Exercise 1. Show that for $a > 0$ we have
\[ \int_0^\infty e^{-(t^a)} \, dt = \Gamma \left( 1 + \frac{1}{a} \right). \]
Make the substitution $u = t^a$ and get
\[ \int_0^\infty e^{-(t^a)} \, dt = \int_0^\infty e^{-u} u^{1/a - 1} du = \frac{1}{a} \Gamma \left( \frac{1}{a} \right) = \Gamma \left( 1 + \frac{1}{a} \right). \]

Exercise 2. Verify the following calculations that we needed in the proof of the integral formula for $\Gamma(z)$:

(a) \[ \int_0^t \frac{u}{n} \left(1 - \frac{u}{n}\right)^{n-1} e^u \, du = 1 - e^t \left(1 - \frac{t}{n}\right)^n. \]

(b) \[ \int_0^1 y^{z-1}(1-y)^n \, dy = \frac{n!}{z(z+1)\cdots(z+n)}. \]

(You cannot use the Beta integral formula here, since we needed the integral formula to derive it!)

For (a), note that the left side is the unique function $f(t)$ such that
\[ f'(t) = \frac{t}{n} \left(1 - \frac{t}{n}\right)^{n-1} e^t, \quad f(0) = 0. \]
On the other hand, the right side vanishes at zero and its derivative is
\[-e^t \left(1 - \frac{t}{n}\right)^n - ne^t \cdot \frac{1}{n} \left(1 - \frac{t}{n}\right)^{n-1} = \frac{t}{n} \left(1 - \frac{t}{n}\right)^{n-1} e^t.\]

For (b), denote the integral by $F_n(z)$ and use integration by parts (and the fact that $\Re z > 0$) to get
\[F_n(z) = \frac{n}{z} \cdot F_{n-1}(z + 1).\]

The result follows from induction on $n$.

**Exercise 3.** Prove Euler’s formula:
\[
\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \cdots \cdot \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}},
\]
using the following steps.

(a) \[\prod_{k=1}^{n-1} e^{ik\pi/n} = i^{n-1}.\]

(b) \[\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = n \cdot 2^{1-n}.\]

(Use $2i \sin z = e^{iz} (1 - e^{-2iz})$.)

(c) \[\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \cdot \prod_{k=1}^{n-1} \frac{\pi}{\sin\left(k\pi/n\right)} = \prod_{k=1}^{n-1} \frac{\pi}{\sin\left(k\pi/n\right)}.
\]

(d) Deduce Euler’s formula above.

Since
\[\sum_{k=1}^{n-1} \frac{ik\pi}{n} = \frac{i\pi}{n} \sum_{k=1}^{n-1} k = \frac{i\pi}{n} \cdot \frac{n(n-1)}{2} = \frac{i\pi(n-1)}{2},\]

we have
\[\prod_{k=1}^{n-1} e^{ik\pi/n} = (e^{\pi i/2})^{n-1} = i^{n-1}.\]
Now
\[
\prod_{k=1}^{n-1} \frac{\pi}{\sin \left(k\pi/n\right)} = \prod_{k=1}^{n-1} \frac{e^{k\pi i/n}(1 - e^{-2k\pi i/n})}{2i} = \frac{i^{n-1}}{(2i)^{n-1}} \prod_{k=1}^{n-1} (1 - \zeta^k),
\]

where \(\zeta = e^{-2\pi i/n}\). These powers of \(\zeta\) run through all roots of \(z^n - 1 = 0\) except \(z = 1\), so
\[
\prod_{k=1}^{n-1} (z - \zeta^k) = \frac{z^n - 1}{z - 1},
\]

which becomes \(n\) when \(z = 1\). Hence
\[
\prod_{k=1}^{n-1} \frac{\pi}{\sin \left(k\pi/n\right)} = \frac{n}{2^{n-1}}.
\]

Finally, the functional equation \(\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z\) gives
\[
\prod_{k=1}^{n-1} \Gamma \left(\frac{k}{n}\right)^2 = \prod_{k=1}^{n-1} \left[ \Gamma \left(\frac{k}{n}\right) \Gamma \left(\frac{n-k}{n}\right) \right] = \prod_{k=1}^{n-1} \frac{\pi}{\sin \left(k\pi/n\right)}.
\]

**Exercise 4.** This exercise relates the value \(\Gamma(1/4)\) to elliptic integrals.

(a) Let \(k \geq 0\). Compute the integral
\[
I_k = \int_0^1 \frac{t^k}{\sqrt{1-t^4}} \, dt
\]

in terms of \(\pi\) and \(\Gamma(1/4)\).

(b) Deduce from the previous exercise that
\[
I_0 \cdot I_2 = \frac{\pi}{4}.
\]

(c) In Calculus, we learn that the arclength \(L\) of the ellipse
\[
x^2 + 2y^2 = 2
\]
is given by the elliptic integral
\[ L = 4 \int_0^{\pi/2} \sqrt{1 - \frac{1}{2} \sin^2 \theta} \, d\theta. \]

Rewrite this integral in terms of \( \cos \theta \), make a substitution, and show that
\[ L = 2\sqrt{2}(I_0 + I_2). \]

Express this elliptic arclength is in terms of the circular arclength \( \pi \) and the value \( \Gamma(\frac{1}{4}) \).

