Math 814 HW 2

September 29, 2007

p. 43: 1,4,6,13,15, p. 54 1, 3 (cos z only). $u(x, y) = x^3 - 3xy^2; u(x, y) = x/(x^2 + y^2)$,

p.43, Exercise 1. Show that the function $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

On the region $U = \mathbb{C} - \{0\}$ we have $\bar{z} = f(z)/z$. If $f(z)$ were analytic at some $w \in U$ then $\bar{z}$, being the product of two functions analytic at $w$, would itself be analytic at $w$, which we know is false.

Consider now $w = 0$. Let $\epsilon > 0$. If $|z| < \epsilon$ then

$$\frac{|f(z) - f(0)|}{|z - 0|} = |z| < \epsilon,$$

so $f(z)$ is analytic at 0.

p.43, Exercise 4. Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.

There are two methods:

$$(\cos z)' = \frac{1}{2}(e^{iz} + e^{-iz})' = \frac{i}{2}(e^{iz} - e^{-iz}) = -\sin z,$$

and

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^n 2n \cdot z^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n - 1)!} = -\sin z.$$ 

It is similar for $(\sin z)'$. 

1
p.43, Exercise 6. Describe the following sets:

\begin{align*}
\{ z : e^z = i \} &= \left( 2\mathbb{Z} + \frac{1}{2} \right) \pi i \\
\{ z : e^z = -1 \} &= \left( 2\mathbb{Z} + 1 \right) \pi i \\
\{ z : e^z = -i \} &= \left( 2\mathbb{Z} - \frac{1}{2} \right) \pi i \\
\{ z : \cos z = 0 \} &= \left( \mathbb{Z} + \frac{1}{2} \right) \pi \\
\{ z : \sin z = 0 \} &= \mathbb{Z} \pi.
\end{align*}

p.43, Exercise 13. Let \( U = \mathbb{C} - \mathbb{R}_{\leq 0} \). Find all analytic functions \( f(z) \) on \( U \) such that \( z = (f(z))^n \).

Every branch of \( \log z \) is of the form \( \log z = \text{Log}(z) + 2k\pi i \) for some \( k \in \mathbb{Z} \), where \( \text{Log}(z) \) is the principal branch. Hence we have

\[ z^{1/n} = e^{\log(z)/n} = e^{(\text{Log}(z)+2k\pi i)/n} = e^{\text{Log}(z)/n} \cdot e^{2k\pi i/n}. \]

The numbers \( e^{2k\pi i/n} \) are precisely the \( n \)th roots of unity; they depend only on the remainder of \( k \) modulo \( n \). They are also the \( n \) distinct powers of \( \zeta = e^{2\pi i/n} \). So the branches of \( z^{1/n} \) on \( U \) are

\[ \zeta^k \cdot e^{\text{Log}(z)/n}, \quad k = 0, 1, \ldots, n - 1 \]

and are all constant multiples of each other.

p.43, Exercise 15. Fix \( r > 0 \). Let \( A \) be the image under \( e^{1/z} \) of the punctured disk \( \{ z : 0 < |z| < r \} \). Describe \( A \).

The image of the punctured disk under \( 1/z \) is the infinite annulus

\[ B = \{ z : r^{-1} < |z| \} \]

and \( A \) is the image of \( B \) under \( e^z \). I claim the image of \( A \) is \( \mathbb{C} - \{ 0 \} \), regardless of \( r \). To see this, we have to prove that for \( w \neq 0 \), the equation

\[ e^z = w \]

has a solution \( z \in B \). Write \( w = |w|e^{i\theta} \). We must find \( z = x + iy \) such that \( x^2 + y^2 > r^{-1} \) and

\[ e^x e^{iy} = |w|e^{i\theta}. \]
So we want
\[ e^x = |w|, \quad y = \theta + 2k\pi, \]
for some \( k \in \mathbb{Z} \). If we take \( x = \log |w| \) and choose \( k \) large enough that
\[ (\log |w|)^2 + (\theta + 2k\pi)^2 > r^{-1}, \]
then \( z = x + iy \) works.

Another way to see this is to write \( z = \rho e^{i\theta} \) and consider the image of rays from the origin with fixed \( \theta \) and \( 0 < \rho < r \). You get spirals that fill up the plane with 0 removed.

