_0 \wedge x_1 \otimes x_2,\ x_0 \wedge x_2 \otimes x_1$.

(iv) u-degree 4: $x_0 \wedge x_4 \otimes x_0, x_0 \wedge x_3 \otimes x_1,\ x_1 \wedge x_2 \otimes x_1,\ x_0 \wedge x_2 \otimes x_2$.

(v) u-degree 5: $x_0 \wedge x_5 \otimes x_0,\ x_0 \wedge x_4 \otimes x_1,\ x_1 \wedge x_3 \otimes x_1,\ x_1 \wedge x_2 \otimes x_2,\ x_0 \wedge x_2 \otimes x_3,\ x_0 \wedge x_1 \otimes x_4$.

(vi) u-degree 6: $x_0 \wedge x_6 \otimes x_0,\ x_0 \wedge x_5 \otimes x_1,\ x_1 \wedge x_4 \otimes x_1,\ x_2 \wedge x_3 \otimes x_1,\ x_0 \wedge x_2 \otimes x_4,\ x_1 \wedge x_2 \otimes x_3,\ x_0 \wedge x_3 \otimes x_3,\ x_0 \wedge x_1 \otimes x_5$.

(vii) u-degree 7: $x_0 \wedge x_6 \otimes x_1,\ x_1 \wedge x_5 \otimes x_1,\ x_1 \wedge x_4 \otimes x_2,\ x_2 \wedge x_3 \otimes x_2,\ x_0 \wedge x_3 \otimes x_4,\ x_1 \wedge x_2 \otimes x_4,\ x_1 \wedge x_3 \otimes x_3,\ x_0 \wedge x_1 \otimes x_6,\ x_0 \wedge x_2 \otimes x_5$.

(viii) u-degree 8: $x_2 \wedge x_6 \otimes x_0,\ x_2 \wedge x_5 \otimes x_1,\ x_2 \wedge x_4 \otimes x_2,\ x_0 \wedge x_4 \otimes x_4,\ x_2 \wedge x_3 \otimes x_3,\ x_1 \wedge x_3 \otimes x_4,\ x_1 \wedge x_2 \otimes x_5,\ x_1 \wedge x_4 \otimes x_3,\ x_0 \wedge x_3 \otimes x_5,\ x_0 \wedge x_2 \otimes x_6$.

(ix) u-degree 9: $x_3 \wedge x_6 \otimes x_0,\ x_3 \wedge x_5 \otimes x_1,\ x_4 \wedge x_5 \otimes x_0,\ x_3 \wedge x_4 \otimes x_2,\ x_1 \wedge x_4 \otimes x_4,\ x_2 \wedge x_3 \otimes x_4,\ x_2 \wedge x_5 \otimes x_2,\ x_1 \wedge x_3 \otimes x_5,\ x_1 \wedge x_2 \otimes x_6,\ x_0 \wedge x_3 \otimes x_6$.

(x) In u-degrees $10 - 18$, the cosyzygies are obtained from those above by the $\mathbb{Z}_2$-involution described in Remark 3.2.

A straightforward but tedious linear algebra calculation, which we omit, shows that $\mathcal{C}^-$ is a monomial basis of cosyzygies for the genus 7 balanced ribbon. The T-state of $\mathcal{C}^-$ is clearly

$$40x_0 + 42x_1 + 42x_2 + 40x_3 + 42x_4 + 42x_5 + 40x_6.$$

We note in closing that both parts of the proposition can also be verified by a direct search using a computer. □

5. Semi-stability of the 1^{st} syzygy point

In this section, we prove that the balanced canonical ribbon R has semi-stable 1^{st} syzygy point, thus obtaining our main result. We begin with two preliminary lemmas.

Lemma 5.1. *For the balanced canonical ribbon R of odd genus $g \geqslant 5$, we have $K_{1,2}(R) = 0$.*

Proof. The claim follows from the existence of a single monomial basis of cosyzygies. We exhibited such a basis in Proposition 4.3. □

Remark 5.2. In the terminology of Bayer and Eisenbud [BE95, p.730], the Clifford index of the genus $2k+1$ balanced ribbon R is k. Therefore, Proposition 5.1 is an immediately consequence of Green's conjecture for R, which is still open to the best of our knowledge.

Recall from Section 4 that we have constructed three monomial bases of cosyzygies for R, namely, $\mathcal{C}^+, \mathcal{C}^-$, and $\mathcal{C}^\star$, with the following T-states:

$$
\begin{aligned}
w_T(\mathcal{C}^+) &= (g^2-1)(x_0+x_k+x_{2k}) + (6g-6)\sum_{i\neq 0,k,2k} x_i\,,\\
w_T(\mathcal{C}^-) &= 40x_0+42x_1+42x_2+40x_3+42x_4+42x_5+40x_6, \qquad \text{if } g=7,\\
w_T(\mathcal{C}^-) &= 56x_0+60x_1+60x_2+60x_3+56x_4+60x_5+60x_6+60x_7+56x_8, \qquad \text{if } g=9,\\
w_T(\mathcal{C}^-) &= (7g-12)(x_0+x_{2k}) + (7g-15)x_k + (9g-18)\sum_{i\neq 0,k,2k} x_i, \qquad \text{if } g\geqslant 11,\\
w_T(\mathcal{C}^\star) &= \frac{15g-29}{2}(x_0+x_{2k}) + (8g-16)x_k + (9g-20)\sum_{i\neq 0,k,2k} x_i\,.
\end{aligned}
$$

