LEVEL-TWO STRUCTURE OF SIMPLY-LACED COXETER GROUPS

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November 2004

Abstract. Let X be a graph, with corresponding simply-laced Coxeter group W. Then W acts naturally on the lattice L spanned by the vertices of X, preserving a quadratic form. We give conditions on X for the form to be nonsingular modulo two, and study the images of $W \longrightarrow O(L/2^k L)$.

Introduction — This paper investigates the tower of 2-power congruence subgroups in a simply-laced Coxeter group, but the story begins with a puzzle for children. We have a pile of stones, and a graph X with n vertices. At most one stone may be placed on a vertex, so a vertex has one of two states: stoned or unstoned. We move by selecting a vertex v having an odd number of stoned neighbors, and then change the state of v. Given an initial configuration of stones on X, we try to reduce the total number of stones as much as possible. How to determine this minimal number of stones from the initial configuration?

A configuration of stones is an element in the $\mathbf{F}_2$-vector space V spanned by the vertices of X. For $v \in V$, let $q(v)$ be the number of vertices plus the number of edges in the support of v, modulo two. Then q is a quadratic form on V (see section 1), and we let $O(\mathbf{F}_2)$ denote the subgroup of $GL_n(\mathbf{F}_2)$ preserving q.

The moves are linear maps on V preserving q, and are the images of simple reflections under the natural homomorphism

$$\rho : W \longrightarrow O(\mathbf{F}_2),$$

where W is the (simply-laced) Coxeter group having X as Coxeter diagram. Our puzzle can, and henceforth will be rephrased as follows: Find the orbits of W on V, determinine the orbit of a given vector, and find a vector in each orbit which is minimal, in the sense of having the fewest number of nonzero coefficients in terms of the vertex basis.

Partially supported by National Science Foundation grant DMS-0207231

We see at once that there are at least two nonzero minimal vectors, namely a single stoned vertex (contained in an orbit with $q = 1$) or two non-adjacent stoned vertices (contained in an orbit with $q = 0$). If q separates the nonzero W-orbits on V, then the puzzle is solved: there are two nonzero orbits, determined by $q = 0$ or $q = 1$, and a minimal vector can be determined by evaluating q on the initial vector. So, when does q separate the nonzero W-orbits on V?

Suppose that q is nonsingular and ρ is surjective. Results of Arf-Witt and Dieudonné (see section 2) imply the following: If n is even, then q separates orbits. If n is odd there is just one additional orbit, consisting of a single W-invariant vector which is easy to spot. This leads us to the main question addressed in the first part of this paper: For which graphs X is q nonsingular and ρ surjective?

If W is a finite irreducible Weyl group, that is, if X is a Dynkin diagram of type ADE, the answer is easily checked in each case (and also follows from the results herein). We find that q is nonsingular and ρ is surjective exactly for types $A_1, A_2, A_4, A_5, E_6, E_7, E_8$. In fact, it is well-known that the map ρ gives isomorphisms

$$S_2/\pm 1 \longrightarrow O_1(\mathbf{F}_2), \quad S_3 \longrightarrow O_2^-(\mathbf{F}_2), \quad S_5 \longrightarrow O_4^-(\mathbf{F}_2), \quad S_6 \longrightarrow O_5(\mathbf{F}_2),$$

$$W(E_6) \longrightarrow O_6^-(\mathbf{F}_2), \quad W(E_7)/\pm 1 \longrightarrow O_7(\mathbf{F}_2), \quad W(E_8)/\pm 1 \longrightarrow O_8^+(\mathbf{F}_2).$$

Here, $O_n(\mathbf{F}_2)$ or $O_n^{\pm}(\mathbf{F}_2)$ denotes the orthogonal group of a nonsingular quadratic form on $\mathbf{F}_2^n$ which for n even is split $(+)$ or nonsplit $(-)$.

The inverses of the three nontrivial type A isomorphisms are given by the permutation action of the orthogonal group on, respectively, the vectors with $q = 1$, vectors with $q = 0$ and $-$ type hyperplanes in $\mathbf{F}_2^5$. Counting arguments (see [B, pp. 242-3]) show that ρ is surjective in type E.

In the finite case, the W-orbits on V have several interpretations; we mention two. First, if we identify vertices in X with simple co-roots in the corresponding simply-connected Lie group G, then a configuration of stones is an involution in a maximal torus of G, and the moves are conjugation by simple reflections in the Weyl group W of G. Our puzzle amounts, for finite W, to determining the conjugacy class of a given involution. Thus, in the seven cases above, we have two conjugacy classes of involutions given by $q = 1$ and $q = 0$, and an additional central involution in A_5, E_7. We remark that the conjugacy classes of all finite order elements in G were classified by Kac, in terms of coefficients in the highest root (see [H, chapter 10]), but it is not always easy to determine the class of a given element.

The second interpretation arises from recent work on the local Langlands correspondence [DR, section 13]. Here, the vectors in V parametrize certain finite sets of irreducible representations of a p-adic group G, and the W-orbits in V correspond to certain rational classes of tori in G. The present paper arose in this context, while trying to understand the particularly nice example of E_8 in terms of its graph, without resorting to counting.

In this paper we consider, in places, an arbitrary graph X, but we mostly restrict to the case where X is a tree. We say the graph X is *nonsingular* if the quadratic form q is nonsingular on V.

In section 4, we give simple graph-theoretic conditions for X to be nonsingular, and nonsingular trees are characterized in terms of "sprouting" and "pruning". Then in section 7, we prove:

Theorem 1. *If X is a non-singular tree, not of type A_n, then the map $\rho : W \longrightarrow O(\mathbf{F}_2)$ is surjective.*

Thus, we find that E_6, E_7, E_8 are rather the norms than the exceptions, for nonsingular trees; unlike A_1, A_2, A_4, A_5, they are not "low dimensional accidents". The branch node makes all the difference.

Theorem 1 leads us to consider the kernel of ρ, which is only interesting for infinite W. More generally, we consider the congruence subgroups

$$W_k = \ker[W \longrightarrow O(\mathbf{Z}/2^k\mathbf{Z})].$$

For $k > 1$ the groups W_k are torsion-free. It follows easily from properties of the Tits cone that the torsion in W_1 consists of a finite number of conjugacy classes of involutions, corresponding to subgraphs of type E_8, and certain subgraphs of type E_7 (see section 8).

The quotients W_k/W_{k+1} are elementary abelian 2-groups, and we show, for "most" even nonsingular trees, that the rank is as large as possible. (Our arguments apply only to even graphs.) To state the result, let $O'(\mathbf{Z}_2)$ denote the kernel of the 2-adic spinor norm

$$\delta_{\mathbf{Q}_2} : O(\mathbf{Z}_2) \longrightarrow \mathbf{Q}_2^\times/\mathbf{Q}_2^{\times 2}.$$

Then $W \subset O'(\mathbf{Z}_2)$ for every nonsingular graph X, since $\delta_{\mathbf{Q}_2} = 1$ on the simple reflections.

In sections 9-12, we prove our second main result:

Theorem 2. *Assume X is a nonsingular even tree containing a non-singular even hyperbolic subtree. Then W is 2-adically dense in $O'(\mathbf{Z}_2)$. Equivalently,*

$$W_k/W_{k+1} \simeq \begin{cases} \mathfrak{so}'(\mathbf{F}_2) & \text{if} \quad k = 1, 2 \\ \mathfrak{so}(\mathbf{F}_2) & \text{if} \quad k \geq 3, \end{cases}$$

where $\mathfrak{so}(\mathbf{F}_2)$ is the Lie algebra of $O(\mathbf{F}_2)$ and $\mathfrak{so}'(\mathbf{F}_2)$ is the commutator subalgebra.

