Exercise 1. Let $k$ be a finite field of cardinality $q$, and let $M$ be the group of affine transformations of a line over $k$. Concretely, $M$ is the semidirect product of the additive group of $k$ and the multiplicative group of $k$, with the latter acting on the former by multiplication. Let $\psi: k \to \mathbb{C}^\times$ be a nontrivial character of the additive group of $k$. Prove the following.

a) The induced representation $\Psi = \text{Ind}_k^G \psi$ is irreducible and its isomorphism class does not depend on the choice of nontrivial character $\psi$ of $k$.

b) If $W$ is an irreducible representation of $G$ and $W$ is not isomorphic to $\Psi$ then $\dim W = 1$ and the action of $G$ on $W$ factors through a character of $G/k = k^\times$.

[Remarks: The group $M$ may be realized inside the upper triangular subgroup of $\text{GL}_2(k)$, where it plays an important role in the rep thy of $\text{GL}_2$. In particular, the “Kirillov model”, used in automorphic forms, is based on restriction of $\text{GL}_2$-representations to $M$.]

Exercise 2. Verify Theorem 4.3 in the Representation Theory notes (which is stated there without proof), for $n = 4$.

Exercise 3. Let $G$ be a finite group, $H$ a subgroup of $G$ and $\chi: H \to \mathbb{C}^\times$ a character of $H$. Prove that

$$\det(g, \text{Ind}_H^G \chi) = \text{sgn}(g) \cdot \chi(T(g)),$$

where $T: G \to H/[H,H]$ is the transfer map and $\text{sgn}(g)$ is the sign of the permutation of $g$ on $G/H$.

Exercise 4. Suppose $(\rho, V)$ and $(\rho', V')$ are irreducible $\mathbb{C}$-representations of groups $G$ and $G'$ respectively.

a) Use characters to prove that the outer tensor product $V \boxtimes V'$ is irreducible for $G \times G'$.

b) Show that every irreducible representation of $G \times G'$ is of the form $V \boxtimes V'$ as in a).

Exercise 5. Prove the Peter-Weyl theorem for finite groups.

Exercise 6. Let $G$ be a finite group acting on a set $X$, let $V_X$ be the permutation representation over $\mathbb{C}$, and let $V_X^0$ be the functions in $V_X$ whose sum over $X$ is zero. Use Mackey’s theorem to prove that $V_X^0$ is irreducible if and only if $G$ is 2-transitive on $X$.

Exercise 7. Prove Prop. 6.2 in the Rep Thy notes.