Lemma 5.3. *Suppose $g \geqslant 5$. Let $\mathcal{C}^+, \mathcal{C}^-$, and $\mathcal{C}^\star$ be the monomial bases of cosyzygies for R, constructed in Section 4. Then the convex hull of the T-states $w_T(\mathcal{C}^+)$, $w_T(\mathcal{C}^-)$, and $w_T(\mathcal{C}^\star)$ contains the barycenter*

$$\frac{3(3g-5)(g-1)}{g}\Big(\sum_{i=0}^{2k} x_i\Big).$$

Proof. Equivalently, we may show that the 0-state is an effective linear combination of $w_T(\mathcal{C}^+)$, $w_T(\mathcal{C}^-)$, and $w_T(\mathcal{C}^\star)$ modulo $\sum_{i=0}^{2k} x_i$. First we deal with the low genus cases. We note that

$$
\begin{aligned}
w_T(\mathcal{C}^+) &= 24\Big(\sum_{i=0}^{2k} x_i\Big) \quad \text{if } g=5,\\
w_T(\mathcal{C}^+) + 6w_T(\mathcal{C}^-) &= 288\Big(\sum_{i=0}^{2k} x_i\Big) \quad \text{if } g=7,\\
w_T(\mathcal{C}^+) + 8w_T(\mathcal{C}^-) &= 528\Big(\sum_{i=0}^{2k} x_i\Big) \quad \text{if } g=9.
\end{aligned}
$$

Assuming $g \geqslant 11$, we have

$$w_T(\mathcal{C}^+) = (g-5)(g-1)(x_0 + x_k + x_{2k}) \pmod{\sum_{i=0}^{2k} x_i},$$

$$w_T(\mathcal{C}^-) = -(2g-6)(x_0 + x_{2k}) - (2g-3)x_k \pmod{\sum_{i=0}^{2k} x_i},$$

$$w_T(\mathcal{C}^\star) = -\frac{3g-11}{2}(x_0 + x_{2k}) - (g-4)x_k \pmod{\sum_{i=0}^{2k} x_i}.$$

Form a positive linear combination L of the last two lines as follows:

$$\begin{aligned} L &:= 6w_T(\mathcal{C}^\star) + (g-3)w_T(\mathcal{C}^-) \\ &= -(2g^2 - 3g - 15)(x_0 + x_k + x_{2k}) \pmod{\sum_{i=0}^{2k} x_i}. \end{aligned}$$

Plainly, the 0-state is a positive linear combination of $w_T(\mathcal{C}^+)$ and L. □

Having established that $\mathrm{CoSyz}_1(R)$ is well-defined in Lemma 5.1, we are now ready to prove our main theorem.

Theorem 5.4. *Let $g \geqslant 5$ be odd. The balanced canonical ribbon R of genus g has SL_g semi-stable 1^{st} syzygy point.*

Proof. Because $\mathrm{H}^0(R, \omega_R)$ is a multiplicity-free representation of $\mathbb{G}_m \subset \mathrm{Aut}(R)$ by Proposition 3.1, it suffices to verify semi-stability of $\mathrm{CoSyz}_1(R)$ with respect to the maximal torus T acting diagonally on the distinguished basis $\{x_0, \dots, x_{2k}\}$ of $\mathrm{H}^0(R, \omega_R)$ described in Proposition 3.1. For a proof of this reduction see [MS11, Proposition 4.7], [AFS13, Proposition 2.4], or [Lun75, Cor. 2 and Rem. 1].

The non-zero Plücker coordinates of $\mathrm{CoSyz}_1(R)$ diagonalizing the action of T are precisely the monomial bases of cosyzygies for R. The T-semi-stability of $\mathrm{CoSyz}_1(R)$ now follows from Lemma 5.3 and the Hilbert-Mumford numerical criterion. □

Corollary 5.5 (Theorem 2.6). *A general canonical curve of odd genus $g \geqslant 5$ has a semi-stable 1^{st} syzygy point.*

Proof. This follows from the fact that R deforms to a smooth canonical curve [Fon93]. □

6. Computer calculations

For any given genus, the semi-stability of any syzygy point of the balanced ribbon can in principle be verified numerically by enumerating all the states and checking that their convex hull contains the trivial state. We did calculations in `Macaulay2` and `polymake` [GS, GJ] that established GIT semi-stability of the 1^{st} syzygy point of the balanced ribbon for $g = 7, 9, 11, 13$ and the 2^{nd} syzygy point for $g = 9, 11$. (Computations for higher genera appear to be impracticable.) The main theorem of this paper (on first syzygies) and these calculations on second syzygies (for small genus) provide the first evidence for feasibility of Farkas and Keel's approach to constructing the canonical model of $\overline{M}_g$.

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