Exercise 1. a) Let \( f \) be a bounded function on a closed interval \([c, d]\). Let
\[
M = \sup_{x \in [c, d]} \{f(x)\}, \quad m = \inf_{x \in [c, d]} \{f(x)\}, \quad M' = \sup_{x \in [c, d]} \{|f(x)|\}, \quad m' = \inf_{x \in [c, d]} \{|f(x)|\}.
\]
Prove that \( M' - m' \leq M - m \).

b) Suppose \( f : [a, b] \to \mathbb{R} \) is integrable. Prove that \(|f|\) is integrable on \([a, b]\) and that
\[
\left| \int_a^b f \right| \leq \int_a^b |f|.
\]

Solution.

Exercise 2. Suppose \( f : [a, b] \to \mathbb{R} \) is continuous except at finitely many points \( c_1, \ldots, c_k \in [a, b] \). Use induction, and the case \( k = 1 \) proved in class, to prove that \( f \) is integrable on \([a, b]\).

Solution.

Exercise 3. Let
\[
f(x) = \begin{cases} 
    \sin(1/x) & \text{for } x \neq 0 \\
    0 & \text{for } x = 0.
\end{cases}
\]

(a) Prove that \( f(x) \) is integrable on \([-1, 1]\).

(b) Prove that there exists a differentiable function \( F(x) \) on \((-1, 1)\) such that \( F'(x) = f(x) \) on \((-1, 1)\).

Solution.

Exercise 4. This exercise is a lemma for part (b) of the next exercise. Suppose \( H : (a, b) \to \mathbb{R} \) is a function satisfying the following conditions.

- \( H \) continuous on \((a, b)\);
- There is a point \( c \in (a, b) \) such that \( H \) is differentiable on \((a, b)\) except possibly at \( c \);
- \( H'(x) = 0 \) for all \( x \in (a, b), x \neq c \).

Prove that \( H \) is constant on \((a, b)\).

Solution.
Exercise 5. (a) Give an example of a pair of functions $f : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ such that $f$ has only finitely many points of discontinuity and $G'(x) = f(x)$ at all points $x \in [a, b]$ where $f$ is continuous, but $\int_a^b f \neq G(b) - G(a)$.

(b) Show that if $f : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ are such that $f$ has only finitely many points of discontinuity, $G'(x) = f(x)$ at all points $x \in [a, b]$ where $f$ is continuous and $G$ is continuous on $[a, b]$ then $\int_a^b f = G(b) - G(a)$.

Solution.