In these exercises, \( g = \mathfrak{sl}_2 \). The root decomposition is \( g = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n} \), where
\[
\begin{align*}
\mathfrak{n} &= \mathbb{C}f, \\
\mathfrak{t} &= \mathbb{C}h, \\
\mathfrak{n} &= \mathbb{C}e
\end{align*}
\]
and the basis vectors \( e, h, f \) satisfy the relations
\[
[h, e] = 2e, \quad [e, f] = h, \quad [f, h] = 2f.
\]

**Exercise 1.** In the enveloping algebra \( U(g) \) we have the Casimir element
\[
c = ef + \frac{h^2}{2} + fe.
\]
Show that \( c \) is in the center of \( U(g) \) and compute the scalar by which \( c \) acts on the Verma module \( M(\lambda) \), for \( \lambda \in \mathfrak{t}^* \).

**Exercise 2.**

a) Show that \( M(\lambda) \) is irreducible unless \( \langle \lambda, h \rangle \in \mathbb{Z}_{\geq 0} \).

b) Suppose \( \langle \lambda, h \rangle \in \mathbb{Z}_{\geq 0} \) and let \( L(\lambda) \) be the unique simple quotient of \( M(\lambda) \). Show that we have an exact sequence
\[
0 \to M(-\lambda - 2\delta) \to M(\lambda) \to L(\lambda) \to 0,
\]
where \( \delta = (1/2)\alpha \) is the fundamental dominant weight.

**Exercise 3.** Let \( W \) be a vector space with basis \( \{ w_i : i \in \mathbb{Z} \} \).

a) Show that for any complex number \( \nu \), the formulas
\[
e \cdot w_i = \left( \frac{\nu + i}{2} \right) w_{i+2}, \quad h \cdot w_i = iw_i, \quad f \cdot w_i = \left( \frac{\nu - i}{2} \right) w_{i-2}
\]
define a representation of \( g \) on \( W \). We denote this representation by \( W(\nu) \).

b) Compute the action of the Casimir operator \( c \) on \( W(\nu) \).

c) The subspaces \( W_+(\nu) = \text{span}\{ w_i : i \text{ even} \} \) and \( W_-(\nu) = \text{span}\{ w_i : i \text{ odd} \} \) are clearly invariant under \( g \). Show that if \( \nu \) is not an integer of parity \( \pm \) then \( W_\pm(\nu) \) is irreducible for \( g \).

Note that the \( W_\pm(\nu) \) are not highest weight modules. They arise from the action of \( \text{SL}_2(\mathbb{R}) \) on the space of smooth functions \( f : \mathbb{R}^2 \setminus (0,0) \to \mathbb{C} \) satisfying \( f(tx, ty) = t^{-\nu}f(x, y) \) for \( t > 0 \). These are the principal series representations of \( \text{SL}_2(\mathbb{R}) \). See David Vogan’s notes www.math.mit.edu/~dav/sl2rev.pdf.

**Exercise 4.** Let \( V = \mathbb{C}[x_1, \ldots, x_n] \) be the polynomial algebra in \( n \) variables and let \( \partial_i = \partial/\partial x_i \) be the partial derivatives. We have the Euler and Laplace differential operators on \( V \), defined by
\[
D = \sum_i x_i \partial_i, \quad \Delta = \sum_i \partial_i^2.
\]
and we let $Q : V \to V$ be the operator of multiplication by the quadratic form $q = x_1^2 + x_2^2 + \cdots + x_n^2$.

a) Find scalars $a$ and $b$ such that

$$e \mapsto aQ, \quad h \mapsto D + \frac{n}{2}, \quad f \mapsto b\Delta$$

defines a representation of $\mathfrak{g}$ on $V$.

Note there are no highest weight vectors in $V$.

b) The special orthogonal group $\text{SO}_n$ (preserving the quadratic form $q$) also acts on $V$ (which is the space of polynomial functions on $\mathbb{C}^n$). Show that this action of $\text{SO}_n$ commutes with the $\mathfrak{sl}_2$ action.

c) The lowest weight vectors for $\mathfrak{g}$ form the subspace $H = \ker \Delta$ of harmonic polynomials, which are graded by degree: $H = \bigoplus H_j$. Show that $\text{SO}_n$ preserves each $H_j$ and that $H_j$ is irreducible for $\text{SO}_n$. What is the highest weight of $H_j$ for the Lie algebra $\mathfrak{so}_n(\mathbb{C})$?

d) Regarding $V$ as a representation of the Lie algebra $\mathfrak{so}_n(\mathbb{C}) \oplus \mathfrak{g}$, the subspaces

$$V_+ = \bigoplus_{i,j \geq 0} q^i H_{2j}, \quad V_- = \bigoplus_{i,j \geq 0} q^i H_{2j+1}$$

of even and odd degree polynomials are invariant. Are $V_\pm$ irreducible for $\mathfrak{so}_n(\mathbb{C}) \oplus \mathfrak{g}$?

For more on this see the book by Roger Howe and Eng-chye Tan, “Harmonic Analysis: Applications of $\text{SL}(2, \mathbb{R})$.”

**Exercise 5.** Take a nonzero $\eta \in \mathbb{C}$, and let $\mathbb{C}(\eta)$ be the one-dimensional $\mathcal{U}(\mathfrak{n})$-module $\mathbb{C}$ on which $e \cdot z = \eta z$. The induced module

$$\mathcal{Y}(\eta) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{n}) \mathbb{C}(\eta)$$

is the analogue of the Gelfand-Graev representation for $\mathcal{U}(\mathfrak{g})$. Now for any $\lambda \in \mathbb{C}$, consider the quotient

$$\mathcal{Y}(\eta, \lambda) := \mathcal{Y}(\eta)/(c - \lambda)\mathcal{Y}(\eta),$$

on which the Casimir element $c$ acts via the scalar $\lambda$. Let $v \in \mathcal{Y}(\eta, \lambda)$ be the image of $1 \otimes 1$ from $\mathcal{Y}(\eta)$.

a) Show that the linear map $\mathbb{C}[t] \to \mathcal{Y}(\eta, \lambda)$ given by $t^i \mapsto h^i \cdot v$ is bijective.

b) By a), the $\mathfrak{g}$-action on $\mathcal{Y}(\eta, \lambda)$ transfers to the space of polynomials $\mathbb{C}[t]$. Find explicit formulas for this action of $e, f, h$ on $\mathbb{C}[t]$.

c) Using the formulas in b), prove that $\mathcal{Y}(\eta, \lambda)$ is an irreducible representation of $\mathcal{U}(\mathfrak{b})$, hence of $\mathcal{U}(\mathfrak{g})$.

d) Show that $h$ has no eigenvectors in $\mathcal{Y}(\eta, \lambda)$.

For extensions of this to general semisimple Lie algebras, see Bert Kostant’s paper “On Whittaker Vectors and Representation Theory” Inventiones mathematicae 48 (1978): 101-184.