MATH 1103 Homework 10
Due Friday April 20 2018

Homework 10 problems to be turned in. Practice problems are on the next page.

1. Let \( p \) and \( q \) be positive constants. Show that

\[
\int_0^1 x^p (1 - x)^q \, dx = \int_0^1 x^q (1 - x)^p \, dx = 2 \int_0^{\pi/2} \cos^{2p+1}(\theta) \sin^{2q+1}(\theta) \, d\theta.
\]

[Hint for the second equality: \( x = \sin^2 \theta \)]

2. Recall our notation \( P_k = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-1}{2k} \), \( P_0 = 1 \).
   a) Show that \( P_k \leq \frac{1}{\sqrt{k\pi}} \). (Examine the proof of Wallis’ formula for \( \pi \).)
   b) Use a) to prove that \( \sum_{k=0}^{\infty} (-1)^k P_k \) converges.
   c) Find the sum of the series in b). (Hint: \( (-1)^k P_k = \frac{(-1/2)^k}{k} \).)

3. In his first argument that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \), Euler proposed (conjectured) the product formula

\[
\frac{\sin \pi x}{\pi x} = \left(1 - \frac{x^2}{1^2}\right) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \cdots
\]

As evidence for the correctness of this formula, Euler first noted the obvious facts that both sides take the same value when \( x \) is an integer. He then observed that both sides take the same value when \( x = \frac{1}{2} \). How did he know this last fact?

Euler’s product formula was later shown to be correct, but it was controversial at the time. To satisfy his critics, Euler found another proof that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \), which relies on Wallis integrals. In the next three problems you will work through this alternative proof.

4. For integer \( k \geq 0 \), compute \( \int_0^1 \frac{x^{2k+1}}{\sqrt{1-x^2}} \, dx \). (Hint: let \( x = \sin \theta \).)

5. Compute \( \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}} \, dx \) in two ways: a) using a substitution and b) using the power series for \( \arcsin x \) along with problem 4.

6. Problem 5 computes the sum \( \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \). Use this to show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).
Practice Problems with solutions (not to be turned in)

**Practice 1.** Compute the following integrals

a) \[ \int_0^1 \frac{x}{x + 1} \, dx. \]

Substitution: \( u = x + 1 \), so \( x = u - 1 \) and \( dx = du \). Limits \( x = 0, 1 \) become \( u = 1, 2 \), so

\[
\int_0^1 \frac{x}{x + 1} \, dx = \int_1^2 \frac{u - 1}{u} = (u - \log u) \Bigg|_1^2 = 1 - \log 2.
\]

b) \[ \int_0^1 x(1 - x)^{100} \, dx. \]

Substitution: \( u = 1 - x \), \( du = -dx \), limits switch, so

\[
\int_0^1 x(1 - x)^{100} \, dx = \int_0^1 (1 - u)u^{100} = \int_0^1 (u^{100} - u^{101}) = \frac{1}{100} - \frac{1}{101}.
\]

c) \[ \int_0^9 \sqrt{4 - \sqrt{x}} \, dx. \]

Substitution: \( u = 4 - \sqrt{x} \), \( \sqrt{x} = 4 - u \), \( du = -\frac{1}{2\sqrt{x}} \, dx \), \( dx = -2(4 - u) \, du \). \( x = 0, 9 \Rightarrow u = 1, 4. \)

\[
\int_0^9 \sqrt{4 - \sqrt{x}} \, dx = -2 \int_4^1 \sqrt{u}(4 - u) \, du = 2 \int_1^4 4u^{1/2} - u^{3/2} \, du = 2 \left[ \frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_1^4 = 188 \frac{2}{15}.
\]

d) \[ \int \frac{dx}{x^2 + c^2} \quad (c \text{ is a constant}). \]

answer: \( \frac{1}{c} \arctan \left( \frac{x}{c} \right) + \text{constant}. \)

e) \[ \int_{-1}^1 \frac{dx}{x^2 + x + 1}. \]

Completing the square gives

\[ x^2 + x + 1 = \left( x + \frac{1}{2} \right)^2 + \frac{3}{4}. \]

Let \( u = x + \frac{1}{2} \), so \( du = dx \) with new limits \(-1/2, 3/2\), so

\[
\int_{-1}^1 \frac{dx}{x^2 + x + 1} = \frac{2}{\sqrt{3}} \arctan \left( \frac{2u}{\sqrt{3}} \right) \bigg|_{-1/2}^{3/2} = \frac{2}{\sqrt{3}} \left( \arctan(\sqrt{3}) + \arctan(1/\sqrt{3}) \right) = \frac{\pi}{\sqrt{3}},
\]

using the identity \( \arctan(x) + \arctan(1/x) = \frac{\pi}{2} \).
f) \( \int_1^\infty \frac{\log t}{t} \, dt, \quad \int_e^{e^2} \frac{1}{t \log t} \, dt \)  
\( u = \log t \) for both. Answers: \((\log x)^2/2, \log 2\).

g) \( \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx, \quad \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \).

First one: \( u = \cos x \), answer: \( \pi/2 \). Second one: \( x = \pi - u \), solve for the integral, answer: \( \pi^2/4 \).

**Practice 2.** Let \( p > 0 \) be a constant. Show that

\[ \int_0^\infty \frac{dx}{(1 + x^2)^{p+1}} = \int_0^{\pi/2} \cos^{2p} \theta \, d\theta. \]

Let \( x = \tan \theta \), so \( dx = \sec^2 \theta \). When \( x = 0 \) we have \( \theta = 0 \), and \( x \to \infty \) corresponds to \( \theta \to \pi/2 \).

\[ \int_0^\infty \frac{dx}{(1 + x^2)^{p+1}} = \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{(1 + \tan^2 \theta)^{p+1}} = \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{(\sec^2 \theta)^{p+1}} = \int_0^{\pi/2} \frac{d\theta}{\sec^{2p} \theta} = \int_0^{\pi/2} \cos^{2p} \theta \, d\theta. \]

**Practice 3.** Compute \( \int_0^\infty xe^{-x^2} \) and show that \( \int_0^\infty x^2 e^{-x^2} = \frac{1}{2} \int_0^\infty e^{-x^2} \)

First one: Substitute \( u = x^2 \), answer: \( 1/2 \). Second one: \( \int \)-by-parts with \( u = x \) and \( v' = xe^{-x^2} \).

**Practice 4:** Use substitution to show that if \( n \) is a positive integer then

\[ \int_0^{\pi/2} \cos^n(x) \, dx = \int_0^{\pi/2} \sin^n(x) \, dx \quad \text{and} \quad \int_0^\pi \cos^n(x) \, dx = \begin{cases} 2 \int_0^{\pi/2} \cos^n(x) \, dx & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \]

First one: use \( \cos(x) = \sin(\pi/2 - x) \) and let \( u = \pi/2 - x \). Second one: \( u = \pi - \theta \).

**Practice 5:** Suppose \( f(x) \) is an odd function and \( \int_0^1 f = \pi \). Compute

a) \( \int_0^{\pi/2} f(x) \cos x \, dx \),

b) \( \int_{-\pi/2}^{\pi/2} f(x) \cos x \, dx \).

**Practice 6:** Show that the Bessel function \( J_0(x) \) is represented by the integral

\[ J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) \, d\theta. \]

Use the power series for cosine and integrate term-by-term.