MT815 Complex Variables Homework IV

Witch 2

Due Friday April 4

Exercise 1. Let $L$ be a lattice in $\mathbb{C}$, let

$$G_k(L) = \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^k}$$

be the corresponding Eisenstein series and let

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \left[ \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right] = \frac{1}{z^2} + \sum_{n=1}^{\infty} (n + 1)G_{n+2}(L)z^n$$

be the Weierstrass $\wp$-function for $L$, which satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where $g_2 = 60G_4(L)$ and $g_3 = 140G_6(L)$.

(a) Prove that if $iL = L$ then $G_k(L) = 0$ unless 4 divides $k$.

(b) Show that $\wp(iz) = -\wp(z)$ if $iL = L$.

(c) Suppose now that $\rho L = L$, where $\rho = e^{2\pi i/3}$. State and prove results analogous to (a) and (b).

For any $\alpha \in \mathbb{C}^\times$ we have $G_k(\alpha L) = \alpha^{-k}G_k(L)$. So if $\alpha L = L$ we have $G_k(L) = 0$ unless $\alpha^k = 1$.

If $iL = L$ then

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (4m - 1)G_{4m}z^{4m-2}. $$
Since each power of $z$ is $\equiv 2 \mod 4$, we have $\wp(iz) = i^2 \wp(z) = -\wp(z)$.

If $\rho L = L$ then

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (6m-1)G_{6m} z^{6m-2}.$$  

Since each power of $z$ is $\equiv 2 \mod 6$, we have $\wp(\rho z) = \rho^{-2} \wp(z)$.

**Exercise 2.** Prove that if $L$ is any lattice in $\mathbb{C}$ closed under complex conjugation, and $x \in \mathbb{R}$ is not in $L$, then $\wp_L(x) \in \mathbb{R}$.

If $\bar{L} = L$ then $G_k(\bar{L}) = G_k(L) = G_k(L)$, so each $G_k(L)$ is real. Hence the Laurent series for $\wp(z)$ at $z = 0$ has real coefficients. This implies that $\wp(x) \in \mathbb{R}$ for $x \in \mathbb{R} - L$.

**Exercise 3.** Let $L = \mu \mathbb{Z}[i]$, where $\mathbb{Z}[i] = \{n + mi : n, m \in \mathbb{Z}\}$ is the lattice of Gaussian integers and $\mu$ is any positive real number. Find the zeros of $\wp = \wp_L$ and then prove that $g_2(L) > 0$.

The differential equation for $\wp$ is

$$(\wp')^2 = 4\wp^3 - g_2(L)\wp.$$  

So the zeros of $\wp$ are among the zeros of $\wp'$, which we know are

$$z_1 = \mu/2, \quad z_2 = \mu i/2, \quad z_3 = \mu(1+i)/2.$$  

Since $iz_3 \equiv z_3 \mod L$, and $\wp(iz) = -i\wp(z)$, we have $\wp(z_3) = 0$. If $\wp(z_1) = 0$, then also $\wp(z_2) = \wp(iz_1) = 0$, which is too many zeros for $\wp$. Hence $\wp$ has a double zero at $z_3$, and $\wp(z_1) \neq 0$. Since $\bar{L} = L$, we know that $\wp(0, \mu) \subset \mathbb{R}$, so $\wp(z_1) \in \mathbb{R}$, hence $\wp(z_2) = -\wp(z_1) \in \mathbb{R}$, and also $g_2(L) \in \mathbb{R}$. Since the polynomial $4x^3 - g_2(L)x$ has the three distinct roots $\wp(z_1), \wp(z_2), \wp(z_3)$, all of which are real, we must have $g_2(L) > 0$.

**Exercise 4.** Let $f$ be an elliptic function with period lattice $L$. View $f$ as a function $f : \mathbb{C}/L \to \mathbb{C}$. The *order* of $f$ is the number $m$ of poles of $f$ in $\mathbb{C}/L$, counting multiplicities. That is, $m$ is the sum of the orders of the poles of $f$ in $\mathbb{C}/L$, or if you prefer, in any shifted period parallelogram whose sides miss all poles and zeros of $f$.