Additional Comment: The result shows that any arbitrarily small punctured neighborhood of 0 is sent by \( e^{1/z} \) to the entire punctured plane. The point \( z = 0 \) is called an “essential singularity” of \( e^{1/z} \) and this is an example of the Great Picard Theorem (p. 300 in the text).

**p.54, Exercise 1.** Find the image of \( \{ z : \Re z < 0, |\Im z| < \pi \} \) under \( e^z \).

We have seen that the line \( x = c \) is sent by \( e^z \) to a circle of radius \( e^c \). Any segment of the line of length \( 2\pi \) is sent to the entire circle. The region is made out of such segments, so its image is the punctured disk
\[ \{ w : 0 < |w| < 1 \}. \]

**p.54, Exercise 3.** Discuss the mapping properties of \( \cos z \).

We have
\[ \cos z = \cos x \cosh y - i \sin x \sinh y = u + iv. \]
First consider the image of the vertical lines \( x = a \in [0, 2\pi) \).
- If \( a = 0 \), the image is \([1, \infty)\).
- If \( a = \pi \), the image is \((-\infty, -1]\).
- If \( a = \pi/2 \) or \( a = 3\pi/2 \), the image is \(i\mathbb{R}\).
- If \( a \) is none of the above, then
\[ \frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = \cosh^2 y - \sinh^2 y = 1. \]
This is a hyperbola with asymptotes \( y = \pm \tan a \).
Next, consider the image of the horizontal lines $y = b$, for any $b \in \mathbb{R}$. If $b = 0$, the image is $[-1, 1]$. If $b \neq 0$, then
\[
\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = \cos^2 x + \sin^2 x = 1.
\]
This is an ellipse with foci at $\pm 1$ and eccentricity $\epsilon = \sech b$. The foci are the same for every $b$. For large $b$ we have $\epsilon \sim 1$ and the ellipse is nearly a circle: the difference between the foci is negligible from far away. These ellipses for $y = \text{constant}$ are perpendicular to the hyperbolas coming from lines $x = \text{constant}$.

So much for the images of horizontal and vertical lines. The inverse images of horizontal and vertical lines are the level curves of $u$ and $v$. Here the picture is identical to that of $\sin z$ drawn in class, but shifted horizontally by $\pi/2$.

**Extra Exercises.** For $u(x, y) = x^3 - 3xy^2$ and $u(x, y) = x/(x^2 + y^2)$,

a) find a harmonic conjugate $v(x, y)$,

b) write the function $u(x, y) + iv(x, y)$ in terms of $z$,

c) sketch the level curves of $u(x, y)$ and $v(x, y)$.

The harmonic conjugates are
\[
v = 3x^2y - y^3, \quad \text{and} \quad v = \frac{y}{x^2 + y^2},
\]
and we have
\[
u + iv = z^3, \quad \text{and} \quad u + iv = \frac{1}{z},
\]
respectively. One brute-force way to find these is to substitute
\[
x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}
\]
into $u(x, y) + iv(x, y)$ and simplify until $\bar{z}$ disappears, which it will, as long as $u, v$ satisfy the Cauchy-Riemann equations.

The level curves of $u = x^3 - 3xy^2$ are obtained as follows. The critical level curve is $u = 0$, which is three lines $x = 0, x = \pm \sqrt{3}y$, dividing the plane into six equal sectors. A level curve for $u = c \neq 0$ consists of three smooth approximations to the sharp corner in alternate sectors. The level curves of $v$ are obtained by rotating the level curves of $u$ by $\pi/2$. Remarkably, when you do this, the rotated curves are orthogonal to the original curves.
The level curves of \( u(x, y) = x/(x^2 + y^2) \) are obtained as follows. First, since \( (1/z)' = -1/z^2 \) has no zeros, there are no critical points, except at \( z = 0 \). The level curve \( u = 1/2c \) is the circle with radius \( |c| \) and center \( (c, 0) \). As \( c \) varies, we get the family of all circles tangent to \( \mathbb{R}i \) at 0. The level curves of \( v \) are the circles tangent to \( \mathbb{R} \) at 0, and are orthogonal to the previous circles.