Here, a tree is *hyperbolic* if its Coxeter group is infinite, and every proper subtree has finite or affine Coxeter group. In fact, there are only two nonsingular even hyperbolic trees, namely E_{10} and $T_{3,3,4}$ (see section 4). To prove Theorem 2 we use Kac's result that $\pm\,\mathrm{Aut}(X)W = O(\mathbf{Z})$ when X is hyperbolic, along with strong approximation, Theorem 1, and the structure of the adjoint representation of $O(\mathbf{F}_2)$.

1. Graphs and quadratic forms — In this paper, a graph X has a finite vertex set $S = S(X)$, and edges are two-element subsets of S. Our graphs have no loops, or multiple edges. If $\{i, j\}$ is an edge we say vertices i, j are adjacent, or are neighbors, and write i—j. The degree of a vertex is the number of edges containing it. Given an ordering on S, let $A = [a_{ij}]$ be the adjacency matrix of X, defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i\text{—}j \\ 0 & \text{otherwise.} \end{cases}$$

If X and Y are two graphs, then $X + Y$ denotes the disjoint union of X and Y. If $J \subset S$, the *full subgraph on* J is the graph $[J]$ with vertex set J and all edges in X between vertices in J.

Given a graph X with vertex set S, let L be a lattice of rank n with basis $\{\alpha_i : \ i \in S\}$, and let $\langle \ , \ \rangle$ be the symmetric bilinear pairing on L with matrix $2I - A$ (for some ordering of the basis $\{\alpha_i\}$). Note that $\langle\lambda,\lambda\rangle \subset 2\mathbf{Z}$ for all $\lambda \in L$. Let q be the quadratic form on L defined by $q(\lambda) = \frac{1}{2}\langle\lambda,\lambda\rangle$. We also write q for the base extension of q to $R \otimes L$, for any commutative ring R.

Many coefficient rings appear later in the paper, but until further notice we take $R = \mathbf{F}_2$.

Let $V = \mathbf{F}_2 \otimes L$. It has a basis $\{e_i = 1 \otimes \alpha_i\}$. If $x = \sum x_i e_i \in V$, then

$$q(x) = \sum_i x_i^2 + \sum_{\substack{i<j \\ i - j}} x_i x_j \in \mathbf{F}_2.$$

This yields the description of q given in the introduction. We can visualize x as a binary coloring of the vertices of X, where i is colored $\bullet$ if $x_i = 1$, and colored $\circ$ if $x_i = 0$. If $[x]$ denotes the full subgraph of X on the $\bullet$ vertices, then $q(x)$ is the Euler characteristic of the 1-complex $[x]$, modulo two. If $[x]$ has no cycles, then

$$q(x) \equiv c(x) \mod 2,$$

where $c(x)$ is the number of connected components of $[x]$.

The associated symplectic form $f : V \otimes V \longrightarrow \mathbf{F}_2$ is given by

$$f(x,y) = q(x+y) + q(x) + q(y).$$

The bilinear form f has matrix $[f(e_i,e_j)] \equiv A \mod 2$. We write

$$\ker_2 X = \{x \in V : \ f(x,V) = 0\},$$

$$\ker_2 q = \{x \in \ker_2 X : \ q(x) = 0\}.$$

Note that $\ker_2 X = \ker A|_V$. We can visualize $\ker_2 X$ as the set of binary vertex colorings of X in which every vertex has an even number of $\bullet$ neighbors. Since f induces a non-degenerate symplectic form on $V/\ker_2 X$, we have

$$\dim \ker_2 X \equiv \dim V \mod 2.$$

A vector $x \in V$ for which $q(x) = 0$ is called *q-isotropic*. We define

$$V(0) := \{x \in V : \ x \neq 0, \ q(x) = 0\}, \qquad V(1) := \{x \in V : \ q(x) = 1\}.$$

The form q and the graph X are called *nonsingular* if $\ker_2 q = 0$.

2. Orthogonal groups over $\mathbf{F}_2$ — Let $O(V)$ denote the automorphism group of the quadratic $\mathbf{F}_2$ vector space V. Many arguments in this paper depend on the parity of n. We collect some known facts needed in each case.

Lemma 2.1. *Suppose $n = 2m+1$ is odd. Then the form q is nonsingular if and only if $\ker_2 X = \{0,u\}$ has two elements, and $q(u) \neq 0$. In this case, we have*

(1) $V(1) = u + V(0)$.
(2) *The group $O(V)$ has four orbits in V, namely,*

$$\{0\}, \quad \{u\}, \quad V(0), \quad V(1).$$

Proof. The restriction of q to $\ker_2 X$ is a linear functional from $\ker_2 X$ to $\mathbf{F}_2$. By definition, this functional is injective if and only if q is nonsingular. Since $\dim \ker_2 X$ is odd, the first assertion is immediate.

Suppose q is nonsingular, and let u be the nonzero element of $\ker_2 X$. For each coset $\{x, x+u\} \in V/\ker_2 X$, we have $q(x+u) = q(x)+1$, hence (1) holds. Assertion (2) follows from the Arf-Witt theorem [D,p.41]. □

If $n = 2m$, there are two equivalence classes of quadratic forms on V, according to the maximal dimension of a subspace on which both q and f vanish identically. This dimension is $m - d$, where $d \in \{0, 1\}$ is called the *defect* of q.

Lemma 2.2. *Suppose $n = 2m$ is even. Then q is nonsingular iff $\ker_2 X = 0$, that is, iff $\det A$ is odd. In this case, the following hold.*

(1) *The defect is given by*

$$d \equiv \frac{D^2 - 1}{8} \mod 2,$$

where $D = \det[2I - A]$.

(2) *The group $O(V)$ has three orbits in V, namely,*

$$\{0\}, \quad V(0), \quad V(1).$$

Proof. Since $\dim \ker_2 X$ is even, the linear functional $q : \ker_2 X \to \mathbf{F}_2$ is injective iff $\ker_2 X = 0$, hence the first assertion.

From the classification of quadratic forms over $\mathbf{Z}_2$ [Ki, 5.2.5,6] it follows that

$$\mathbf{Z}_2 \otimes L \simeq (m-d) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \perp d \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

as quadratic spaces. This implies (1). Assertion (2) again follows from the Arf-Witt theorem. □

Return now to arbitrary n. A *transvection* is an element of $O(V)$ of the form $x \mapsto x + f(x,y)y$ for some $y \in V(1)$. It follows from Lemmas 2.1(2) and 2.2(2) that the transvections form a single conjugacy class in $O(V)$.

The next result is our main tool in proving surjectivity.

Lemma 2.3. *Suppose X is a nonsingular graph, not isomorphic to $A_2 + A_2$. Let G be a subgroup of $O(V)$ containing a transvection. Then $G = O(V)$ if and only if G is transitive on $V(1)$.*

Proof. If G contains one transvection, and is transitive on $V(1)$, then G contains all transvections. We will have $G = O(V)$ if the transvections generate $O(V)$. It is known [D, Prop. 14] that, for nonsingular q, the transvections do indeed generate $O(V)$, except if $n = 4$ and $d = 0$.

There are four nonsingular graphs with four vertices, having $D = -27, -3, 5, 9$. From Lemma 2.2(1), the respective defects are $d = 1, 1, 1, 0$, the latter coming from $D = 9$ for $A_2 + A_2$. □

3. Nonsingularity conditions for graphs — Let X be a graph with vertex set $S = \{1, \ldots, n\}$, and quadratic form q on $V = \mathbf{F}_2 \otimes L$, as above. In this section we translate the nonsingularity of q into conditions on the graph X.