(a) Prove that for all $w \in \mathbb{C}$, the function $f$ takes the value $w$ exactly $m$ times in $\mathbb{C}/L$, counting multiplicities.
(b) What is the order of \( \wp = \wp_L \)? Given \( z \in C - L \), find \( z^* \) in \( C - L \) such that
\[
\wp(z^*) = \wp(z) \quad \text{and} \quad \wp'(z^*) = -\wp'(z).
\]

(c) Use (a) and (b) to prove that \((\wp, \wp')\) maps \( C - L \) onto the curve
\[
C = \{(u, v) \in \mathbb{C}^2 \mid v^2 = 4u^3 - g_2u - g_3\}.
\]

The elliptic function \( g(z) - w \) has the same poles as \( f \), hence also has order \( m \), so \( g(z) \) has \( m \) zeros. The simplest value for \( z^* \) is \( z^* = -z \). If \( z = t\lambda_1 + s\lambda_2 \) is inside a period parallelogram \( \Pi \) spanned by \( \lambda_1 \) and \( \lambda_2 \), and we also want to have \( z^* \in \Pi \), then we must take
\[
z^* = (1 - t)\lambda_1 + (1 - s)\lambda_2.
\]

Now let \((u, v)\) be a point on the curve \( C \). By part (a) there exists \( z \in C - L \) such that \( \wp(z) = u \). Since
\[
\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4u^3 - g_2u - g_3 = v^2,
\]
we have \( \wp'(z) = \pm v \). Replacing \( z \) by \( z^* \) if necessary, we can arrange \( \wp(z) = v \).

**Exercise 5.** Using exercise 4, show that there is a unique \( \mu > 0 \) so that \( g_2(L) = 4 \), so that \( \wp = \wp_L \) has differential equation \((\wp')^2 = 4(\wp^3 - \wp)\). Then show that \( \wp'' = 6\wp^2 - 2 \) and use this to draw (or describe) the graph of \( \wp(x) \) for \( x \in (0, \mu) \).

We have
\[
g_2(L) = \mu^{-4}g_2(Z[i]) > 0,
\]
so \( g_2(Z[i]) > 0 \) and there is a unique \( \mu > 0 \) such that
\[
4\mu^4 = g_2(Z[i]),
\]
making \( g_2(L) = 4 \) and we have the equation \((\wp')^2 = 4(\wp^3 - \wp)\). Differentiating both sides gives \( \wp'' = 6\wp^2 - 2 \). At \( \mu/2 \), we have
\[
\wp(\mu/2) = 1, \quad \wp'(\mu/2) = 0, \quad \wp''(\mu/2) = 4 > 0,
\]
and there are no other zeros of \( \wp' \) in the interval \((0, \mu)\). It follows that \( \wp \) is concave up in \((0, \mu)\), with a minimum at \( \mu/2 \).
Exercise 6. Let $\mu$ be as in the previous exercise. For $x \in [0, \mu/2]$, consider the function $F(x) = \int_{\varphi(x)}^{\infty} \frac{du}{\sqrt{4u^3 - 4u}}$, where the integral is over the real interval $[\varphi(x), \infty)$ and the square-root is chosen to be positive, except at $u = 1$. (Why can we do this?)

(a) Show that $F(x) = x$. (Use the Fundamental Theorem of Calculus!)

(b) Compute $\int_{1}^{\infty} (u^3 - u)^{-1/2}du$. (Let $t = 1/u^2$.)

(c) Compute $\mu$.

(d) Compute $G_4(\mathbb{Z}[i])$.

