p. 87, no. 6. Let $f$ be analytic on $D = B(0, 1)$ and suppose $|f(z)| \leq 1$ on $D$. Show that $|f'(0)| \leq 1$.

**Proof:** Let $0 < r < 1$ and let $\gamma_r(t) = re^{it}$ for $0 \leq t \leq 2\pi$. By the Cauchy Integral Formula, we have

$$f'(0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w^2} \, dw,$$

so

$$|f'(0)| \leq \frac{1}{2\pi} \cdot \frac{1}{r} \cdot 2\pi r = \frac{1}{r}.$$

Taking the limit as $r \to 1$, we have $|f'(0)| \leq 1$. ■

p. 87, no. 7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ and let $n \in \mathbb{N}$. Find

$$\int_{\gamma} \left( \frac{z}{z - 1} \right)^n \, dz.$$

Apply C.I.F. to $f(z) = z^n$, to get

$$\int_{\gamma} \left( \frac{z}{z - 1} \right)^n \, dz = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(1) = 2n\pi i.$$
You can also do this without C.I.F., by computing directly:

\[
\int_\gamma \left( \frac{z}{z-1} \right)^n \, dz = \int_0^{2\pi} \left( \frac{1 + e^{it}}{e^{it}} \right)^n \cdot i e^{it} \, dt \\
= i \int_0^{2\pi} (1 + e^{-it})^n \cdot e^{it} \, dt \\
= i \int_0^{2\pi} \left( 1 + ne^{-it} + \left( \frac{n}{2} \right) e^{-2it} + \cdots \right) \cdot e^{it} \, dt \\
= 2n\pi i,
\]

since \( \int_0^{2\pi} e^{kit} \, dt = 0 \) for \( k \) a nonzero integer.

**p.96, no. 8a.** We must integrate \((z - a)^{-1}\) and \((z - b)^{-1}\) over the path \( \gamma \), which can be written as a sum of six paths, two of which are closed and have \( a, b \) in their \( \infty \)-components, hence have zero integral and two pairs of non-closed paths. One pair starts at the leftmost crossing point, each goes around \( a \) in opposite directions, and they meet at the middle crossing point. The other one pair starts at the middle crossing point, each goes around \( b \) in opposite directions, and they meet at the rightmost crossing point.

Integrating over the paths around \( a \) is the same as integrating \((z - a)^{-1}\) over \( \gamma_1 - \gamma_2 \), where \( \gamma_1(t) = a + re^{it} \) for \( 0 \leq t \leq \pi \) and \( \gamma_2(t) = a + re^{it} \) for \( \pi \leq t \leq 2\pi \), for some small \( r > 0 \). One computes

\[
\int_{\gamma_1} \frac{dz}{z - a} = \pi i = \int_{\gamma_2} \frac{dz}{z - a},
\]

hence \( n(\gamma, a) = 0 \). Similarly, \( n(\gamma, b) = 0 \).

**p.96, no. 10.** Compute \( \int_\gamma (1 + z^2)^{-1} \) for all closed paths not passing through \( \pm i \).

\[
\int_\gamma \frac{dz}{1 + z^2} = \frac{1}{2i} \int_\gamma \left( \frac{1}{z - i} - \frac{1}{z + i} \right) \, dz \\
= \frac{1}{2i} \cdot 2\pi i \cdot (n(\gamma, i) - n(\gamma, -i)) \\
= \pi \cdot (n(\gamma, i) - n(\gamma, -i)).
\]

**p.96, no. 11.** The Cauchy integral formula can be written

\[
\int_\gamma \frac{f(z)}{(z - a)^{n+1}} \, dz = 2\pi i \cdot n(\gamma, a) \cdot \frac{f^{(n)}(a)}{n!}.
\]
Since \((e^z - e^{-z})'''(0) = 2\), we have
\[
\int_\gamma \frac{e^z - e^{-z}}{z^4} \, dz = \frac{2\pi i}{3} \cdot n(\gamma, 0),
\]
giving the answers
a) \(\frac{2\pi i}{3}\),
b) \(\frac{4\pi i}{3}\),
c) \(\frac{4\pi i}{3}\).

p.110, no. 1.