Let

$$\det(X) := \det A = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) a_{1,\sigma 1} a_{2,\sigma 2} \cdots a_{n,\sigma n} \in \mathbf{Z},$$

where $\epsilon(\sigma)$ is the sign of the permutation $\sigma \in \mathcal{S}_n$. The σ-term is nonzero iff σ belongs to the set $\mathcal{A}(X)$ of those permutations that move every vertex to one of its neighbors, so

$$\det(X) = \sum_{\sigma \in \mathcal{A}(X)} \epsilon(\sigma).$$

Let $Z(X)$ be the set of n-vertex subgraphs $U \subset X$ whose components are either segments $\circ\!\!-\!\!\circ$ or cycles. Let $z(U)$ be the number of components of U which are cycles. The orbits of $\sigma \in \mathcal{A}(X)$ on S define an element $U(\sigma) \in Z(X)$, and $U(\sigma) = U(\sigma')$ iff σ' is obtained from σ by inverting some k-cycles in σ, for $k \geq 3$. This implies that $\epsilon(\sigma) = \epsilon(\sigma')$, and that

$$\det(X) = \sum_{U \in Z(X)} \epsilon(U) 2^{z(U)}, \tag{3a}$$

where $\epsilon(U)$ is the sign of any permutation $\sigma \in \mathcal{A}(X)$ with $U(\sigma) = U$.

An n-vertex subgraph Y of X is called a *maximal matching* if Y contains $\lfloor \frac{n}{2} \rfloor$ connected components each of which is a segment, and if n is odd, one additional isolated vertex, denoted $j(Y)$. Let $M(X)$ be the set of all maximal matchings in X.

3.1 Lemma. *If $n = 2m$ then X is nonsingular if and only if X has an odd number of maximal matchings.*

Proof. If $n = 2m$, then a maximal matching is an element $Y \in Z(X)$ with $z(Y) = 0$, so the claim follows from (3a). □

For any $j \in S$, let X_j be the full subgraph of X supported on $S - \{j\}$. If n is odd, the segments in a maximal matching Y of X form a maximal matching in $X_{j(Y)}$. Consider the vector

$$u := \sum_{j \in S} \det(X_j) e_j = \sum_{Y \in M(X)} e_{j(Y)}. \tag{3b}$$

3.2 Lemma. *If $n = 2m + 1$, then the following hold.*

(1) *$u \in \ker_2 X$.*
(2) *$u \neq 0$ if and only if $\ker_2 X = \{0, u\}$.*
(3) *X is nonsingular if and only if $q(u) = 1$.*

Proof. We first show that $u \in \ker_2 X$. We must show that every $i \in S$ has an even number of neighbors j with $\det X_j$ odd. Let $M(i)$ be the set of all maximal matchings Y in X such that $j(Y)$—i. Then $M(i)$ is the disjoint union of the sets of maximal matchings in X_j, for j—i. From the even case just proved, we get

$$|M(i)| \equiv \sum_{j - i} \det(X_j) \mod 2.$$

Hence it suffices to show that $|M(i)|$ is even. To this end we construct a fixed-point free involution on $M(i)$. Let $Y \in M(i)$, and let $j = j(Y)$. Since $i \neq j$, there is a unique edge in Y meeting i, say $\{i, j'\}$. Replace this edge by $\{i, j\}$, and keep the remaining edges in Y. This gives a new maximal matching Y' which again belongs to $M(i)$, since $j(Y') = j'$—i. Clearly $Y' \neq Y$. Repeating the procedure with Y' will give Y again. This completes the proof of (1).

For (2), note that each $\det X_i$ is a minor of X. Hence $u \neq 0$ iff $\dim \ker_2 X = 1$. Finally, if $q(u) = 1$, then (2) holds, so X is nonsingular. The converse is clear. □

3.3 Definition. We call the vector u defined in (3b) the *kernel vector* of an odd graph X, and the vertices i in the support of u (i.e., those with $\det X_i \neq 0$) the *kernel vertices* of X.

3.4 Example. If X is a Dynkin diagram of finite type, then the kernel vector corresponds to a central involution (see the Introduction).

3.5 Example. Consider the complete graph K_n. By induction, we find there are $1{\cdot}3 \cdots (2m-1)$ maximal matchings in K_{2m}, and $1{\cdot}3 \cdots (2m+1)$ maximal matchings in K_{2m+1}. Thus, K_{2m} is nonsingular for all m, by 3.1. For $X = K_{2m+1}$, we have $X_j \simeq K_{2m}$ for all j, so

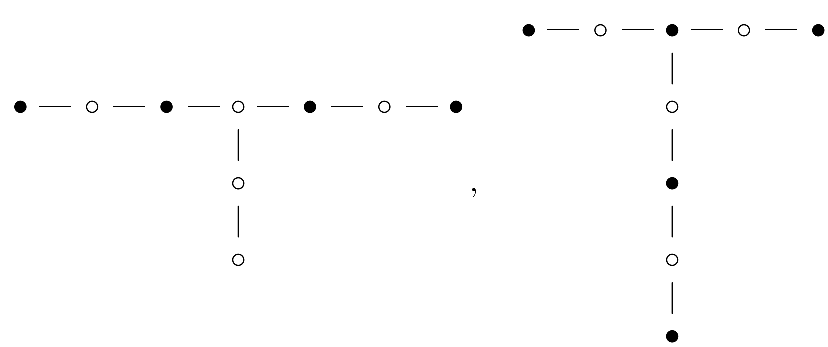
$$u = \sum_{j=1}^{2m+1} e_j,$$

and

$$q(u) \equiv 2m + 1 + \binom{2m+1}{2} = (m+1)(2m+1) \equiv m + 1 \mod 2.$$

Thus, 3.2 shows that K_{4m+1} is nonsingular, while K_{4m+3} is singular.

3.6 Example. We indicate the kernel vector in the following graphs by using • for the kernel vertices. It shows that the first is singular, and the second is nonsingular.



If the nonsingular graphs with $2m$ vertices are known, one can use 3.2 to determine the nonsingular graphs with $2m+1$ vertices, by determining whether each X_j is singular or not, thereby constructing the kernel vector u, and then evaluating $q(u)$.

For example, there are four nonsingular graphs with four vertices, and 21 connected graphs with five vertices. Examining the X_j's in each of the latter, one finds exactly six nonsingular connected graphs with five vertices. We leave this as an exercise.

This method does not seem practical for larger graphs. For trees, one can do better. Lemma 4.8 below contains a much simpler constructive procedure for finding all nonsingular trees.

4. Trees — In the previous section we have seen that odd complete graphs can have an odd number of maximal matchings and still be singular. In this section, we find that this cannot happen if X is a tree. This is a by-product of the construction of all nonsingular trees by "sprouting", as will be explained.

We adopt the standard terminology: A *tree* is a connected graph without cycles. A *leaf* in a graph X is a vertex contained in a unique edge of X. A *branch node* is a vertex of degree at least three.

The first step is a variation on the well-known result "trees have leaves".

4.1 Lemma. *If X is a tree then at least one of the following holds.*

(1) *X consists of a single vertex.*
(2) *There is a vertex in X adjacent to two or more leaves.*
(3) *There is a leaf in X adjacent to a vertex of degree two.*

Proof. Suppose (1-3) all fail to hold. Then every vertex is adjacent to at most one leaf, and every leaf is adjacent to a branch node. We will get a contradiction by constructing a cycle in X. Pick a leaf v_0, and let v_1 be a branch node adjacent to v_0. Proceed away from v_1 on an edge other than v_0, v_1. The next vertex cannot be a leaf, since v_1 is already adjacent to the leaf v_0. If the next vertex has degree 2 then proceed on a new edge. Since no leaf is adjacent to a degree two vertex, we eventually arrive at new branch node v_2. Since $\deg v_2 \geq 3$, and v_2 is adjacent to at most one leaf, we can exit v_2 on a new edge which does not end in a leaf. In this way we visit an unlimited number of branch nodes, so we eventually visit the same branch node twice. □

Note that if X is any nonsingular tree, then 4.1(2) cannot hold, for if i, j are leaves adjacent to the same vertex, then $e_i + e_j \in \ker_2 X$ and $q(e_i + e_j) = 0$. This can also be seen from (3a), since $Z(X)$ is empty.

