MATH 2202 Homework 7 with Solutions
Due Friday March 15

Homework to be turned in

1. Let \( f(x, y) = \frac{1}{1 + x^2 + y^2} \).
   (a) Find the critical point of \( f \)
   (b) Compute the degree-two Taylor expansion of \( f \) at the critical point.
   (c) Draw the graphs of \( f \) and its degree-two Taylor expansion.

2. For the functions \( f \) below, find all of the critical point(s) and classify them according to local min, local max, saddle, or degenerate.
   (a) \( f = (\cosh x)(\sin y) \)
   (b) \( f = \cos(xy) \)

3. The two functions below have \((0, 0)\) as a critical point. Show that the Hessian gives no information about this critical point and find another way to decide if \((0, 0)\) is a maximum, minimum or neither.
   (a) \( f(x, y) = x^3y^3 \)
   (b) \( g(x, y) = 4 - 3x^2y^2 \).

4. The functions below have \((0, 0)\) as a critical point. Without taking any partial derivatives, compute quadratic term
   \[
   \frac{1}{2} \left[ f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2 \right]
   \]
   in the Taylor expansion of \( f \) and use it to classify the critical point \((0, 0)\).
   (a) \( f = (1 + x^2 + y^2)^2 \)
   (b) \( g = e^{xy} \)
   (c) \( h = (\cosh x)(\cos y) \).

5. Consider the function \( f(x, y) = x^2 - 4xy + y^2 \). On the line \( y = 0 \) we have \( f(x, 0) = x^2 \) which has a minimum at \( x = 0 \). On the line \( x = 0 \) we have \( f(0, y) = y^2 \) which has a minimum at \( y = 0 \). It may seem plausible that \( f \) has a local minimum at \((0, 0)\), but it does not.
   (a) Explain this by drawing some level curves of \( f \), including the critical curve, along with the lines \( x = 0 \) and \( y = 0 \).
   (b) Explain this by computing the Hessian \( H_f(0, 0) \).
6. Decide whether the following vector fields $F : U \to \mathbb{R}^2$ are conservative, either by showing that $F$ is not conservative or by finding a potential function for $F$. (See practice problem 4.)

(a) $F = (ax + by)i + (cx + dy)j$ ($b \neq c$)

(b) $F = (ax + by)i + (bx + cy)j$

(c) $F = \nabla \phi$, for any smooth positive function $\phi$ on $U$.

7. Suppose $F$ is a vector field on an open set $U$ in the Plane and that $\phi$ and $\psi$ are two potential functions for $F$.

Assuming $U$ is connected, show that $\phi = \psi + c$ for some constant $c$. [Hint: See practice problem 7 and the definition prior to it.]

8. Let $T$ be an equilateral triangle in the Plane and let $P$ be a point inside $T$. Prove that the sum of the distances from $P$ to the sides of the triangle is the same for all interior points $P$.

The sum of the distances to the sides is the same for all points

[Hint: Let $U$ be the interior of $T$ and let $f : U \to \mathbb{R}$ be the function $f(P) = |P\ell_1| + |P\ell_2| + |P\ell_3|$ where $\ell_1, \ell_2, \ell_3$ are the sides of $T$. Use practice problems 6 and 7.]

Practice Problems: (not to be turned in)

1. Find and classify the critical points of $f(x, y) = (x^2 - y^2 - 1)^2 + 4x^2y^2$.

Solution: We compute $f_x = 4x(x^2 + y^2 - 1)$ and $f_y = 4y(x^2 + y^2 + 1)$, so the critical points are $(0, 0)$ and $(\pm 1, 0)$. Continuing, we get

\[
\begin{align*}
f_{xx} &= 4(3x^2 + y^2 - 1) \\
f_{xy} &= 8xy \\
f_{yy} &= 4(x^2 + 3y^2 + 1).
\end{align*}
\]
At these points the Hessian determinants are

\[ H_f(0, 0) = -16 \quad H_f(\pm 1, 0) = 64. \]

So \((0, 0)\) is a saddle and since \(f_{xx}(\pm 1, 0) = 8\), the points \((\pm 1, 0)\) are local minima.

2. Let \(U\) be the part of the Plane with the hyperbola \(xy = 1\) removed, and let \(f : U \to \mathbb{R}\) be the function \(f(x, y) = \frac{1}{1 - xy}\).

(a) Find the critical point of \(f\).

(b) Without taking second partial derivatives, find the degree-two Taylor expansion of \(f\) at the critical point.

(c) Use the Taylor expansion to classify the critical point.

**Solution:**

(a) \(f_x = yf^2\) and \(f_y = xf^2\), and \(f\) is never zero, so \((0, 0)\) is the critical point.

(b) Using the geometric series \(\frac{1}{1 - xy} = 1 + xy + (xy)^2 + \cdots\), so the quadratic term is \(xy\).

(c) \(xy\) is a saddle.

3. Let \(f(x, y) = xy\), defined on the right half-plane \(x > 0\). Find the critical points of \(f\), and classify them according to local min, local max, saddle, or degenerate. Also give the degree two Taylor expansion at the critical point.

