Exercise 1. (a) Prove that the two-sphere $S^2 \subset \mathbb{R}^3$ is not homeomorphic to the plane $\mathbb{R}^2$.

(b) Prove that $S^2$ minus the north pole $(0,0,1)$ is homeomorphic to $\mathbb{R}^2$. [Hint: Consider all lines through $(0,0,1)$.]}

Solution. The next exercise will guide you to a proof of the Continuous Extension Theorem, which says that a uniformly continuous function on a non-closed interval $(a,b)$, $(a,b]$ or $[a,b)$ extends uniquely to a continuous function on $[a,b]$. For simplicity we take the interval to be $(0,1]$.

Exercise 2. (a) Suppose $f : M \to N$ is uniformly continuous. Let $(a_n)$ be a Cauchy sequence in $M$. Prove that $(f(a_n))$ is a Cauchy sequence in $N$.

(b) (A digression) Use (a) to prove that the function $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ is not uniformly continuous.

(c) (Now back to the proof) Suppose $f : (0,1] \to \mathbb{R}$ is uniformly continuous. Let $(a_n) \subset (0,1]$ be any sequence converging to 0 in $\mathbb{R}$. Prove that $f(a_n)$ converges to some $b \in \mathbb{R}$.

(d) Define $g : [0,1] \to \mathbb{R}$ by

$$
g(x) = \begin{cases} 
f(x) & \text{if } x \in (0,1] \\
 b & \text{if } x = 0,
\end{cases}
$$

where $b$ is the number from part (c). Prove that $g(x)$ is continuous on $[0,1]$.

(e) Suppose $h : [0,1] \to \mathbb{R}$ is continuous and $h = f$ on $(0,1]$. Prove that $h = g$ on $[0,1]$.

Solution. □

Discussion: The bisection method. For the next two problems, suppose $f : [a,b] \to \mathbb{R}$ is a continuous real-valued function on a closed interval $[a,b]$, such that $f(a) < 0 < f(b)$. The IVT tells us there exists $c \in (a,b)$ such that $f(c) = 0$, but it does not tell us how to find $c$. The bisection method is an algorithm for approximating $c$.

Set $a_0 = a$, $b_0 = b$ and let $c_0$ be the midpoint of $[a,b]$. If $f(c_0) = 0$, we’ve found a solution so we’re done. If $f(c_0) \neq 0$, define a new interval $[a_1,b_1]$ by:

$$
[a_1,b_1] = \begin{cases} 
[a_0,c_0] & \text{if } f(c_0) > 0 \\
[c_0,b_0] & \text{if } f(c_0) < 0
\end{cases}
$$

so that we again have $f < 0$ at the left endpoint and $f > 0$ at the right endpoint. Let $c_1$ be the midpoint of $[a_1,b_1]$, and repeat. We may eventually get a zero of $f$ at one of the midpoints $c_n$, in which
case we have our solution and the process stops. From now on assume that \( f(c_n) \neq 0 \) for all \( n \). We then have a nested sequence of closed intervals

\[
[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots .
\]

By the *Nested Intervals Theorem* (Thm 34 p.82) the intersection \( \bigcap_{n=0}^{\infty} [a_n, b_n] \) is nonempty.

**Exercise 3.** (a) Prove that the intersection of these intervals consists of just one point:

\[
\bigcap_{n=0}^{\infty} [a_n, b_n] = \{c\},
\]

and that \( c_n \to c \).

(b) Prove that \( f(c) = 0 \). Hint: Assume \( |f(c)| = \epsilon > 0 \) and use uniform continuity of \( f \) on \([a, b]\).

(c) How can you use the algorithm to approximate a fixed-point of a continuous function \( f : [0, 1] \to [0, 1] \)?

**Solution.**

**Exercise 4.** In the bisection method, assume \( a, b \in \mathbb{Q} \).

(a) Prove that \( c_n \in \mathbb{Q} \) for all \( n \in \mathbb{N} \).

(b) Show by example that \( c \) need not be in \( \mathbb{Q} \).

(c) Find \( n \in \mathbb{N} \) such that the rational number \( c_n \) approximates \( c \) to within ten decimal places, for any \( f \).

**Solution.**

**Exercise 5.** Find all of the isometries of \( \mathbb{R} \) (with respect to the standard metric on \( \mathbb{R} \)).

**Solution.**