4.2 Definition. If the tree X is obtained by attaching i—j— to some vertex k in a tree X', we say X is obtained from X' by *sprouting* at k, and X' is obtained from X by *pruning* at k.

Our eventual aim is to show how all nonsingular trees may be obtained by sprouting. We begin with even trees.

4.3 Lemma. *A tree X with $2m$ vertices is nonsingular if and only if it has a maximal matching, in which case the maximal matching is unique, and* $\det(X) = (-1)^m$. *The set of nonsingular even trees is preserved under sprouting and pruning. All such trees are obtained by starting with the segment* ∘—∘ *and sprouting at arbitrary vertices.*

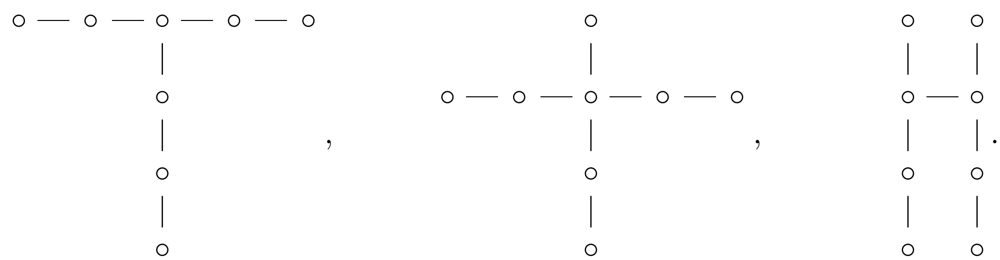
Proof. If X has no maximal matchings, then it is singular by 3.1. Suppose X has a maximal matching. Then 4.1(1,2) cannot hold, so X has a subgraph i—j—k with no other edges in X meeting i or j. Prune at k to obtain a new graph X'. Since any maximal matching in X must contain $\{i, j\}$, we have

$$\det(X) = -\det(X').$$

The lemma follows by induction. □

4.4 Example. The nonsingular even trees with $n \leq 8$ are $A_2,\ A_4,\ A_6,\ E_6,\ E_8,$

$T_{3,3,4} :=$



To prepare for odd trees, we first consider the support of vectors in $\ker_2 X$. Recall from section 1 that $\ker_2 X$ consists of those binary vertex colorings of X in which every vertex has an even number of $\bullet$ neighbors.

4.5 Lemma. *Suppose X is any tree, and $0 \neq x \in \ker_2 X$. Then the connected components of $[x]$ are single vertices. At least one vertex in $[x]$ is a leaf in X.*

Proof. Any component c of $[x]$ is a tree. If c has more than one vertex, then c has a leaf i, adjacent to a unique vertex j in c. Hence j is the only $\bullet$ neighbor of i, which contradicts $x \in \ker_2 X$.

For the second assertion suppose X has n vertices, and the assertion is true for trees with fewer than n vertices. Write $x = \sum_{i=1}^{n} x_i e_i$. Choose any leaf i in X, and let j be the neighbor of i. If $x_i = 1$ we have found the desired leaf. If $x_i = 0$, remove the vertex i and the edge $\{i, j\}$ to obtain the tree X_i. We have $x \in \ker_2 X_i$. Note that $x_j = 0$ since i had an even number of (hence zero) $\bullet$ neighbors in x. By induction, there is a leaf ℓ in X_i with $x_\ell \neq 0$. Since $x_j \neq x_\ell$, we must have $j \neq \ell$, so ℓ is also a leaf in X. $\square$

For X odd, we can apply 4.5 to the kernel vector

$$u = \sum_{j=1}^{n} \det(X_j) e_j \in \ker_2 X$$

and find that $q(u)$ is the number of nonsingular X_j's modulo two. By the uniqueness in 4.3, this implies the following uniform nonsingularity condition for trees with any number of vertices.

4.6 Lemma. *Let X be a tree. Then X is nonsingular if and only if X contains an odd number of maximal matchings.*

The construction of odd nonsingular trees by sprouting has an extra wrinkle, because one must also consider singular trees with $|\ker_2 X| = 2$.

4.7 Lemma. *If X is an odd tree, then the following are equivalent:*

(1) $|\ker_2 X| = 2$.
(2) *The kernel vector u is nonzero.*
(3) *There exists a maximal matching in X.*

Proof. We have seen in 3.2 that (1) and (2) are equivalent for any odd graph. If $u \neq 0$, then some X_i is nonsingular, so has a maximal matching Y_i, by 4.3. Adding the vertex i to Y_i gives a maximal matching in X. Conversely, if Y is a maximal matching in X, then removing the isolated vertex $j = j(Y)$ gives a maximal matching in X_j, so $\det(X_j) \neq 0$, so $u \neq 0$. $\square$

4.8 Lemma. *The set of odd trees X with $|\ker_2 X| = 2$ is preserved under sprouting and pruning. Every such tree is obtained by a sequence of sproutings, starting with a single vertex. Suppose X is obtained from X' by sprouting i—j— at the vertex k. Then i is a kernel vertex in X if and only if k is a kernel vertex in X'. If this holds then X is nonsingular if and only if X' is singular, and $u = u' + e_i$. Otherwise (i.e., if k is not a kernel vertex in X'), X is singular if and only if X' is singular, and $u = u'$. (Here u and u' are the kernel vectors of X, X'.)*

Proof. We first claim that if $n > 1$ and $|\ker_2 X| = 2$ then 4.1(3) holds. For if not, then by 4.1(2) there are at least three distinct leaves i, i', j with i, i' adjacent to the same vertex, and a leaf j' (which may be one of i, i') adjacent the same vertex as j. Then $e_i + e_{i'}$ and $e_j + e_{j'}$ are two linearly independent vectors in $\ker_2 X$.

Now, if X' is obtained by pruning the sprout i—j— from 4.1(3), then X has a maximal matching if and only if X' does. This proves the first two sentences in the lemma.

If k is a kernel vertex in X', there exists a maximal matching Y in X'_k. When we add the edge $\{j, k\}$ to X we get an even tree X'' which is nonsingular, since it contains the maximal matching $Y'' = Y \cup \{j, k\}$. Hence i is a kernel vertex in X. Conversely, if i is a kernel vertex in X, then $X'' = X_i$ is nonsingular and $\{j, k\}$ belongs to the unique maximal matching in X''. Removing this edge gives a maximal matching in X_k, hence k is a kernel vertex in X'.

If k is a kernel vertex in X', the number of maximal matchings in X is, on account of Y'', one more than the number of maximal matchings in X'. Hence X is nonsingular if and only if X' is singular.

If k is not a kernel vertex in X' then every maximal matching in X is obtained by adding $\{i, j\}$ to a maximal matching of X'. Hence X and X' have the same number of maximal matchings. □

4.9 Example: We illustrate the method of sprouting/pruning with the family of graphs $T_{p,q,r}$. Here $p, q, r \geq 2$ and $T_{p,q,r}$ is the graph with $n = p + q + r - 2$ vertices consisting of subgraphs A_p, A_q, A_r each joined at one end in a vertex of degree three. For example, $E_8 = T_{2,3,5}$, and $T_{3,3,4}$ was shown in 4.4 above.

Assume first that n is even. If p, q, r are all even, we can prune $T_{p,q,r}$ down to $T_{2,2,2} = D_4$, which is singular, hence $T_{p,q,r}$ is singular. If, say p is even and q, r are odd, we can prune down to $T_{2,3,3} = E_6$ which is nonsingular, so $T_{p,q,r}$ is nonsingular.

Suppose now that n is odd. If $p = 2k + 1,\ q = 2\ell + 1,\ r = 2m + 1$, then $T_{p,q,r}$ is obtained from the singular graph $T_{3,3,3} = \widetilde{E}_6$ by $k - 1 + \ell - 1 + m - 1$ sproutings at kernel vertices, so $T_{2k+1,2\ell+1,2m+1}$ is nonsingular if and only if $k + \ell + m$ is even. If $p = 2k,\ q = 2\ell,\ r = 2m + 1$, then $T_{p,q,r}$ is obtained from the singular graph $T_{2,2,r} = D_{r+2}$ by $k - 1 + \ell - 1$ sproutings at kernel vertices, so $T_{2k,2\ell,2m+1}$ is nonsingular if and only if $k + \ell$ is odd.