Since $\varphi(0, \mu/2) = (1, \mathbb{R})$, on which the function $u^3 - u \geq 0$, we can choose the positive real square root. Then $F'(x)^2 = (\varphi'(x)^2 \cdot (4\varphi(x)^3 - 4\varphi(x))^{-1} = 1,$
so $F'(x) = \pm 1$ is a constant. Since $\phi(x)$ is decreasing on $(0, \mu/2)$, the sign is + and since $\phi(0) = \infty$ we have $F(0) = 0$, so $F(x) = x$. Setting $x = \mu/2$, we have

$$
\mu = 2 \int_1^\infty \frac{du}{\sqrt{4u^3 - 4u}} = \int_1^\infty \frac{du}{\sqrt{u^3 - u}}
$$

$$
= \frac{1}{2} \int_0^1 t^{-3/4} (1 - t)^{-1/2} dt
$$

$$
= \frac{\Gamma(1/4)\Gamma(1/2)}{2 \cdot \Gamma(3/4)} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}
$$

$$
= \varpi.
$$

Alternatively, we can turn the integral for $\mu$ directly into the Lemniscatic integral by the substitution $u = v^{-2}$, which gives

$$
\int_1^\infty \frac{du}{\sqrt{u^3 - u}} = 2 \int_0^1 \frac{dv}{\sqrt{1 - v^4}} = 2 \cdot \varpi = \varpi.
$$

From the equation $g_2(\mu\mathbf{Z}[i]) = 4$, we have

$$
\mu^{-4}60G_4(\mathbf{Z}[i]) = 4,
$$

or

$$
G_4(\mathbf{Z}[i]) = \frac{\varpi^4}{15}.
$$

**COMMENTS:** It’s a Swiss Thing

Recall Euler’s result from the mid 1700’s that

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(2\pi)^{2k} B_k}{2 \cdot (2k)!}
$$

which we could write as

$$
\sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^{2k}} = \frac{(2\pi)^{2k} B_k}{(2k)!},
$$

where

$$
\pi = 2 \int_0^1 \frac{dt}{\sqrt{1 - t^2}}
$$
and the Bernoulli numbers $B_k$ are defined as the coefficients in the expansion of the meromorphic function

$$\cot z = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{2k} \frac{z^{2k-1}}{(2k-1)!}$$

(2)

whose period group is $\pi \mathbb{Z}$.

In 1897 the Swiss mathematician Adolf Hurwitz (following his Swiss predecessors Euler and the Bernoullis) found an analogous formula for the sum

$$G_{4k}(Z[i]) = \sum_{0 \neq \lambda \in Z[i]} \frac{1}{\lambda^{4k}},$$

with the cotangent function replaced by the Weierstrass function $\wp$ of that lattice $L$ for which

$$(\wp')^2 = 4\wp^3 - 4\wp.$$

(3)

In the exercise above, you found that

$$L = \mathcal{C} Z[i],$$

where

$$\mathcal{C} = 2 \int_0^1 \frac{dt}{\sqrt{1 - t^4}}.$$

Proceeding analogously to the expansion (2), Hurwitz defines numbers that we will call $H_k$, by the expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{2^{4k}H_k}{4k} \frac{z^{4k-2}}{(4k-2)!}.$$

Substituting this into the equation (3), Hurwitz found the recursion formula

$$H_k = \frac{3}{(2k - 3)(16k^2 - 1)} \sum_{j=1}^{k-1} (4j - 1)(4k - 4j - 1) \binom{4k}{4j} H_j H_{k-j}$$

whose first few values are

$$H_1 = \frac{1}{10}, \quad H_2 = \frac{3}{10}, \quad H_3 = \frac{3^4 \cdot 7}{10 \cdot 13}.$$
Since
\[
\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (4k - 1)G_{4k}(\varpi Z[i])z^{4k-2},
\]

It follows that
\[
G_{4k}(Z[i]) = \frac{(2\varpi)^{4k}}{(4k)!}H_k,
\]
in analogy with Euler’s result (1).

We have not discussed the number theory of Bernoulli numbers, in particular the von Staudt-Clausen Theorem on the denominators of $B_k$ (see Ireland-Rosen A Classical Introduction to Modern Number Theory). Hurwitz found analogues of these results as well. His paper (in German, but readable by us thanks to consistency of notation over the years) is on the course website.