Math 210 Linear Algebra
Exam II Solutions

1. (10) Find the kernel of the matrix \( A = \begin{bmatrix} 0 & 4 & 2 \\ 6 & 2 & 7 \\ 3 & 4 & 5 \end{bmatrix} \).

Solution: \( \mathbb{R}(2, 1, -2) \).

2. (15) Find an eigenvector for the matrix \( A = \begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 4 & -1 & -1 \end{bmatrix} \).

Solution: \( P_A(x) = (x - 1)^3 \), evec: \((1, -2, 3)\). Up to scalar, this is the only evec.

3. (15) (a) Find the inverse of the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \).

(b) Use \( A^{-1} \) to solve the system of equations
\[
\begin{align*}
x + 2y + 3z &= 1 \\
x + y + 2z &= 0 \\
x + y + z &= 1.
\end{align*}
\]

Solution: \( A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \) and the solution to the system is \((0, 2, -1)\).

4. (15) Let \( A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \).

(a) Compute \( \ker(A) \).

(b) Determine the rank of \( A \).

Solution: \( \ker A \) is the intersection of the hyperplanes \( x + y + z + 2w = x + y + z + w = 0 \). This can be simplified to the intersection \( x + y + z = w = 0 \).

5. (15) (a) Find a nonzero vector on the intersection of the two planes
\[
\begin{align*}
x + 2y + 2z &= 0 \\
x + 2y + z &= 0.
\end{align*}
\]

Solution: \((2, -3, 2)\)

(b) Find the equation of the plane spanned by \( \mathbf{u} = (1, 2, 2) \) and \( \mathbf{v} = (2, 2, 1) \).

Solution: \( 2x - 3y + 2z = 0 \).

(c) What is the geometric relation between the plane in part (b) and the two planes in part (a)?
**Solution:** The plane in (b) is perpendicular to both of the planes in (a).

6. (15) Find the $3 \times 3$ matrix that rotates by 180 degrees about the line through $(1, -1, 1)$.

**Solution:** A unit vector on the line is $u = 3^{-1/2}(1, -1, 1)$. The matrix of cross-product-by $u$ is

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

whose square is

$$U^2 = \frac{1}{3} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

Since $\sin(\theta) = 0$ and $\cos(\theta) = -1$, the desired matrix is

$$A = I - 2U^2 = -\frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}.$$

7. (15) Let $A$ be the reflection about a plane in $\mathbb{R}^3$ with normal vector $n$, and let $B$ be a $3 \times 3$ rotation matrix. Show that $BAB^{-1}$ is reflection about the plane with normal vector $Bn$.

**Solution:** $BAB^{-1}$ is a product of three orthogonal matrices, hence is orthogonal. Since $An = -n$ we have

$$BAB^{-1}(Bn) = BA(n) = B(-n) = -Bn.$$

If $u$ is in the plane perpendicular to $Bn$ then $B^{-1}u$ is in the plane perpendicular to $u$, and the same calculation shows that $BAB^{-1}u = u$. So $BAB^{-1}$ fixes the plane perpendicular to $Bn$ and negates $Bn$. It follows that $BAB^{-1}$ is reflection about $Bn$. 