5. Coxeter groups — Given a graph X with vertex set S, let $W = W(X)$ be the group with generators $\{\sigma_i :\ i \in S\}$, and relations

$$\begin{aligned} \sigma_i^2 &= 1, \\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j, \quad \text{if } i\text{—}j \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \quad \text{otherwise.} \end{aligned}$$

The group W acts linearly on L by

$$\sigma_i(\lambda) = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i, \tag{5a}$$

preserving the form $\langle\ ,\ \rangle$.

5.1 Lemma. *Let k be any field, let $V_k = k \otimes L$, and let V_k^0 be the radical of the form on V_k induced by $\langle\ ,\ \rangle$. Then V_k^0 coincides with the space of invariants of W in V_k. If X is connected then W acts irreducibly on V_k/V_k^0.*

Proof. The first assertion is clear from (5a). Suppose $x \in V_k - V_k^0$. Then $\sigma_i(x) \neq x$ for some i. Hence $(1 - \sigma_i)(x) = c\alpha_i$, for some $c \in k^\times$. If i—j then $\sigma_i\sigma_j(\alpha_i) = 1\alpha_j$. If X is connected, this shows there are no proper subspaces of V_k/V_k^0. $\square$

In the next two sections $k = \mathbf{F}_2$, and V is the quadratic space over $\mathbf{F}_2$ constructed from a graph X, as in section 1. The induced action of W on V preserves q, and gives a homomorphism

$$\rho : W \longrightarrow O(V).$$

Each $s_i := \rho(\sigma_i)$ is a transvection

$$s_i(x) = x + f(x, e_i)e_i.$$

Visually, if i has an odd number of $\bullet$ neighbors in x, then $s_i(x)$ is obtained from x by changing the state of i. If i has an even number of $\bullet$ neighbors in x, then $s_i(x) = x$.

By 2.1, the homomorphism ρ is surjective if and only if W is transitive on $V(1)$, and $X \neq A_2 + A_2$. If i—j then $s_is_j(e_i) = e_j$, so all e_i in a connected component of X belong to the same W-orbit. Since each $e_i \in V(1)$, the surjectivity of ρ amounts to having $V(1) = We_i$ for some (any) i.

6. Surjectivity for complete graphs — The case $X = K_n$ is particularly simple. Recall from 3.4 that K_n is nonsingular when n is even or $n \equiv 1 \mod 4$, and $u = \sum_{i=1}^n e_i$ in the latter case. If $0 \neq x \in V$, then $[x] \simeq K_r$ for some $r \leq n$, so

$$q(x) \equiv r + \frac{r}{2}(r-1) = \frac{r}{2}(r+1) = \begin{cases} 0 & \text{if } r = 4j,\ 4j-1 \\ 1 & \text{if } r = 4j+1,\ 4j+2. \end{cases}$$

Every $\bullet$ vertex in x has $r-1$ $\bullet$ neighbors and every $\circ$ vertex in x has r $\bullet$ neighbors. If r is even we can only alter the $\bullet$ vertices; in this case if i is a $\bullet$ vertex, we have $[s_i(x)] = K_{r-1}$. Likewise if r is odd, we have $[s_i(x)] = K_{r+1}$. It follows that the orbit decomposition of $V(1)$ under W is

$$V(1) = U_1 \sqcup U_5 \sqcup \cdots \sqcup U_{4\ell+1},$$

where $U_r = \{x \in V(1) :\ [x] \simeq K_r \text{ or } K_{r+1}\}$, and $\ell = \lfloor \frac{n-2}{4} \rfloor$. By 2.1, this proves

6.1 Proposition. *For nonsingular complete graphs K_n, the reduction map $\rho : W \longrightarrow O(V)$ is surjective if and only if $n = 1, 2, 4, 5$.*

7. Surjectivity for trees — In this section X is a tree. We abbreviate $wx := \rho(w)x$. The vector $x = \sum_{i=1}^n e_i$ belongs to $V(1)$. As a first step toward surjectivity, we note that $x \in We_i$ for any i. Indeed, choose a leaf i, so that $s_ix = x - e_i$. The tree X is replaced by the tree X_i, and x is replaced by its analogue $x - e_i$. Repeating, we find $x \in We_i$ for any i.

Now let $x \in V$ be arbitrary and nonzero. Recall that $c(x)$ denotes the number of components of $[x]$. Applying the previous argument to each component of $[x]$ proves the following.

7.1 Lemma. *Let $x \in V$ be nonzero. Then there is $w \in W$ such that the components of $[wx]$ are isolated vertices. Moreover, $c(wx) = c(x)$.*

The key case for proving surjectivity is the tree $T_{p,q,r}$ whose nonsingularity was discussed in 4.9.

7.2 Lemma. *Suppose $X = T_{p,q,r}$ is nonsingular. Then $\rho : W \longrightarrow O(V)$ is surjective.*

Proof. Let $x \in V(1)$. By 7.1, we may assume that $[x]$ consists of $c(x)$ isolated vertices, and $c(x)$ is odd. It suffices to find $w \in W$ so that $c(wx) < c(x)$. For this, it is enough to achieve the following "triad" configuration in the neighborhood of the branch node.

$$\begin{array}{ccccc} \ldots\bullet & — & \circ & — & \bullet\ldots \\ & & | & & \\ & & \bullet & & \\ & & \vdots & & \end{array}$$

For, the branch reflection will then reduce the number of components by two.

Since $s_i s_j(e_i) = e_j$ if i—j, we can move stones along branches as follows

$$\cdots\bullet — \circ — \circ\cdots \quad \mapsto \quad \cdots\circ — \bullet — \circ\cdots .$$

With these moves, we first "pack down" the stones in each branch. That is, if any branch has stones, we move the stone closest to the leaf onto the leaf, the next closest stone to the penultimate spot away from the leaf, and so on, until each branch with stones looks like

$$\bullet — \circ — \bullet — \circ\cdots$$

and no stones on any branch can be moved towards the leaf on that branch. These moves do not change $c(x)$.

If there is a stone on the branch vertex, and we cannot move it onto one of the branches, our configuration is W-invariant, contradicting $x \in V(1)$. Hence we can ensure that the branch vertex has no stone, and this move also does not change $c(x)$.

If all three branches now have stones, we can move those stones closest to the branch vertex and achieve the triad. If only one branch A has stones, then A has at least three stones, and since $X \neq D_n$ (by the nonsingularity assumption), some other branch B has at least two vertices. We move one stone from A onto the leaf b of B. Since b is not adjacent to the branch vertex, we can move another stone from A onto the leaf of the third branch C. This takes us back to the previous case.

The remaining possibility is that some branch A has at least two stones, some other branch B has at least one stone, and the third branch C has no stones. If a stone in B prevents us from moving a stone from A onto C, then the branch neighborhood looks as follows.

$$\begin{array}{ccccc} \bullet & — & \circ & — & \bullet \\ & & | & & \\ & & \circ & & \end{array}$$

Since no stones can be moved towards the leaves of A or B, this vector is again W-invariant. This contradiction completes the proof. $\square$

Now we can prove the more general result (Theorem 1 of the Introduction).

7.3 Theorem. *Suppose X is a nonsingular tree, not of type A_n. Then $\rho : W \longrightarrow O(V)$ is surjective.*

Proof. By 7.2 we may assume X is not of the form $T_{p,q,r}$. We first suppose X has an even number $n = 2m$ of vertices, and argue by induction on m.

