MT845 Homework 1

Don’t forget to type Your Name!

Due Monday, September 27

Exercise 1. This exercise shows how to view quaternions as $2 \times 2$ complex matrices. View $\mathbb{C}$ as the subring $\{t + xi : t, x \in \mathbb{R}\}$ of $\mathbb{H}$, and let $M_2(\mathbb{C})$ denote the ring of $2 \times 2$ complex matrices. Show the following.

a) Every quaternion $q \in \mathbb{H}$ can be uniquely expressed as $q = \alpha + j\beta$ for some $\alpha, \beta \in \mathbb{C}$.

b) The map $\rho : \mathbb{H} \to M_2(\mathbb{C})$ given by $\rho(\alpha + j\beta) = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}$ is an injective ring homomorphism.

c) Show that the image of $\rho$ is the subring of $M_2(\mathbb{C})$ consisting of matrices $A$ for which $AJ = J\bar{A}$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\bar{A}$ is the matrix obtained by complex-conjugating all the entries in $A$.

d) Show that $\rho(S^3)$ is the group $SU_2(\mathbb{C}) := \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \in GL_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 \right\} \leq SL_2(\mathbb{C})$.

Proof.

Exercise 2. Give an algebraic proof of the formula

$$e^{k\theta/2}e^{j\phi/2} \cdot k \cdot e^{-j\phi/2}e^{-k\theta/2} = (\sin \phi \cos \theta)i + (\sin \phi \sin \theta)j + (\cos \phi)k,$$

which was demonstrated geometrically in class.

Proof.

Exercise 3. Consider the circle $T_c = \{t + xi \in \mathbb{H} : t^2 + x^2 = 1\}$ in $S^3$. Prove that if $s \in T$ and $s \neq \pm 1$ then the centralizer of $s$ in $S^3$ is exactly $T$.

Proof.

Exercise 4. Compute the square of a general quaternion $q = t + xi + yj + zk$. Then use this to show that the conjugacy class $C_0 = \{xi + yj + zk \in \mathbb{H} : x^2 + y^2 + z^2 = 1\}$ consists precisely of the elements of order four in $S^3$.

Proof.

Exercise 5. Let $G = SO_3(\mathbb{R}) = \{g \in GL_3(\mathbb{R}) : {}^tgg = I \text{ and } \det g = 1\}$. Let $T$ be the subgroup of $G$ fixing the first basis vector $e_1$ in $\mathbb{R}^3$. Show the following.

a) $T \simeq SO_2(\mathbb{R})$ and $N_G(T) \simeq O_2(\mathbb{R})$.

b) $G$ acts transitively on the unit sphere in $\mathbb{R}^3$. 

Proof.
c) Every element of \( G \) is rotation about some axis and is conjugate to an element of \( T \). [Hint: show first that \( \det(I - g) = 0 \) for all \( g \in G \).]

d) Two elements \( s, t \in T \) are conjugate in \( G \) if and only if \( s = t^{\pm 1} \).

e) The elements of order two in \( G \) form a single conjugacy class. Describe these elements geometrically.

Proof.

Exercise 6. The group \( G = SL_2(\mathbb{R}) \) of real \( 2 \times 2 \) matrices whose determinant = 1 has subgroups

\[
K = SO_2(\mathbb{R}) \cong S^1, \quad A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\} \cong \mathbb{R}^*_+, \quad N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R}.
\]

Show that the map \( K \times A \times N \to G \) is a bijection.

[This is called the \textbf{Iwasawa decomposition}. It shows that \( SL_2(\mathbb{R}) \) is homeomorphic to \( S^1 \times \mathbb{R}^2 \). In particular, \( SL_2(\mathbb{R}) \) is a connected, non-compact group.]

Exercise 7. Use the previous exercise to prove that the group \( GL_2(\mathbb{R})^+ = \{ g \in GL_2(\mathbb{R}) : \det g > 0 \} \) is connected. Deduce from this that \( GL_2(\mathbb{R}) \) has exactly two connected components, according to the sign of the determinant.

Exercise 8. Prove that \( S^3 = T_iT_jT_k \). That is, prove that every element \( q \in S^3 \) may be written as

\[
q = e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma}
\]

for some \( \alpha, \beta, \gamma \in [0, 2\pi] \).

Proof.