MT310 Homework 5

Solutions

Due Friday, March 12 by 5:00 pm

Exercise 1. Suppose G has two subgroups H, K with $K \triangleleft G$. Let $HK = \{hk : h \in H, k \in K\}$. Prove that HK is a subgroup of G.

Proof. Let h_1k_1 and h_2k_2 be two elements of HK. Then

$$h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h_2^{-1}h_2k_1k_2^{-1}h_2^{-1}.$$

We have $h_1h_2^{-1} \in H$ and since $K \triangleleft G$ we have $h_2k_1k_2^{-1}h_2^{-1} \in K$. Hence $h_1k_1(h_2k_2)^{-1} \in HK$, so HK is a subgroup of G.

□

Exercise 2. For each of the groups $G = S_3, A_4, S_4$, find subgroups H, K with $K \triangleleft G$, such that $G = HK$. Prove your claims.

Solution.

G	H	K
S_3	$\langle(12)\rangle$	$\langle(123)\rangle$
A_4	$\langle(123)\rangle$	K_4
S_4	S_3	K_4
S_4	$\langle(12)\rangle$	A_4

(There are two possible answers for S_4.) □

Exercise 3. This exercise shows how to recognize direct products. Let G be a group with normal subgroups $H \triangleleft G$ and $K \triangleleft G$. Assume that $H \cap K = \{e\}$ and $HK = G$. Prove that $G \simeq H \times K$.

Hint: First show that $hkh^{-1}k^{-1} = e$ for all $h \in H, k \in K$. Then show that the function $f : H \times K \to G$ given by $f(h, k) = hk$ is a group isomorphism.

Proof. For all $h \in H, k \in K$, we have

$$hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = h(kh^{-1}k^{-1}) \in H \cap K,$$

so $hkh^{-1}k^{-1} = e$. This means that $hk = kh$.

Now prove that f is an isomorphism, as follows. For $h, h' \in H$ and $k, k' \in K$, we compute:

$$f(h, k) \cdot f(h', k') = (hk)(h'k') = h(kh')k' = h(h'k)k' = (hh')(kk') = f(hh', kk') = f((h, k) \cdot (h', k')).$$

Hence f is a homomorphism. Since $HK = G$, any element $g \in G$ may be written as $g = hk$ for some $h \in H, k \in K$, and $f(h, k) = hk = g$, so f is surjective. Finally, if $(h, k) \in \ker f$, then $hk = e$, so $h = k^{-1} \in H \cap K = \{e\}$, so $(h, k) = (e, e)$. Hence f is injective. □

Exercise 4. Let G be a nonabelian group of order $2n$, where $n \geq 3$. Suppose there exist elements $a, b \in G$ such that a has order n, b has order 2, and $bab^{-1} = a^{-1}$. Prove that $G \simeq D_n$.

Hint: First show that $G = \{a^ib^j : 0 \leq i < n, 0 \leq j \leq 1\}$, then find elements in D_n analogous to a, b and use all this to define an isomorphism $f : G \to D_n$.

Proof. The subgroup $\langle a\rangle$ has index two in G. It does not contain b, lest $a = bab^{-1} = a^{-1}$, contradicting $n \geq 3$. Hence

$$G = \langle a\rangle \cup \langle a\rangle b = \{a^ib^j : 0 \leq i < n, 0 \leq j \leq 1\}.$$

In D_n, let r be a rotation by $2\pi/n$ and let s be a reflection. Then r has order n and s has order 2. Using a picture, you can check that $srs^{-1} = r^{-1}$. Hence we also have

$$D_n = \{r^i s^j : \ 0 \le i < n, 0 \le j \le 1\}.$$

Define $f : G \to D_n$ by $f(a^i b^j) = r^i s^j$. We must show that it is a group homomorphism. Now

$$a^i b^j a^k b^\ell = \begin{cases} a^{i+k} b^\ell & \text{if } j = 0 \\ a^{i-k} b^{\ell+1} & \text{if } j = 1. \end{cases}$$

So

$$\begin{aligned} f(a^i b^j a^k b^\ell) &= \begin{cases} r^{i+k} s^\ell & \text{if } j = 0 \\ r^{i-k} s^{\ell+1} & \text{if } j = 1 \end{cases} \\ &= r^i s^j r^k s^\ell \\ &= f(a^i b^j) f(a^k b^\ell). \end{aligned}$$

Therefore f is a group homomorphism, as claimed. It is clear that f is a bijection, so it is an isomorphism. □

Exercise 5. Suppose that G is a nonabelian group of order $2p$, where $p > 2$ is a prime.
Prove that $G \simeq D_p$.

Hint: Find elements a, b as in the previous exercise.

Proof. Since G has even order, it contains an element b of order 2. (Proved in exam 1 study problems.) Since G is nonabelian, it cannot have all elements of order 2 (proved in hw), nor can it have an element of order $2p$, lest it be cyclic. Hence G has an element a of order p. The subgroup $\langle a \rangle$ has index two in G, hence is normal, so $bab^{-1} = a^k$ for some k. Conjugating again by b, we have

$$a = b^2 ab^{-2} = (a^k)^k = a^{k^2}.$$

hence $p \mid k^2 - 1$. Since p is prime, we have $p \mid k-1$ or $p \mid k+1$. But if $p \mid k-1$ then $bab^{-1} = a^k = a$ so G would be abelian, which it is not. Hence $p \mid k+1$, so $bab^{-1} = a^{-1}$. By the result in the previous problem, we have $G \simeq D_{2p}$. □

Exercise 6. Let G be a finite group, let $a \in G$, and let A be the conjugacy class of a in G, and let $H = C_G(a)$ be the centralizer of a in G. Prove that

$$|A| = \frac{|G|}{|H|}.$$

Hint: Show that the function $f : G/H \to A$ given by $f(gH) = gag^{-1}$ is a well-defined bijection.

Proof. We first show that f is well-defined. Suppose $gH = kH$. Then $g = kh$ for some $h \in H$. We have $f(gH) = gag^{-1} = khah^{-1}k^{-1}$. Since $h \in H = C_G(a)$, we have $hah^{-1} = a$. Therefore $f(gH) = kak^{-1} = f(kH)$, so f is well-defined. By definition, $A = \{gag^{-1} : \ g \in G\}$. So if $b \in A$ there is $g \in G$ such that $b = gag^{-1}$, and $f(gH) = gag^{-1} = b$. Hence f is surjective. Finally, suppose $f(gH) = f(kH)$, for some $g, k \in G$. Then $gag^{-1} = kak^{-1}$, so $k^{-1}gag^{-1}k^{-1} = a$. This means $k^{-1}g \in H$, so $kH = gH$. Hence f is injective. Since $f : G/H \to A$ is a bijection, we have $|A| = |G/H| = |G|/|H|$. □

Comment: Note that H is not necessarily normal in G. The set G/H is not a group, and f is not a group homomorphism. That's why we could not prove injectivity by showing $\ker f$ is trivial.

Exercise 7. Use the first isomorphism theorem to prove that $S_4/K_4 \simeq S_3$.

Proof. It suffices to find a surjective homomorphism $f : S_4 \to S_3$ with $\ker f = K_4$. We have found such a homomorphism in Exercise 5 of hw 3 (see also Exercise 6 of hw 3). By the first isomorphism theorem, it follows that the map

$$\bar{f} : S_4/K_4 \to S_3$$

given by $\bar{f}(\sigma K_4) = f(\sigma)$, for $\sigma \in S_4$, is an isomorphism. □