By 4.2, X is obtained by sprouting i—j— at some vertex of a nonsingular tree X' with $n-2$ vertices. Let W' and V' be the analogues of W, V for X'. Note X' is not of type A_{n-2}, since X is not of type $T_{p,q,r}$. Hence $\rho' : W' \longrightarrow O(V')$ is surjective, by the induction hypothesis.

Let $x \in V(1)$, so that $c(x)$ is odd; we assume $c(x) \geq 3$. It suffices to find $y \in Wx$ such that $c(y) < c(x)$.

We may assume, by 7.1, that the components of $[x]$ are isolated vertices. If $[x] \subset X'$ we are done by induction. By moving a stone from j to i, if necessary, we may assume that $x = e_i + x'$, where $[x'] \subset X'$, and $x' \neq 0$.

Since $X' \neq A_{n-2}$, there is a vertex k in X' of degree ≥ 3. We choose k as near as possible to j. Hence k is the branch node in a subgraph of type D_4 with neighboring vertices a, b, c, and a, say, is on the path from k to j. Now $c(x')$ is even, so $x' \in V'(0)$. Also $e_b + e_c \in V'(0)$, because X' is a tree. By surjectivity for W', there is $w' \in W'$ such that $w'(x') = e_b + e_c$, so $w'(x) = e_i + e_b + e_c$. Hence we can achieve the triad by moving the stone on i along the path toward k. This completes the proof in the even case.

Now suppose X is an odd nonsingular tree, with kernel vector $u = \sum \det(X_i)e_i$. By 4.5 , there is a leaf i in X such that X_i is nonsingular. Note that X_i is an even tree, not of type A_{n-1}.

Let $x \in V(1)$, and assume the components of $[x]$ are single vertices. If $x_i = 0$ then $[x] \subset X_i$ and we are reduced to the even case. Hence we may assume $x_i = 1$ and $x_j = 0$, where j is the neighbor of i. Note that $q(x+e_i) = 0$. Since $X_i \neq A_{n-1}$, there are at least two leaves a, b in X_i, other than j, and $q(e_a + e_b) = 0$. From the surjectivity in the even case, there is $w \in W(X_i)$ such that $w(x + e_i) = e_a + e_b$, so $w(x) = e_i + e_a + e_b$. Now j cannot be adjacent in X to any leaf but i, since X is nonsingular. Hence j is not adjacent to a or b. It follows that $s_i s_j w(x) = e_j + e_a + e_b$ and we are again reduced to the even case. This completes the proof in the odd case. □

8. Involutions at level one — We turn now to the higher level congruence subgroups W_k of W. These groups are torsion-free for $k \geq 2$. Here, we analyze the torsion in W_1.

For any lattice L, and subgroup $\Gamma \subset GL(L)$, define

$$\Gamma_k := \ker[GL(L) \longrightarrow GL(L/2^k L)], \qquad k \geq 1.$$

The quotient Γ_k/Γ_{k+1} has a Lie algebra structure over $\mathbf{F}_2$, induced by the commutator, and the map

$$\partial_k : \Gamma_k/\Gamma_{k+1} \longrightarrow \mathrm{End}(L/2L), \qquad \partial_k(\gamma) = 2^{-k}(\gamma - I),$$

is a Lie algebra isomorphism. If $\gamma \in \Gamma_k$ then $\gamma^2 \in \Gamma_{k+1}$ and

$$\partial_{k+1}(\gamma^2) = \begin{cases} \partial_k(\gamma) + \partial_k(\gamma)^2 & \text{if} \quad k = 1 \\ \partial_k(\gamma) & \text{if} \quad k > 1. \end{cases} \tag{8a}$$

8.1 Lemma. *Γ_2 is torsion-free, and the torsion elements of Γ_1 are involutions.*

Proof. If $\gamma \in \Gamma_1$ has order ab, with $a = 2^c$ and b odd, then $\partial_k(\gamma^a)$ has odd order in the abelian group $\mathrm{End}(L/2L)$, for all $k \geq 1$. Hence $\gamma^a \in \Gamma_k$ for all k, so $\gamma^a = I$ and $b = 1$. If $\gamma \in \Gamma_k$ with $k \geq 2$, then $\partial_k(\gamma) = \partial_{k+c}(\gamma^a) = 0$, so $\gamma \in \Gamma_{k+1}$, again forcing $\gamma = I$. If $\gamma \in \Gamma_1$, then γ^2 is a torsion element in Γ_2, hence $\gamma^2 = I$. $\square$

An involution $\gamma \in \Gamma_1$ is of the form $\gamma = I + 2D$, where $D = \partial_1(\gamma)$ and $D^2 = -D$. It gives a splitting

$$L = DL \oplus (I + D)L,$$

and DL, $(I + D)L$ are the $-1, +1$ eigenspaces of γ, respectively.

Thus, for $\Gamma = GL(L)$, taking the $-1, +1$ eigenspaces gives a bijection between involutions in Γ_1 and ordered pairs (L', L'') of sublattices of L such that $L = L' \oplus L''$.

If $\Gamma = O(L)$ is the orthogonal group of a symmetric form $\langle\ ,\ \rangle : L \times L \longrightarrow \mathbf{Z}$, the involutions in Γ_1 correspond to orthogonal pairs (L', L'').

Suppose $\Gamma = W$ is the Coxeter group obtained from a connected graph X, and L is the associated root lattice, with symmetric form $\langle\ ,\ \rangle$ as in section 1. For $J \subset S$, let $W(J)$ be the subgroup of W generated by $\{\sigma_j :\ j \in J\}$, and let L_J be the $\mathbf{Z}$-span of $\{\alpha_j :\ j \in J\}$. By a theorem of Tits (c.f. [K, Prop. 3.12]), every finite subgroup of W can be conjugated into a finite subgroup $W(J)$, for some $J \subset S$. It is easy to see that

$$W_k \cap W(J) = \prod_i W_k \cap W(J_i),$$

where $[J_1], \ldots, [J_c]$ are the connected components of $[J]$.

To classify the level one involutions $w \in W_1$, we may therefore assume $w \in W_1 \cap W(J)$, where $[J]$ is connected with $W(J)$ finite, and moreover that w cannot be conjugated into $W(I)$, for $I \subsetneq J$. Then w must be the unique element $w_J \in W(J)$ acting by -1 on L_J. In particular, $[J]$ must have one of types A_1, D_{2m}, E_7, E_8. We will see that the first two cases cannot occur in nonsingular trees.

If $i \notin J$, we have

$$w_J \alpha_i = \alpha_i + \sum_{j \in J} n_{ij} \alpha_j,$$

with all $n_{ij} \geq 0$. It follows that L_J is the whole -1 eigenspace of w_J in L. Hence we have an orthogonal splitting

$$L = L_J \oplus L_J^{\perp},$$

and $L_J^{\perp} = \{\lambda \in L :\ w_J \lambda = \lambda\}$. If $J \subset K \subset S$, then replacing W by $W(K)$ shows that L_J must be an orthogonal summand of L_K. If X has more than one vertex, then J cannot have type A_1, since A_1 is not an orthogonal summand of A_2.

For any $J \subset S$ such that $\langle\ ,\ \rangle$ is nonsingular on $\mathbf{Q} \otimes L_J$, there exists, for each $k \in S \setminus J$, a vector $\omega_k \in \mathbf{Q} \otimes L_J$ defined by

$$\langle \omega_k, \alpha_j \rangle = -\langle \alpha_k, \alpha_j \rangle, \quad \text{for all} \quad j \in J.$$

For example, if k is not adjacent to any vertex in J then $\omega_k = 0$.