b) \(\cos z/z = 1/z + \text{(higher powers)}\) has a simple pole at \(z = 0\).
c) \(f(z) = (\cos z - 1)/z = -z/2 + \text{(higher powers)}\) has a removable singularity at \(z = 0\) and \(f(0) = 0\).
h) \(1/(1 - e^z) = -1/z + \text{(higher powers)}\) has a simple pole at \(z = 0\).
i) Since \(\sin z/z\) is entire, the function \(z \sin(1/z)\) has an essential singularity at \(z = 0\). We consider the function \(g(z) = \sin z/z\) for \(z\) large. We first invoke the Little Picard Theorem, which asserts that in any neighborhood \(|z| > R\) of \(\infty\), we have either \(g(U) = \mathbb{C}\) or \(g(U) = \mathbb{C} \cup \{w\}\), for some \(w \in \mathbb{C}\). I claim that in this case it is the former. Assume there is such a \(w\). Then since \(g(\bar{z}) = \overline{g(z)}\), it follows that \(w\) is real. We have
\[
g(x+iy) = u+iv = \frac{x \sin x \cosh y + y \cos x \sinh y}{x^2 + y^2} + i \frac{x \cos x \sinh y - y \sin x \cosh y}{x^2 + y^2}.
\]
It is easy to see that \(g(\mathbb{R}) = [-2/3\pi, 1]\) and \(g(i\mathbb{R}) \supset [1, \infty]\). Hence this hypothetical \(w\) must lie in \((-\infty, -2/3\pi]\). Now, \(g\) is real on the curve
\[
x \cos x \sinh y - y \sin x \cosh y = y \left(\frac{x \cos x \sinh y}{y} - \sin x \cosh y\right) = 0.
\]
Since we already know \(g(\mathbb{R})\), we set the factor in \(\ldots\) equal to zero, and get the curve
\[
C : \ y \coth y = x \cot x.
\]
This curve \(C\) meets the \(x\)-axis at the solutions of \(\tan x = x\), which form a sequence \(z_n, n \in \mathbb{Z}\), such that \(z_n \to (n - \frac{1}{2})\pi\) as \(|n| \to \infty\). On \(C\), we have
\[
g|_C = u|_C = \frac{\sin x \cosh y}{x} = \frac{\cos x \sinh y}{y}.
\]
For $n$ a large positive integer, let $x_n = 3\pi/4 - 2n\pi$ and let $y_n$ satisfy $y_n \coth y_n = -x_n$. (Note that $y \coth y$ is unbounded, so such $y_n$ exists.) Then since $\cot x_n = -1$, the point $(x_n, y_n)$ lies on $C$. Let $A_n$ be the arc on $C$ from $(x_n, y_n)$ to the $x$-axis and let $(z_n, 0)$ be the point where $A_n$ meets the $x$-axis. Since $A_n$ is connected, the set $g(A_n)$ is an interval. We have

$$u(x_n, y_n) = \cosh y_n \sqrt{2} x_n = -\sinh y_n \sqrt{2} y_n,$$

and

$$u(z_n, 0) = \cos z_n \approx \cos(\pi/2 + n\pi) = 0.$$

Hence $u(A_n)$ contains the interval $[-\sinh y_n/\sqrt{2} y_n, \epsilon]$, where $\epsilon > 0$ is small. For large enough $n$, this will overlap with our previously obtained interval $[-2/3\pi, \infty)$. So for large $n$ we now have $[-\sinh y_n/\sqrt{2} y_n, \infty)$ in the image of $g$. Since $y_n \to \infty$, we have $-y_n^{-1} \sinh y_n \to -\infty$, so the entire negative real axis is covered. Whew!

**p.110, no. 5.** Let $a_n = \pi/2 + n\pi$. Since $\tan z = \sin z / \cos z$ and $(\cos z)' = -\sin z$, it follows that $\tan z$ has a simple pole at each $a_n$, with residue $-1$. Hence the singular part of $\tan z$ at $a_n$ is $-1/(z - a_n)$.

**p.110, no. 13.** a) If $f(z)$ is entire and $\lim_{z \to \infty} f(z)$ exists and is finite, then $f$ is bounded, so $f$ is constant, by Liouville’s theorem.

b) If $f(z)$ is entire and has a pole of order $m$ at $\infty$, then $f(1/z)$ has a pole of order $m$ at 0. Hence $f(z) = z^{-m} g(z)$, where $g(z)$ is analytic and nonzero at 0, so $f(z) = z^{-m} g(1/z)$ and $g(1/z)$ is bounded, for $|z|$ large, say $|g| \leq M$. Then $|f(z)| \leq M |z|^{-m}$ for $|z|$ large, so $f$ is a polynomial of degree $m$, by the extension of Liouville from the first exam.

c) A rational function $P(z)/Q(z)$, with $P, Q \in \mathbb{C}[z]$, is bounded for $|z|$ large if and only if $\deg P \leq \deg Q$.

d) A rational function $P(z)/Q(z)$ has a pole of order $m$ at $\infty$ iff $P(z)/Q(z) = z^m g(z)$, where $g(z)$ is a rational function bounded near $\infty$. This means $\deg P = m + \deg Q$. 
