Affine-Pinned automorphisms of a simple Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbf{C}$ with adjoint group $G = \mathrm{Aut}(\mathfrak{g})^{\circ}$. Let N, T be the normalizer and centralizer in G of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and let $W = N/T$ be the Weyl group. In general, there is no subgroup of N projecting isomorphically onto W.

Let R be the set of roots of T in $\mathfrak{g}$ and choose a base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of R. Let α_0 be the lowest root of R with respect to Δ and set $\Pi = \{\alpha_i : \ i \in I\}$, where $I = \{0, 1, \dots, \ell\}$. The subgroup of W preserving Π,

$$W_\Pi = \{w \in W : \ w(\Pi) = \Pi\},$$

is isomorphic to the fundamental group of G. It appears in [3] as the image in W of an alcove stabilizer in the extended affine Weyl group, under the gradient mapping. I will show that there is a subgroup of N projecting isomorphically onto W_Π. [1]

Choose a Chevalley lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ spanned by the co-root lattice in $\mathfrak{t}$ and root vectors for T. An *affine pinning* is a set $\widetilde{\Pi} = \{E_0, E_1, \cdots, E_\ell\}$ consisting of nonzero root vectors $E_i \in \mathfrak{g}_{\alpha_i} \cap \mathfrak{g}(\mathbb{Z})$ for each $i \in I$. Let $N(\mathbb{Z})$ be the stabilizer of $\mathfrak{g}(\mathbb{Z})$ in N, and consider the subgroup

$$N_{\widetilde{\Pi}} = \{n \in N(\mathbb{Z}) : \ n(\widetilde{\Pi}) = \widetilde{\Pi}\}.$$

Proposition 1. *For any affine pinning $\widetilde{\Pi}$, the subgroup $N_{\widetilde{\Pi}}$ of N projects isomorphically onto W_Π.*

Proof. Let $f : N_{\widetilde{\Pi}} \to W$ be the restriction to $N_{\widetilde{\Pi}}$ of the projection $N \to W$. It is clear that $f(N_{\widetilde{\Pi}}) \subset W_\Pi$. An element in $\ker f$ lies in T and fixes each root vector E_i, hence lies in the center of G, which is trivial since G is adjoint. Hence f is injective.

Let $1 \neq w \in W_\Pi$ and let σ be the permutation of I defined by $w\alpha_i = \alpha_{\sigma(i)}$. By [7], the projection $N(\mathbb{Z}) \to W$ is surjective. Hence there exists $n' \in N(\mathbb{Z})$ projecting to w. For each $i \in I$ we have $n' \cdot E_i = c_i E_{\sigma(i)}$, for some $c_i = \pm 1$.

Let $\check{\omega}_1, \dots, \check{\omega}_\ell \in X_*(T)$ be the fundamental co-weights of T dual to $\alpha_1, \dots, \alpha_\ell$. The element $t = \prod_{i=1}^{\ell} \check{\omega}_i(c_i)$ lies in $T(\mathbb{Z}) (= T \cap N(\mathbb{Z}))$ and the new lift $n = n't$ of w satisfies $n \cdot E_i = E_{\sigma(i)}$ for $1 \leq i \leq \ell$.

The proposition will be proved if it is shown that $n \in N_{\widetilde{\Pi}}$. Since $n \cdot E_0 = c_0 E_{\sigma(0)}$, it suffices to show that $c_0 = +1$.

Consider the principal nilpotent element

$$e = \sum_{i=1}^{\ell} E_i.$$

There exists an $\mathfrak{sl}_2$-triple (e, h, f), where f belongs to the span of the negative root spaces. Let $C_{\mathfrak{g}}(f) = \{z \in \mathfrak{g} : [z, f] = 0\}$ be the centralizer of f. As shown by Kostant [5], the affine space

$$\mathcal{K} := e + C_{\mathfrak{g}}(f)$$

has the property that each G-orbit in $\mathfrak{g}$ meets $\mathcal{K}$ in at most one point. Since α_0 is the lowest root and f is in the span of negative root spaces, we have $E_0 \in C_{\mathfrak{g}}(f)$, so the elements $x := e + E_0$ and $x' := e + c_0 E_0$ both lie in $\mathcal{K}$. It therefore suffices to show that x and x' belong to the same G-orbit.

Since $w \neq 1$ it follows that $w\Delta \neq \Delta$, which in turn means that $\sigma(0) \neq 0$. Let $j, k \in I - \{0\}$ be such that $\sigma(j) = 0$ and $\sigma(0) = k$. We have

$$n(x') = \sum_{i \in I - \{0, j\}} E_{\sigma(i)} + n(E_j) + c_0 n(E_0) = \sum_{i \in I - \{0, k\}} E_i + E_0 + c_0^2 E_k = x,$$

since $c_0^2 = 1$. The proposition is proved.

□

[1] This result is stated as Lemma 16 in [6] but the argument presented there has a serious gap, as was pointed out to me by Jiu-Kang Yu.

References

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