8.2 Lemma. *Let $J \subset S$ be such that $\langle\ ,\ \rangle$ is nonsingular on $\mathbf{Q} \otimes L_J$. Then L_J is an orthogonal summand of L if and only if $\omega_k \in L_J$ for all $k \in S \setminus J$.*

Proof. If $L = L_J \perp U$, then for each $k \in S \setminus J$ write α_k as $\alpha_k = \lambda_k + u_k$, with $\lambda_k \in L_J$ and $u_k \in U$. Then $-\lambda_k$ satisfies the equations defining ω_k, so $\omega_k = -\lambda_k \in L_J$. Conversely, if $\omega_k \in L_J$ for all $k \in S \setminus J$, then set $u_k = \alpha_k + \omega_k$, and let $U = \sum_{k \in S \setminus J} \mathbf{Z} u_k$. Clearly $U \subseteq L_J^{\perp}$, and it suffices to show equality. Since $\alpha_k = u_k - \omega_k$, we have $L = L_J + U$. Write $\lambda \in L_J^{\perp}$ as $\lambda = \omega + u$, with $\omega \in L_J$ and $u \in U$. Then $0 = \langle \alpha_j, \lambda \rangle = \langle \alpha_j, \omega \rangle$ for all $j \in J$, so $\omega \in L_J \cap L_J^{\perp} = 0$, since L_J is nonsingular over $\mathbf{Q}$. $\square$

8.3 Lemma. *If $J \simeq E_8$, then L_J is an orthogonal summand of L, and $w_J \in W_1$.*

Proof. This is immediate from 8.2, since E_8 is a unimodular lattice. $\square$

8.4 Lemma. *Suppose X is a nonsingular tree. If $J \simeq D_{2m}$, $m \geq 2$, then L_J is not an orthogonal summand of L.*

Proof. Number the vertices of J as

$$\begin{array}{c} 1—2—3\ \cdots\ (2m-2)—(2m-1) \\ \qquad\qquad\qquad | \\ \qquad\qquad\qquad 2m\ . \end{array}$$

For $j \in J$, define $\lambda_j \in \mathbf{Q} \otimes L_J$ by

$$\langle \lambda_j, \alpha_i \rangle = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and let L_J^+ be the $\mathbf{Z}$-lattice spanned by $\{\lambda_j :\ j \in J\}$. Then L_J^+ contains L_J and $L_J^+/L_J \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. In the latter quotient, we have the relations

$$\lambda_1 = \lambda_3 = \cdots = \lambda_{2m-3} = \lambda_{2m-1} + \lambda_{2m}, \qquad \lambda_2 = \lambda_4 = \cdots = \lambda_{2m-2} = 0.$$

For any $k \in S \setminus J$, we have

$$\omega_k = \sum_{j \in J_k} \lambda_j,$$

where $J_k = \{j \in J :\ k—j\}$.

Since X is nonsingular, the vertices $2m-1, 2m$ cannot both be leaves in X, by the remark prior to 4.2. Hence there exists $k \in S \setminus J$ such that at least one of $\{2m-1, 2m\}$ belongs to J_k. But if $\omega_k \in L_J$, the above relations then force both $2m-1$ and $2m$ to be in J_k. Hence there is a 4-cycle in X, with vertices $\{2m-2, 2m-1, k, 2m\}$. This contradicts our assumption that X is a tree. $\square$

I do not know if it is necessary to assume that X is a tree in Lemma 8.4.

Now take $J \simeq E_7$, and let $u_J \in V$ be the kernel vector of $[J]$. Visually,

$$\begin{array}{l} u_J = \circ — \circ — \circ — \bullet — \circ — \bullet \\ \qquad\qquad\quad\ | \\ \qquad\qquad\quad\ \bullet \end{array} .$$

Let J_0 be the set of $\bullet$ vertices in the above subgraph. These are the kernel vertices in J. If n is odd we can compare u_J with the kernel vector $u = \sum \det(X_i) e_i$ of X.

8.5 Lemma. *If $J \simeq E_7$ then L_J is an orthogonal summand of L if and only if $u_J \in \ker_2 X$. If this holds and X is nonsingular then n is odd and $u_J = u$.*

Proof. Let $\lambda_j \in \mathbf{Q} \otimes L_J$, $j \in J$, and J_k be as in the proof of 8.4. Viewed as an involution, u_J generates the center of the simply connected Lie group E_7, so we have $\sum c_j \lambda_j \in L_J$ if and only if $\sum_{j \in J_0} c_j$ is even. Hence $\omega_k \in L_J$ exactly when $|J_k \cap J_0|$ is even. This holds for all $k \in S \setminus J$ iff $u_J \in \ker_2 X$. The last assertion is an immediate consequence. $\square$

Now 8.1-5 yield the following.

8.6 Proposition. *Suppose X is a nonsingular tree. Then every conjugacy class of involutions in W_1 contains a commuting product $w = \prod_J w_J$, where J runs over full subgraphs of X of type E_7 or E_8. If some factor of type E_7 occurs then n is odd and there are three vertices i, j, k in X such $u = e_i + e_j + e_k$ and $J_0 = \{i, j, k\}$ for every factor w_J of type E_7 occuring in an involution $w \in W_1$.*

We illustrate 8.6 with $X = T_{2,4,5}$, labelled as shown.

```
1—2—3—4—5—6—7—8
        |
        9
```

This graph is nonsingular, with $u = e_6 + e_8 + e_9$. One involution in W_1 comes from the unique E_8 subdiagram. There are two E_7 subdiagrams, but only $J = S - \{1, 2\}$ gives an involution. Explicitly, the vectors

$$\theta_7 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 + 2\alpha_9,$$
$$\theta_8 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 10\alpha_5 + 7\alpha_6 + 4\alpha_7 + \alpha_8 + 5\alpha_9$$

belong to $L_J^\perp$ and $L_{E_8}^\perp$, respectively. Since α_2 has coefficient=1 in θ_7 and α_8 has coefficient=1 in θ_8, we have orthogonal decompositions

$$L = L_J \oplus \mathbf{Z}\{\alpha_1, \theta_7\} = L_{E_8} \oplus \mathbf{Z}\{\theta_8\}$$

and w_J, w_{E_8} represent the two conjugacy classes of involutions in W_1.

9. Density in orthogonal groups — This is the only section where it is not essential to have $p = 2$. We aim to prove a version of p-adic density for certain hyperbolic Coxeter groups. This is a combination of known results, and will be applied in the 2-adic case to more general Coxeter groups. This section takes place in characteristic zero, and we will recycle some of our earlier notation.

We begin a free $\mathbf{Z}$-module L of rank n, and a symmetric bilinear map $f : L \times L \longrightarrow \mathbf{Z}$ such that $f(x, x) \in 2\mathbf{Z}$ for all $x \in L$. Let $D = \det f$, and set $q(x) = \frac{1}{2} f(x, x)$. We assume $n \geq 5$ and that f is nondegenerate over $\mathbf{Q}$ and indefinite over $\mathbf{R}$.

For any integral domain R, let $GL(R)$ denote the group of R-linear automorphisms of $R \otimes L$, $O(R)$ the subgroup of $GL(R)$ preserving the extension of q to $R \otimes L$, and let $SO(R) = \{g \in O(R) : \det g = 1\}$. Any ring homomorphism $R \longrightarrow R'$ induces natural group homomorphisms

$$GL(R) \longrightarrow GL(R'), \quad O(R) \longrightarrow O(R'), \quad SO(R) \longrightarrow SO(R').$$

If $x \in R \otimes L$ and $q(x)$ is a unit in R, we have the "reflection" $r_x \in O(R)$, defined by

$$r_x(y) = y - q(x)^{-1} f(x, y) x.$$

Let F be the quotient field of R, and assume F has characteristic zero. Then $O(F)$ is generated by reflections [D, Prop. 8], and the spinor norm

$$\delta_F : O(F) \longrightarrow F^\times / F^{\times 2}$$

is a group homomorphism determined by the rule $\delta_F(r_x) = q(x)$. More generally, if $g \in O(F)$ is an involution and E_g is the -1-eigenspace of g in $F \otimes L$, then $\delta_F(g)$ is the discriminant of the restriction of f to E_g.

