MT310 Exam 2 study guide/ problems

Definitions to know:

- group
- subgroup
- order of a group and element of a group
- conjugacy class and centralizer of element in a group
- left and right coset, index of a subgroup.
- quotient group
- ring
- ideal
- unit in a ring
- quotient ring
- isomorphism, homomorphism, kernel, image (for groups and rings)

Groups to know in detail: (Order of the group and its elements, all of the subgroups, conjugacy classes, homomorphisms to other groups, various incarnations.)

- $\mathbb{Z}_n$, cyclic groups
- $\mathbb{Z}_p \times \mathbb{Z}_p$
- S_3 and S_4
- D_4 and Q_8
- A_4.
- Groups of order $p, p^2, 2^3$

Practice problems:

1. Prove that any group of order ≤ 5 is abelian.
2. Let H, K be subgroups of a group G, with $K \triangleleft G$. Prove that $HK < G$, that $K \triangleleft HK$, that $H \cap K \triangleleft H$ and that $HK/K \simeq H/H \cap K$.
3. List all groups of order 8 and give conditions for recognizing each group. Apply your answers to determine the isomorphism class of the group

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{Z}_2 \right\}.$$

4. Prove that A_4 and the dihedral group D_6 are not isomorphic to each other.

5. Prove from scratch that if $|G| = p^2$, where p is a prime then G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.

6. Prove that if F is a field then F has no ideals other than $\{0\}$ and F.

7. Let R be a commutative ring and let I be an ideal in R. Suppose I contains a unit in R. Prove that $I = R$.

8. Prove that a finite integral domain R is a field. (Hint: Let $a \in R$ and show that the map $L_a : R \to R$ given by $L_a(r) = ar$ is a bijection.)

9. Find all irreducible monic polynomials of degrees $2, 3, 4$ in $\mathbb{Z}_3[x]$.

10. Summarize our methods for proving irreducibility of polynomials in $\mathbb{Q}[x]$.

11. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$:

$$x^4 + 1, \qquad x^3 + x^2 - 2x - 1, \qquad x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1, \qquad x^4 - x^3 - 4x^2 + 4x + 1.$$

These are the minimal polynomials of $\sqrt{i}, \cos(2\pi/7), \cos(2\pi/11)$ and $\cos(2\pi/15)$, respectively.

12. Eisenstein's criterion does not apply directly to the polynomials above. Replace x by $x+1$, $x-2$, $x-2$, $x-1$ respectively, and prove irreducibility using Eisenstein's criterion.

13. Find the minimal polynomial in $\mathbb{Q}[x]$ of the golden ratio $\tau = (1+\sqrt{5})/2$ and use it to prove that $\tau^n = F_n\tau + F_{n-1}$, where F_n is the n^{th} Fibonacci number.