Exercise 5. Give the power series expansion of \( \log z \) about \( z = i \) and find its radius of convergence.

For any nonzero \( a \in \mathbb{C} \), we have

\[
\frac{1}{z} = \frac{1}{a} \cdot \frac{1}{1 + \frac{z-a}{a}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n,
\]

with radius of convergence \( |a| \). Take \( a = i \), antidifferentiate, and remember that \( i^2 = -1 \), \( \log i = \frac{i\pi}{2} \). You get

\[
\log z = \frac{i\pi}{2} - \sum_{n=0}^{\infty} \frac{i^{n+1}}{n+1} (z-i)^{n+1} = \frac{i\pi}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n},
\]

with radius of convergence \( |i| = 1 \).

Exercise 6. Give the power series expansion of \( \sqrt{z} \) about \( z = 1 \) and find its radius of convergence.

There are two branches of \( \sqrt{z} \), differing by a sign, which can be detected from the value \( \pm 1 \) at \( z = 1 \). Choose the branch \( f(z) \) such that \( f(1) = 1 \). For \( n > 0 \), we have (CORRECTED VERSION)

\[
f^{(n)}(1) = (-1)^{n-1} \frac{(2n-2)!}{2^{n-1}(n-1)!}.
\]

(Note: This is better than writing

\[
f^{(n)}(1) = (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{2^n},
\]
since the latter is ambiguous at \( n = 1 \). We get
\[
f(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!n!} (z-1)^n = 1 - 2 \sum_{n=1}^{\infty} \frac{1}{2n-1} \binom{2n-1}{n} \left( \frac{1-z}{4} \right)^n
\]
and the radius of convergence \( R \) is the distance to the nearest nonanalytic point, which is \( z = 0 \), so \( R = 1 \).

**Exercise 7.** In problems 7,9, let \( \gamma_0(t) = e^{it} \), for \( t \in [0, 2\pi] \).

a) \[
\int_{\gamma_0} \frac{e^{iz}}{z^2} \, dz = 2\pi i \cdot f'(0),
\]
where \( f(z) = e^{iz} \). So the integral is \( 2\pi i \cdot i = -2\pi \).

b) \[
\int_{a+r\gamma_0} \frac{dz}{z-a} = 2\pi i.
\]

c) \[
\int_{\gamma_0} \frac{\sin z}{z^3} \, dz = 2\pi i \cdot f''(0),
\]
where \( f(z) = \sin z \). Hence the integral is \( -2\pi i \cdot \sin 0 = 0 \).

d) The integral
\[
\int_{1+\frac{1}{2}\gamma_0} \frac{\log z}{z^n} \, dz
\]
is zero, since \( z^{-n} \log z \) is analytic in a disk containing the path.

**Exercise 9.**

c) First,
\[
\frac{1}{z^2 + 1} = \frac{1}{2i} \left[ \frac{1}{z-i} - \frac{1}{z+i} \right].
\]
Both \( (z \pm i)^{-1} \) integrate to \( 2\pi i \) around \( 2\gamma_0 \). Hence
\[
\int_{2\gamma_0} \frac{dz}{z^2 + 1} = \frac{1}{2i} [2\pi i - 2\pi i] = 0.
\]
Alternatively, note that
\[
M(z) = \frac{z-i}{z+i}
\]
maps \( \mathbb{C} - [-i, i] \) to \( \mathbb{C} - \mathbb{R}_{\leq 0} \), so \( \text{Log} M(z) \) is analytic on the region \( \mathbb{C} - [-i, i] \) containing \( 2\gamma_0 \). Moreover,

\[
(\text{Log} M(z))' = \frac{1}{z - i} - \frac{1}{z + i}.
\]

Hence the integral is zero.

d)

\[
\int_{\gamma_0} \frac{\sin z}{z} \, dz = 2\pi i \cdot \sin(0) = 0.
\]

Alternatively, note that \( \frac{\sin z}{z} \) is entire, hence has zero integral over every closed path in \( \mathbb{C} \).

**Exercise 12.** Since \( \sec z \) is even and \( \sec 0 = 1 \), it follows that

\[
\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k},
\]

where the radius of convergence is the distance from 0 to the nearest non-analytic point(s) of \( f(z) = \sec z \), which is \( \pi / 2 \), and \( E_{2k} = f^{2k}(0) \).

Multiplying the series for \( \sec z \) and \( \cos z \), we get

\[
1 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{n-k} \frac{E_{2k}}{(2k)! (2n - 2k)!} \right) z^{2n}.
\]

Comparing coefficients of \( z^{2n} \) and multiplying by \( (2n)! \), we get the recursive formula

\[
\sum_{k=0}^{n} (-1)^{n-k} E_{2k} \binom{2n}{2k} = 0.
\]

We have

\[
E_0 = 1, \quad E_2 = 1, \quad E_4 = 5, \quad E_6 = 61, \quad E_8 = 1385.
\]

**Exercise 13.** We have

\[
\frac{e^z - 1}{z} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}.
\]
with infinite radius of convergence. The series
\[ f(z) = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \]
has radius of convergence \( R \) equal to the distance from 0 to the nearest zero(s) of \( e^z - 1 \), which are \( \pm 2\pi i \), so \( R = 1 \).

Multiplying these two series, we get
\[ 1 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{a_k}{k!(n-k+1)!} \right) z^n. \]
Comparing coefficients of \( z^n \) and multiplying by \((n + 1)!\), we get the recursive formula
\[ \sum_{k=0}^{n} a_k \binom{n+1}{k} = 0. \]
Taking \( n = 1 \) and using \( a_0 = f(0) = 1 \), we find that \( a_1 = -\frac{1}{2} \). The function
\[ \tilde{f}(z) = f(z) + \frac{z}{2} = \frac{z(e^z + 1)}{2(e^z - 1)} = 1 + \sum_{k=2}^{\infty} \frac{a_k}{k!} z^k \]
is even, so \( a_k = 0 \) for \( k \) odd, \( k > 1 \). Let \( B_{2n} = (-1)^{n-1} a_{2n} \), so that
\[ \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} z^{2n}. \]
We have
\[ B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66}. \]

**Exercise 14.** Find the power series of \( \tan z \) in terms of Bernoulli numbers. Replace \( z \) by \( 2iz \) in the function \( \tilde{f}(z) \) of the previous problem. We get
\[ \tilde{f}(2iz) = \frac{iz(e^{2iz} + 1)}{e^{2iz} - 1} = \frac{iz(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}} = z \cot z. \]
Replacing \( z \) by \( 2iz \) in the power series for \( \tilde{f}(z) \), we get
\[ z \cot z = 1 - \sum_{n=1}^{\infty} 4^n \frac{B_{2n}}{(2n)!} z^{2n}. \]
Now,

\[
\cot 2z = \frac{\cos 2z}{\sin 2z} = \frac{\cos^2 z - \sin^2 z}{2 \sin z \cos z} = \frac{1}{2} (\cot z - \tan z),
\]

so

\[
z \tan z = z \cot z - 2z \cot 2z = \left[ 1 - \sum_{n=1}^{\infty} \frac{4^n B_{2n} z^{2n}}{(2n)!} \right] - \left[ 1 - \sum_{n=1}^{\infty} \frac{4^{2n} B_{2n} z^{2n}}{(2n)!} \right] = \sum_{n=1}^{\infty} \frac{4^n (4^n - 1) B_{2n}}{(2n)!} z^{2n}.
\]

Hence we get

\[
\tan z = \sum_{n=1}^{\infty} \frac{4^n (4^n - 1) B_{2n}}{(2n)!} z^{2n-1}.
\]