**Solution:** \(f_x = (y/x)f, f_y = (\log x)f\). Since \(f > 0\) the unique critical point is \((1, 0)\) and \(f(1, 0) = 1\). Continuing,

\[ f_{xx} = (y^2 - y)\frac{f}{x^2} \quad f_{xy} = (1 + y \log x)\frac{f}{x} \quad f_{yy} = (\log x)^2 f \]

so the Hessian is \(0^2 - 1 = -1\), meaning the critical point is a saddle. The Taylor expansion at \((1, 0)\) is \(f(x + 1, y) \approx 1 - xy\).

4. Decide whether the following vector fields \(\mathbf{F} = fi + gj\) are conservative, either by showing that \(\mathbf{F}\) is not conservative or by finding a potential function for \(\mathbf{F}\).

(a) \(\mathbf{F} = (x + 2y)i + (x - 2y)j\)

(b) \(\mathbf{F} = (x + 2y)i + (2x - y)j\)
(c) \( \mathbf{F} = \frac{y}{x} \mathbf{i} + \log x \mathbf{j} \), where \( x > 0 \).

**Solutions:**

(a) is not conservative because \( g_x - f_y = 1 - 2 \neq 0 \).
(b) is conservative with potential function \( \varphi = \frac{x^2}{2} + 2xy - \frac{y^2}{2} \).
(c) is conservative with potential function \( \varphi = y \log x = \log(x^y) \).

5. Compute the curl of the vector fields

\[ \mathbf{F} = \frac{-yi + xj}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \mathbf{G} = \frac{-yi + xj}{x^2 + y^2} \]

**Solution:** \( \nabla \times \mathbf{F} = \frac{1}{\sqrt{x^2 + y^2}} \), and \( \nabla \times \mathbf{G} = 0 \).

6. Suppose \( U \) is the half-Plane on one side of a line \( \ell \). Let \( f : U \to \mathbb{R} \) be the function \( f(P) = |\ell P| \) giving the distance from \( P \) to \( \ell \). Prove that \( \nabla f = \mathbf{u} \) (constant), where \( \mathbf{u} \) is the unit vector normal to \( \ell \) pointing towards \( U \).

**Solution:** The level curves of \( f \) are lines parallel to \( \ell \), so \( \mathbf{u} \) is normal to the level curves of \( f \). Hence \( \nabla f \) is parallel to \( \mathbf{u} \) at each point \( P \) in \( U \). This means that \( f(P) = \lambda(P)\mathbf{u} \) for some scalar \( \lambda(P) \). Since \( \mathbf{u} \) is a unit vector we have

\[ f_u(P) = \nabla f(P) \cdot \mathbf{u} = \lambda(P)\mathbf{u} \cdot \mathbf{u} = \lambda(P). \]

On the other hand, \( \nabla f(P) \cdot \mathbf{u} = f_u(P) \) is the directional derivative, and by definition

\[ f_u(P) = \lim_{t \to 0^+} \frac{f(P + t\mathbf{u}) - f(P)}{t}. \]

But \( f(P + t\mathbf{u}) = f(P) + t \) since \( \mathbf{u} \) is normal to \( \ell \). So

\[ f_u(P) = \lim_{t \to 0^+} \frac{f(P) + t - f(P)}{t} = 1. \]

Therefore \( \lambda(P) = 1 \) so \( \nabla f = \mathbf{u} \), as claimed. (A similar argument is in example 2 of section 6.4. Note we don’t need coordinates as we did in hw 1 because now we have the Gradient Theorem.)

**Definition:** We say an open subset \( U \) of the Plane is **connected** if any two points can be joined by a differentiable path in \( U \), that is, if for any two points \( P \) and \( Q \) in \( U \)
there is a differentiable function $\gamma : [a, b] \to U$ such that $\gamma(a) = P$ and $\gamma(b) = Q$. For example the whole Plane is connected, the Plane with finitely many points removed is connected, and the Plane with a line removed is not connected.

7. Suppose $U$ is an open connected set in the plane. Prove that if $f : U \to \mathbb{R}$ is differentiable and $\nabla f$ is zero everywhere on $U$ then $f$ is constant. [Hint: Apply the Mean-Value Theorem to the function $f(\gamma(t))$.

**Solution:** Let $P$ and $Q$ be two points in $U$. Since $U$ is connected there is differentiable path $\gamma : [a, b] \to U$ such that $\gamma(a) = P$ and $\gamma(b) = Q$. Applying the Mean-Value Theorem to the function $\phi(t) = f(\gamma(t))$, there is a number $c$ between $a$ and $b$ such that $\phi(b) - \phi(a) = \phi'(c)(b - a)$. Note that $\phi(b) = f(\gamma(b)) = f(Q)$ and $\phi(a) = f(\gamma(a)) = f(P)$. So $f(Q) - f(P) = \phi'(c)(b - a)$.

By the Gradient Chain Rule, we have $\phi'(c) = \nabla f(\gamma(c)) \cdot \gamma'(c)$. This is zero because $\gamma(c)$ is in $U$ and $\nabla f$ is zero everywhere on $U$. So $f(Q) - f(P) = 0$, as was to be shown.