1. Let $F$ be the vector field $F = xj$.
   (a) Draw a picture of $F$.
   (b) Express the integral $\int_C F \cdot T$ integral explicitly for an arbitrary curve $C$ with oriented parameterization $\gamma(t) = (x(t), y(t))$, where $a \leq t \leq b$. Here “explicitly” means in terms of $x, y, x', y'$.

2. Continue with $F = xj$. Use your formula from 1(b) to compute the line integral $\int_C F \cdot T$ over the following ccw curves, then compute the area inside each curve.
   (a) $C$ is the triangle $(0, 0)$ to $(b, 0)$ to $(a, h)$ back to $(0, 0)$, where $b, h$ are positive and $a$ is arbitrary.
   (b) $C$ follows the graph of $y = x^2$ from $(0, 0)$ to $(1, 1)$, then follows the graph of $x = y^2$ from $(1, 1)$ back to $(0, 0)$.

3. Let $C$ be the oriented curve parametrized by $\gamma(t) = (1 + t, 1 - t^2)$ for $0 \leq t \leq 1$. Let $F = -yi + xj$.
   (a) Draw $C$ and the arrows for $F$ at the points on $C$ where $x = 1, \frac{3}{2}, 2$. Use this picture to determine the sign of $\int_C F \cdot T$ without computing the line integral.
   (b) Compute $\int_C F \cdot T$.

4. Let $C$ be the first-quadrant part of the unit circle centered at $(0, 0)$, from $(1, 0)$ to $(0, 1)$. Consider the vector fields
   \[ F = (e^x \sin y)i + (e^x \cos y)j \quad \text{and} \quad G = (e^x \cos y)i + (e^x \sin y)j. \]
   Compute either $\int_C F \cdot T$ or $\int_C G \cdot T$, your choice.
5. In $xy$ coordinates with $O = (0, 0)$, the Vortex vector field is

$$
V_O = \frac{-y \mathbf{i} + x \mathbf{j}}{x^2 + y^2}.
$$

Consider the inward spiral parametrized by $\gamma(t) = (e^{-t} \cos t, e^{-t} \sin t)$, for $t \geq 0$.

(a) Compute the function $V_O(\gamma(t)) \cdot \gamma'(t)$.

(b) Use (a) to compute the line integral of $\int_C V_O \cdot \mathbf{T}$ where $C$ consists of any complete inward turn of the spiral.

6. Use a potential function to compute the line integral of $V_O \cdot \mathbf{T}$ over any curve $C$ travelling from the positive $x$-axis to the negative $x$-axis while staying in the upper half-plane $y \geq 0$.

7. Let $U$ be the Plane with a point $O$ removed. Find the gradient $\nabla \varphi$ of the function $f(P) = |OP|$ (Hint: follow the method of practice problem 6 on hw 7.)

8. The two-dimensional electric field caused by a unit point charge at $O$ is the vector field

$$
E_O(P) = -\frac{\overrightarrow{OP}}{|OP|^2}.
$$

Find a potential function for $E_O$ on the Plane with $O$ removed. (Hint: previous problem and hw 7 problem 4(c).)
Practice Problems: (not to be turned in)

1. Compute \( \int_C \mathbf{F} \cdot \mathbf{T} \) where
   
   (a) \( \mathbf{F} = x \mathbf{i} + y \mathbf{j} \) and \( C \) is the line segment from \((1,0)\) to \((0,2)\)
   
   (b) \( \mathbf{F} = x \mathbf{i} + y \mathbf{j} \) and \( C \) is parametrized by \( \gamma(t) = (\cos t, 2 \sin t), \ 0 \leq t \leq \frac{\pi}{2} \).
   
   (c) \( \mathbf{F} = -y \mathbf{i} + x \mathbf{j} \) and \( C \) as in (a).
   
   (d) \( \mathbf{F} = -y \mathbf{i} + x \mathbf{j} \) and \( C \) as in (b).