(d) Deduce from the previous two exercises that \( I_0 \) and \( I_2 \) are the roots of the polynomial
\[ 4x^2 - \sqrt{2}Lx + \pi = 0. \]

Make the substitution \( u = t^4 \) to get
\[ I_k = \frac{1}{4} \int_0^1 u^{(k-3)/4}(1-u)^{-1/2} \, du = \frac{\Gamma\left(\frac{k+1}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{4 \cdot \Gamma\left(\frac{k+3}{4}\right)} = \frac{\Gamma\left(\frac{k+1}{4}\right) \cdot \sqrt{\pi}}{4 \cdot \Gamma\left(\frac{k+3}{4}\right)}. \]

Both \( \Gamma \) values can be reduced to \( \Gamma(1/4) \). For example, we have
\[ \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right) = \pi \csc(\pi/4) = \pi \sqrt{2}, \]
so
\[ I_0 = \frac{\Gamma(1/4) \sqrt{\pi}}{4 \Gamma(3/4)} = \frac{\Gamma(1/4)^2}{4 \sqrt{2\pi}} \]
and
\[ I_2 = \frac{\Gamma(3/4) \sqrt{\pi}}{\Gamma(5/4)} = \frac{\Gamma(3/4) \sqrt{\pi}}{\Gamma(1/4)} = \frac{\pi \sqrt{2\pi}}{\Gamma(1/4)^2}, \]
so
\[ I_0 \cdot I_2 = \frac{\pi}{4}. \]

Turning to the ellipse, writing \( \sin^2 \theta = 1 - \cos^2 \theta \) gives
\[ L = 2\sqrt{2} \int_0^{\pi/2} \sqrt{1 + \cos^2 \theta} \, d\theta. \]

The substitution \( u = \cos \theta \) gives
\[ L = 2\sqrt{2} \int_0^1 \frac{\sqrt{1 + u^2}}{\sqrt{1 - u^4}} \, du = 2\sqrt{2} \int_0^1 \frac{1 + u^2}{\sqrt{1 - u^4}} \, du = 2\sqrt{2}(I_0 + I_2). \]
Using the calculations of $I_0$ and $I_2$ from problem 4, we find that the arclength of the ellipse is given by

$$L = 2\sqrt{2} \left[ \frac{\Gamma(1/4)^2}{4\sqrt{2}\pi} + \frac{\pi\sqrt{2}\pi}{\Gamma(1/4)^2} \right] = \sqrt{\pi} \left[ \frac{\Gamma(1/4)^2}{2\pi} + \frac{4\pi}{\Gamma(1/4)^2} \right].$$

Now $I_0$ and $I_2$ are the zeros of the polynomial

$$(x - I_0)(x - I_2) = x^2 - (I_0 + I_2)x + I_0I_2,$$

which leads to the equation $4x^2 - \sqrt{2}Lx + \pi = 0$. It is perhaps nicer to say that $2\sqrt{2}I_0$ and $2\sqrt{2}I_2$ are the zeros of

$$x^2 - Lx + 2\pi,$$

a polynomial whose coefficients are the arclengths of the ellipse and the circle.

**Exercise 5.** In analogy with the basic formula

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2},$$

where $\pi$ is the circular constant, we write

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \varpi,$$

where $\varpi$ (“varpi” in TeX) is the *Lemniscatic Constant*. The lemniscate has equation $|z^2 - \frac{1}{2}| = \frac{1}{2}$, or when squared, $(x^2 + y^2)^2 = x^2 - y^2$.

First, more remarks on the Lemniscate: A *Cassini Oval* is the locus of points $z$ whose distance from two given points $a$ and $b$ is a constant. If we take $b = -a$ on the real axis, then the equation of a Cassini oval is

$$|z^2 - a^2| = \text{constant}.$$

For various values of the constant, the Cassini ovals look like

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1. 1625-1712, Italian Astronomer, name of NASA probe currently nearing Saturn.
2. Picture stolen from MathWorld.
The Lemniscate, the figure-eight in the middle, is the Cassini oval when the constant equals $a^2$. In this problem, we have $a^2 = 1/2$.

(a) Use the parametrization:

$$x(t) = \sqrt{\frac{t^2 + t^4}{2}}, \quad y(t) = \sqrt{\frac{t^2 - t^4}{2}},$$

to show that the arclength of the lemniscate is $2\wp$.

(b) Express $\wp$ in terms of $\pi$ and $\Gamma(1/4)$ (see previous exercise).

(c) Let $a_1 = \sqrt{2}, b_1 = 1$. Define $(a_n, b_n)$ recursively by

$$a_n = (a_{n-1} + b_{n-1})/2, \quad b_n = \left(a_{n-1} \cdot b_{n-1}\right)^{1/2}.$$

compute $(a_6, b_6)$ and $\pi/\wp$ to several decimal places.

As $t$ varies from 0 to 1, the curve $\gamma(t) = x(t) + iy(t)$ traces out the part of the Lemniscate in the quadrant $x > 0, y > 0$, which is one fourth of the total arclength. We have

$$x'(t)^2 + y'(t)^2 = \frac{1}{1-t^4},$$

so the total arclength is

$$4 \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt = 4 \int_0^1 \frac{dt}{1-t^4} = 4 \cdot \frac{\wp}{2} = 2\wp.$$
From the calculation of $I_0$ in the previous problem, we have
\[
\varpi = 2I_0 = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} = 2.622057554292119810464840\ldots,
\]
so
\[
\frac{\pi}{\varpi} = 1.198140234735592207439922\ldots.
\]
Meanwhile the Arithmetic-Geometric Mean between 1 and $\sqrt{2}$ gives

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