Exercise 1. (1.20 from text) Show that limits are unique. That is, if \((a_n)\) is a sequence of real numbers that converges to a real number \(b\) and also converges to a real number \(b'\) then \(b = b'\).

Solution.

Exercise 2. Suppose \((a_n)\) is a sequence which is bounded above and which satisfies \(a_n \leq a_{n+1}\) for all \(n\). Prove that \((a_n)\) converges.

Solution.

Exercise 3. Suppose \((a_n)\) is a sequence contained in the closed interval \([c, d]\), and that \(a_n \to b\). Complete the proof that \(b \in [c, d]\). (In class we proved that \(b \leq d\), so you only have to prove that \(c \leq b\)).

Solution.

Exercise 4. Show that \(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}} \ldots\) converges and find the limit. (Hint: find a recursive formula and then use induction to find a closed formula for the \(n^{th}\) term.)

Solution.

Exercise 5. Let \(f : [a, b] \to \mathbb{R}\) be a continuous function. Suppose \(c \in [a, b]\) and that \((x_n)\) is a sequence of real numbers contained in \([a, b]\) such that \(x_n \to c\). Prove that \(f(x_n) \to f(c)\).

Solution.

Exercise 6. Consider the continuous function \(f : \mathbb{R} \to \mathbb{R}\) defined by \(f(x) = \frac{2x}{x^2 + 1}\).

(a) Prove that \(f([0, 1]) = [0, 1]\).

(b) Prove that there exists \(c \in [0, 1]\) such that \(f(c) = c\), and compute \(c\).

(c) Define a sequence \((x_n)\) of rational numbers by \(x_1 = f(1/2), x_{n+1} = f(x_n), \) for \(n \geq 1\). Prove that \(x_n \to c\).

Solution.

Exercise 7. Let \((a_n)\) be a sequence such that there are constants \(C > 0\) and \(\lambda \in (0, 1)\) such that \(|a_n - a_{n+1}| < C\lambda^n\) for all \(n\). Prove that \((a_n)\) is a Cauchy sequence, and hence converges.

Solution.