Exercise 1. (1.9) Let \( x = A \mid B, x' = A' \mid B' \) be cuts in \( \mathbb{Q} \). We defined \( x + x' = (A + A') \mid \text{rest of } \mathbb{Q} \).

(a) Show that although \( B + B' \) is disjoint from \( A + A' \), it happens in certain cases that \( \mathbb{Q} \neq (A + A') \cup (B + B') \).

(b) Use (a) to explain why defining \( x + x' = (A + A') \mid (B + B') \) would be incorrect.

(c) Why did we not define \( x \cdot x' = (A \cdot A') \mid \text{rest of } \mathbb{Q} \)?

Solution. put your solution here

Exercise 2. Prove that for each cut \( x \) we have \( x + (-x) = 0^* \). [This is not entirely trivial.]

Solution.

Exercise 3. A multiplicative inverse of a nonzero cut \( x = A \mid B \) is a cut \( y = C \mid D \) such that \( x \cdot y = 1^* \).

(a) If \( x > 0^* \) what are \( C \) and \( D \)?

(b) If \( x < 0^* \) what are \( C \) and \( D \)?

(c) Prove that the multiplicative inverse of \( x \) is unique.

Solution.

Exercise 4. (a) Prove that there does not exist a smallest positive real number.

(b) Is there a smallest positive rational number? (Prove your answer.)

(c) Given a real number \( x \) does there exist a smallest real number \( y \) such that \( y > x \)? (Prove your answer.)

Solution.

Exercise 5. Let \( b = \text{l.u.b.}(S) \), where \( S \) is a bounded nonempty subset of \( \mathbb{R} \).

(a) Given \( \epsilon > 0 \) show that there exists an \( s \in S \) with \( b - \epsilon < s \leq b \).

(b) Can \( s \in S \) always be found so that \( b - \epsilon < s < b \)?

(c) If \( x = A \mid B \) is a cut in \( \mathbb{Q} \), show that \( x = \text{l.u.b.}(A) \).

Exercise 6. Let \( x = A \mid B \) be the cut in \( \mathbb{Q} \) with

\[
A = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}.
\]

Prove that \( x^2 = 2 \). [Hint: Use the previous exercise to show that \( x^2 \) can be neither < 2 nor > 2.]

Solution.