Answers:

(a) \( \frac{3}{2} \) (b) \( \frac{3}{2} \) (c) \( 2 \) (d) \( \pi \).

Could you have predicted that (a) and (b) would be the same but (c) and (d) might not be?

2. Compute \( \int_C \mathbf{F} \cdot \mathbf{T} \) where
   
   (a) \( \mathbf{F} = (x + 2y + 1) \mathbf{i} + (2x - y + 1) \mathbf{j} \), \( C \) is the graph of \( y = x^2 \) from \((0,0)\) to \((1,1)\).
   
   (b) \( \mathbf{F} = \cos x \mathbf{i} + \sin y \mathbf{j} \), \( C \) is the ccw unit circle centered at \((0,0)\).
   
   (c) \( \mathbf{F} = ye^x \mathbf{i} + xe^y \mathbf{j} \), \( C \) is the line segment from \((0,0)\) to \((1,2)\).
   
   (d) \( \mathbf{F} = (x + 2y + 1) \mathbf{i} + (2x - y + 1) \mathbf{j} \) and \( C \) is a squiggly path from \((0,1)\) to \((2,0)\).

Solutions: (a) \( 4 \) (b) A potential function is \( \varphi = \sin x - \cos y \), so the integral over any closed curve is zero.
   
   (c) \( \int_0^1 (2te^t + 2te^{2t}) \, dt = (e^2 + 5)/2 \).
   
   (d) A potential function is
   
   \[ \varphi = \frac{x^2}{2} + 2xy - \frac{y^2}{2} + x + y \]

   so the line integral is \( \varphi(2,0) - \varphi(0,1) = 4 - (1/2) = 7/2 \).

3. Let \( C \) be the oriented line segment from \((1,2)\) to \((2,1)\) and let \( \mathbf{F} = -yi + xj \).
   
   (a) Draw \( C \) and the arrows for \( \mathbf{F} \) at the points on \( C \) where \( x = 1, \frac{3}{2}, 2 \). Use this picture to predict the sign of \( \int_C \mathbf{F} \cdot \mathbf{T} \) without computing the line integral.
   
   (b) Compute \( \int_C \mathbf{F} \cdot \mathbf{T} \).
**Solution:** (a) The arrows are pointing generally northwest, while $C$ heads southwest. So the integral should be negative.

(b) $\int_C F \cdot T = -3$.

4. Compute $\int_C F \cdot T$ where

$$F = (\cos x \cos y)i - (\sin x \sin y)j,$$

and $C$ is the graph of $y = x^2$ from $(0,0)$ to $(1,1)$.

**Solution:** Check that $\nabla \times F = 0$. Then look for a potential function $\varphi$. We need

$$\varphi_x = \cos x \cos y \quad \varphi_y = -\sin x \sin y.$$

The first says $\varphi = \sin x \cos y + c(y)$ where $c(y)$ is a function of $y$ alone. Then the second equation becomes $\varphi_y = -\sin x \sin y + c'(y)$. So we can take $c = 0$ and $\varphi = \sin x \cos y$. Now evaluate at the endpoints to get

$$\int_C F \cdot T = \varphi(1,1) - \varphi(0,0) = (\sin 1)(\cos 1).$$

5. In $xy$ coordinates with $O = (0,0)$, the Vortex vector field is

$$V_O = \frac{-yi + xj}{x^2 + y^2}.$$

Use parametrizations to compute the integral $\int_C V_O \cdot T$ over the following paths:

(a) $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$.

(b) The line segment from $(1,0)$ to $(1,1)$ followed by the line segment from $(1,1)$ to $(0,1)$

(c) $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi$.

(d) $\gamma(t) = (\cos t, -\sin t)$ for $0 \leq t \leq \pi/2$.

**Answers:** (a) $\pi/2$  (b) $\pi/2$  (c) $\pi$  (d) $-\pi$.

You can check these by instead computing the line integrals using a potential function on the Plane with the negative $y$-axis removed.