1. You want to land your rocketship on the circle with equation $x^2 + y^2 = 1$. Unfortunately you bought the cheapest rocketship, which can only fly in a straight line with unit speed. It also has plastic wheels, so you will need the smoothest possible landing. Therefore you must fly on one of the tangent lines to the circle. Your present location is the point $P = (a, 0)$ for some $a > 1$.

(a) What should your velocity vector be?
(b) Where will you land?
(c) After how much elapsed time $T$ will you land on the circle?

[Hint: This seems like the natural order for asking the questions, but it might be easier to answer them in reverse order. Also there are two answers for a) and b).]

So that everyone uses the same notation, let $O = (0, 0)$ be the center of the circle and let $Q = (x, y)$ on the circle be your (as yet unknown) landing point(s).

2. Let
$$\gamma(t) = \left( \frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right), \quad -\infty < t < \infty$$

(a) Show that $\gamma(t)$ lies on the circle $C$ with equation $x^2 + y^2 = 1$, for every $t$.
(b) Compute the velocity $\gamma'(t)$ and speed $|\gamma'(t)|$.
(c) Use $\gamma(t)$ to compute the circumference of $C$. (You already know the answer. The point is to show that calculating with $\gamma(t)$ gives the correct answer. )

3. Let $C$ be the curve parametrized by $\gamma(t) = (\sin t, \sin t \cos t)$ for $0 \leq t \leq 2\pi$.

(a) Find an equation for $C$ in square coordinates $xy$.
(b) Plot $\gamma(t)$ for $t = \frac{n\pi}{4}, \ n = 0, 1, 2, 3, 4, 5, 6, 7.$ and sketch the graph of $C$. 

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4. Continuing with $C$ from the previous problem.
   
   (a) Compute the velocity vector $\gamma'(t)$ and the speed $|\gamma'(t)|$.
   
   (b) Write down the integral for the arclength of $C$. Do not try to compute the integral.

5. Let $C$ be the curve parametrized by $\gamma(t) = (e^{-t}\cos t, e^{-t}\sin t)$, for $0 \leq t < \infty$.
   
   (a) Draw a picture of $C$, explaining what happens as $t \to \infty$.
   
   (b) Find the total length of $C$.

6. Find the length of the curve with equation $x^{2/3} + y^{2/3} = 1$, $x \geq 0, y \geq 0$.

7. Find the average $x$-coordinate of one complete turn of the spiral in problem 3 (the time interval is $0 \leq t \leq 2\pi$).

8. We know from one-variable calculus that

   $$\int_0^{2\pi} \cos^{2n} t \, dt = \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} 2\pi.$$ 

   Use this formula to compute the average of the function $x^{2n}$ over the unit circle centered at $(0, 0)$.

9. Compute the center of mass of the arc of a unit circle from the angle zero to a given angle $\alpha$.

10. A function on the Plane is called Harmonic if its average over any circle equals its value at the center of the circle. Which of these functions is harmonic?
    
    (a) $f(x, y) = xy$  
    
    (b) $f(x, y) = x^2$.

    (Hint: Use Practice problem 1.)
Practice Problems: (not to be turned in)

1. Find a parametrization of an arbitrary circle, using square coordinates, then modify it to get a unit-speed parametrization of the same circle.

**Solution:** Let the circle have radius \((a, b)\) and center \(r\). Its equation is 
\[(x - a)^2 + (y - b)^2 = r^2,\]
so it is parametrized by
\[\gamma(t) = (a + r \cos t, b + r \sin t), \quad 0 \leq t \leq 2\pi.\]
This parametrization has speed \(|\gamma'(t)| = r\). To get unit speed we can use instead
\[\gamma(t) = (a + r \cos(t/r), b + r \sin(t/r)), \quad 0 \leq t \leq 2\pi r\]

2. Let \(C\) be the ellipse with equation \(\frac{x^2}{4} + y^2 = 1\) and let \(P\) be the point on \(C\) with coordinates \(P = (\sqrt{3}, 1/2)\).

(a) Parametrize the tangent line to \(C\) at \(P\).

(b) Find an equation for the tangent line to \(C\) at \(P\).

**Solutions:** (a) First parametrize \(C\), say by
\[\gamma(t) = (2 \cos t, \sin t), \quad \gamma'(t) = (-2 \sin t)i + (\cos t)j.\]
We have \(\gamma(\pi/6) = P\), so the vector \(v = \gamma'(\pi/6) = -i + (\sqrt{3}/2)j\) is tangent to \(C\) at \(P\). Therefore the tangent line is parametrized by
\[P + tv = \left(\sqrt{3} - t, \frac{1}{2}(1 + t\sqrt{3})\right) .\]

(b) The vector \(n = (\sqrt{3}/2)i + j\) is perpendicular to \(v\), so the tangent line has equation \((\sqrt{3}/2)x + y = c\) for some \(c\). Taking \((x, y) = (\sqrt{3}, 1/2)\) we get \(c = 2\), so the tangent line has equation \((\sqrt{3}/2)x + y = 2\).
3. Compute the length of the line parametrized by $\gamma(t) = P + tv$, for $a \leq t \leq b$.

**Solution:** The velocity vector is $v$, and its speed $v = v$ is constant. So

$$L = \int_{a}^{b} v \, dt = v(b - a).$$

4. You throw a ball in the $xy$ plane. Its coordinates at time $t$ are

$$x(t) = \frac{\sqrt{3}}{2} t \quad y(t) = \frac{t}{2} - \frac{t^2}{4}.$$

How long is the path of the ball before it lands?

**Solution:** Ignore this.

5. As a wheel rolls on a straight line, a fixed point $P$ on the edge of the wheel will trace out a *cycloid*. If the wheel has unit radius, the location of $P$ is parametrized by $\gamma(t) = (t - \sin t, 1 - \cos t)$. Find the distance travelled by $P$ during one rotation of the wheel.

**Solution:** One rotation happens in the time interval $0 \leq t \leq 2\pi$. the velocity vector of $P$ is $\gamma'(t) = (1 - \cos t)i + (\sin t)j$, and the speed of $P$ is

$$v(t) = \sqrt{(1 - \cos t)^2 + (\sin t)^2} = \sqrt{2 - 2\cos t}.$$

From the half-angle formula $\sin^2(t/2) = \frac{1 - \cos t}{2}$, we get

$$v(t) = 2|\sin(t/2)| = 2\sin(t/2),$$

since $\sin(t/2) \geq 0$ when $0 \leq t \leq 2\pi$. So

$$L = \int_{0}^{2\pi} 2\sin(t/2) \, dt = 8.$$