MT310 Exam 1 study guide/ problems

Definitions to know:

- group
- subgroup
- order of a group and element of a group
- homomorphism
- isomorphism, homomorphism, kernel, image
- conjugacy class, centralizer
- left and right coset, index of a subgroup.

Groups to know in detail: (Order of the group and its elements, all of the subgroups, conjugacy classes, homomorphisms to other groups, various incarnations.)

- $\mathbb{Z}_n$, cyclic groups
- $\mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_n^\times$, also known as $U(n)$ in the text, and $(\mathbb{Z}/n\mathbb{Z})^\times$ elsewhere.
- S_3 and S_4
- D_4
- A_4.

Permutation Groups: Know how to:

- Calculate products and conjugacy classes in S_n
- Construct and analyze homomorphisms from $f : G \to S_n$, arising from permutation actions of a group G on a set with n-elements.

Practice problems:

1. Prove that any group of order ≤ 5 is abelian.
2. Let H, K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G. Give an example of a specific G, H, K for which $H \cup K$ is not a subgroup of G.
3. Prove that D_3, the symmetry group of an equilateral triangle, is isomorphic to the symmetric group S_3.
4. Find two groups of order 6 which are not isomorphic to each other and prove that they are not isomorphic to each other.
5. Prove that if $|G| = p$ is prime then G is isomorphic to $\mathbb{Z}_p$.
6. The *center* of a group G is defined as $Z(G) = \{a \in G : \; ab = ba \; \forall b \in G\}$. Prove that $Z(G)$ is a subgroup of G. Find $Z(G)$ for all the groups you know.

7. List as many non-isomorphic groups of order 8 as you can, and give conditions for recognizing each of them.

8. Prove that if $|G|$ is even then G has an element of order 2.

9. Prove that if $a, b \in G$ commute and have relatively prime orders m, n, then the order of ab is mn.

10. Suppose $a \in G$ is an element of odd order m. Prove that a^2 also has order m.

11. Suppose $\sigma \in S_n$ is a cycle of odd length m. Prove that σ^2 is also a cycle of length m.

12. If $\sigma \in S_n$ is a cycle of even length $2k$, what is the cycle type of σ^2?

13. Define $f : \mathbb{Z}_9^\times \to \mathbb{Z}_9^\times$ by $f(a) = a^3$, for any $a \in \mathbb{Z}_9^\times$. Prove that f is a well-defined homomorphism and compute $\ker f$ and $\operatorname{im} f$.

14. A subgroup $H < S_n$ is called *transitive* if for all $i, j \in \{1, 2, \ldots, n\}$, there exists $h \in H$ such that $h(i) = j$. Find a transitive subgroup $H < S_6$ which is isomorphic to S_4 and list the cycle types in H. (Hint: S_4 acts on the cube.)