In these exercises, \( g \) is a simple complex Lie algebra with Cartan subalgebra \( t \), positive roots \( R^+ \) and simple roots \( \{\alpha_1, \ldots, \alpha_n\} \subset R^+ \). An \( \mathfrak{sl}_2 \)-\textbf{triple} in \( g \) is a triple \((e, h, f)\) of elements in \( g \) satisfying the relations 
\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\]
For each \( i = 1, \ldots, n \), let \((e_i, h_i, f_i)\) be the \( \mathfrak{sl}_2 \)-triple arising from the embedding \( \phi_{\alpha_i} : \mathfrak{sl}_2 \rightarrow g \). For each integer \( k \geq 0 \), \( V_k \) is the irreducible \( \mathfrak{sl}_2 \)-representation of dimension \( k + 1 \).

**Exercise 1. (The principal \( \mathfrak{sl}_2 \))**

1. Let \( h \in t \) be defined by the condition \( \langle \alpha_i, h \rangle = 2 \) for \( i = 1, \ldots, n \). Write \( h = \sum r_i h_i \), and set \( e = \sum_{i=1}^n e_i \) and \( f = \sum r_i f_i \). Prove the following.
   a) The triple \((e, h, f)\) is an \( \mathfrak{sl}_2 \)-triple. [Hint: the difference of simple roots is never a root.]
   b) The restriction to \( s \) of the adjoint representation of \( g \) has the form 
   \[
   g|_s = \bigoplus_{i=1}^n V_{2m_i}
   \]
   where the \( m_i \) are positive integers whose sum \( \sum m_i = |R^+| \). (Hint: consider the centralizer of \( h \) in \( g \).)
   c) An irreducible self-dual representation \( V \) of \( g \) is orthogonal if \( \langle \lambda, h \rangle \) is even, and is symplectic if \( \langle \lambda, h \rangle \) is odd. For which \( g \) is every irreducible representation orthogonal?
   d) \( e \) is contained in exactly one Borel subalgebra of \( g \). (See Kostant, thm. 5.6.)
   e) Let \( a \) be the centralizer of \( e \) in \( g \). Then \( \dim a = n \) (easy) and \( a \) is abelian (not so easy-see Kostant, thm 5.7).
   f) For \( g = \mathfrak{sl}_{n+1} \), show that \( e \) is conjugate by a diagonal matrix in \( \text{SL}_{n+1} \) to the Jordan matrix sending each ordered basis vector to the previous one. Compute \( a \) for this Jordan matrix.
   g) If \( g \) has type \( B_k \) or \( C_k \), the embedding \( \mathfrak{sl}_2 \rightarrow s \subset g \) is given by the irreducible representaton \( V_{2k+1} \) or \( V_{2k} \), respectively.

---

1 For much more information, see chap. 5 in Kostant: *The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group* Amer. J. Math. published during the month I was born.

2 The integers \( m_1, \ldots, m_n \) are called the \textbf{exponents} of \( g \).
Exercise 2. Let $G$ be the simply connected group with Lie algebra $\mathfrak{g}$ and let $T \subset G$ be the maximal torus whose Lie algebra is $\mathfrak{t}$. Let $\varphi : \text{SL}_2(\mathbb{C}) \to G$ be the map whose derivative sends $\mathfrak{sl}_2 \to \mathfrak{s}$. Using the results from 1, show the following.

a) $\varphi(-I)$ is contained in the center of $G$.

b) If $V$ is a self-dual irreducible representation of $G$ then $\varphi(-I)$ acts on $V$ by $+1$ if $V$ is orthogonal and $-1$ if $V$ is symplectic.

c) $\varphi \left( \begin{array}{cc} \star & 0 \\ 0 & \star \end{array} \right)$ is contained in a unique maximal torus $T$ of $G$.

d) $\varphi \left( \begin{array}{cc} \star & \star \\ 0 & \star \end{array} \right)$ is contained in a unique Borel subgroup of $G$.

e) $\varphi \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ normalizes $T$ and its image in $W$ is the longest element $w_0$.

Exercise 3. For the irreducible representation $L(\delta)$, the Weyl dimension formula gives $\dim(L(\delta)) = 2^{\#R^+}$. This suggests there should be a natural bijection between subsets of $R^+$ and the weights of $L(\delta)$. Indeed, use the Weyl character formula to prove that the character $\chi_\delta$ of $L(\delta)$ is given by

$$\chi_\delta = e_\delta \prod_{\alpha \in R^+} (1 + e_{-\alpha}).$$

Hint: $e_{w(\delta+\delta)}(t) = e_{w\delta}(t^2)$. Your proof should generalize easily to $L(k\delta)$, for any integer $k \geq 1$.

Exercise 4. For each $\alpha \in R^+$, let $c(\alpha) = \sum c_i$, where $\alpha = \sum c_i \alpha_i$. For each $S \subset R^+$, let $c(S) = \sum_{\alpha \in S} c(\alpha)$, and let $c = C(R^+)$. Show that the restriction of $L(\delta)$ to the principal $\mathfrak{sl}_2$ is given by

$$L(\delta)|_\mathfrak{s} = \sum_{i=0}^{\lfloor c/2 \rfloor} (n_i - n_{i-1})V_{c-2i},$$

where $n_i$ is the number of subsets $S \subset R^+$ with $c(S) = i$ and $\lfloor c/2 \rfloor$ is the greatest integer $\leq c/2$. Find the numbers $n_i$ for types $A_2$, $B_2$ and $G_2$.

\footnote{It can be shown that $c = \sum_{i=1}^{m_i} \binom{m_i+1}{2}$, where the $m_i$ are the exponents of $\mathfrak{g}$, cf. exercise 1b.}