We set

$$O'(R) = O(R) \cap \ker \delta_F, \qquad SO'(R) = SO(R) \cap \ker \delta_F.$$

The group $SO'(F)$ is the image of $\mathrm{Spin}(F)$ under the two-fold cover $\mathrm{Spin}(F) \longrightarrow SO(F)$. If $F = \mathbf{Q}_p$, the spinor norm is surjective, and we have an exact sequence [S,III.3.2]

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(F) \longrightarrow SO(F) \xrightarrow{\delta_F} F^\times / F^{\times 2} \longrightarrow 1.$$

From the commutative diagram

$$\begin{array}{ccc} O(\mathbf{Q}) & \longrightarrow & O(\mathbf{Q}_p) \\ {\scriptstyle \delta_{\mathbf{Q}}} \downarrow & & \downarrow {\scriptstyle \delta_{\mathbf{Q}_p}} \\ \mathbf{Q}^\times / \mathbf{Q}^{\times 2} & \longrightarrow & \mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2} \end{array} \tag{9a}$$

it follows that

$$O'(\mathbf{Q}) \subset O'(\mathbf{Q}_p), \qquad O'(\mathbf{Z}) \subset O'(\mathbf{Z}_p).$$

Fix a prime p such that q is nonsingular on $\mathbf{F}_p \otimes L$.

9.1 Lemma. *The image of $SO(\mathbf{Z}_p)$ under $\delta_{\mathbf{Q}_p}$ is $\mathbf{Z}_p^\times / \mathbf{Z}_p^{\times 2}$.*

Proof. This follows from the results in [Ki, 5.5]. □

Let $\hat{\mathbf{Q}}$ denote the finite adeles of $\mathbf{Q}$, and let

$$SO'(\hat{\mathbf{Q}}) = \{ g = (g_p) \in SO(\hat{\mathbf{Q}}) : \ g_p \in SO'(\mathbf{Q}_p) \quad \text{for all } p \}.$$

From strong approximation for $\mathrm{Spin}/\mathbf{Q}$, it follows that the diagonal embedding of $SO'(\mathbf{Q}) \hookrightarrow SO'(\hat{\mathbf{Q}})$ has dense image [Kn]. Hence, for any prime p, and integer $k \geq 1$, we have

$$SO'(\hat{\mathbf{Q}}) = SO'(\mathbf{Q}) U'_k K^p,$$

where $U'_k = \ker[SO'(\mathbf{Z}_p) \longrightarrow SO(\mathbf{Z}/p^k\mathbf{Z})]$, and

$$K^p = \prod_{\ell \neq p} SO'(\mathbf{Z}_\ell).$$

It follows easily that

$$SO'(\mathbf{Z}_p) = SO'(\mathbf{Z})U'_k. \tag{9b}$$

(Write $g \in SO'(\mathbf{Z}_p)$ in the form $g = \gamma u \kappa$, with $\gamma \in SO'(\mathbf{Q}),\ u \in U'_k,\ \kappa \in K^p$. Then γ is integral at all primes, hence belongs to $SO'(\mathbf{Z})$, and $\gamma\kappa_\ell = 1$ for all $\ell \neq p$, so $g = \gamma u$.)

Since q is not identically zero on $\mathbf{F}_p \otimes L$, there is $\lambda \in \mathbf{Z}_p \otimes L$ such that $q(\lambda) \in \mathbf{Z}_p^\times$. From this and (9b), it follows that

$$O'(\mathbf{Z}_p) = O'(\mathbf{Z})U'_k \quad \Leftrightarrow \quad O'(\mathbf{Z}) \xrightarrow{\det} \{\pm 1\} \quad \text{is surjective.} \tag{9c}$$

This holds if there is $\lambda \in L$ with $q(\lambda) = 1$.

Suppose now that (L, q) arises from a graph X, as in section 1. Let Ω be the subgroup of $O(\mathbf{Z})$ generated by $\mathrm{Aut}(X)$ and $-I$. Then Ω normalizes W, and we have the subgroup

$$\Omega W \subseteq O(\mathbf{Z}).$$

In fact, $W \subset O'(\mathbf{Z})$ since $\delta_{\mathbf{Q}}(\sigma_i) = 1$, so

$$\Omega' W \subseteq O'(\mathbf{Z}),$$

where $\Omega' = \ker \delta_{\mathbf{Q}} \cap \Omega$.

The graph X is called *hyperbolic* if $W(J)$ is finite or affine for every proper subset $J \subset S$, but W is not itself finite or affine. This implies that the quadratic form q on $\mathbf{R} \otimes L$ has signature $(n-1, 1)$ [B,p.141].

9.2 Lemma. *If X is hyperbolic, then $O'(\mathbf{Z}) = \Omega' W$.*

Proof. This follows from [K, Cor. 5.10b] and the fact that $W \subset O'(\mathbf{Z})$. □

Since $\det(W) = \{\pm 1\}$, 9.2 and (9c) yield the following.

9.3 Corollary. *Suppose $n \geq 5$, and X is hyperbolic. Then $O'(\mathbf{Z}_p) = \Omega' W U'_k$ for every $k \geq 1$.*

Actually, we will only need two examples of 9.3.

9.4 Example. Let $X = E_{10} (= T_{2,3,7})$. Then L is the unique even unimodular hyperbolic lattice in dimension 10. There are no diagram symmetries, so $\Omega = \{\pm I\}$. The discriminant is -1, so $\delta_{\mathbf{Q}}(-I) = -1$, so $\Omega' = 1$, and $O'(\mathbf{Z}) = W$. Hence for all $k \geq 1$, we have $O'(\mathbf{Z}_p) = W U'_k$.

9.5 Example. Let $X = T_{3,3,4}$. Here $\Omega = \{\pm I, \pm \omega\}$, where ω is the nontrivial diagram symmetry. The -1 eigenspace of ω in $\mathbf{Q} \otimes L$ is a hyperbolic plane, so

$$\delta_{\mathbf{Q}}(-I) = -3, \quad \delta_{\mathbf{Q}}(\omega) = -1.$$

Again $\Omega' = 1$, so $O'(\mathbf{Z}_p) = W U'_k$ for all $k \geq 1$.

10. Adjoint representation over $\mathbf{F}_2$ — This section has nothing to do with graphs, and its assertions are surely known. For lack of an adequate reference, we recall here the structure of the adjoint representation of orthogonal groups in characteristic 2. We assume from now on that n is even.

Let F be a field of characteristic 2, with dual numbers $F[\epsilon] = F[x]/(x^2)$. Let V be a vector space over F of dimension $n = 2m$, and let q be a nonsingular quadratic form on V, with associated bilinear form $f(x, y) = q(x + y) + q(x) + q(y)$.

For $A \in \mathrm{End}(V)$, we have $I + \epsilon A \in O(F[\epsilon])$ iff

$$q(v) = q(v + \epsilon Av) = q(v) + \epsilon^2 q(Av) + \epsilon f(v, Av) = q(v) + \epsilon f(v, Av),$$

so the scheme-theoretic Lie algebra of $O(F)$ is

$$\mathfrak{so}(F) := \{A \in \mathrm{End}(V) : \ f(v, Av) = 0 \text{ for all } v \in V\}. \tag{10a}$$

In particular, the Lie algebra depends only on f, not on q. Choose a basis for V such that the matrix of f has the form $\begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}$ for some symmetric matrix $Q \in GL_m(F)$. (It will be convenient to avoid a specific choice for Q.) For $M \in \mathfrak{gl}_m(F)$, let $M^* = Q^{-1}(M^t)Q$, and let

$$\mathfrak{m}(F) = \{M \in \mathfrak{gl}_m(F) : \ M^* = M, \ (QM)_{ii} = 0 \ \text{ for } i = 1, \dots, m\}.$$

In other words, $\mathfrak{m}(F)$ is the set of $m \times m$ matrices M such that QM is symmetric with zero diagonal. Then a matrix calculation with (10a) shows that

$$\mathfrak{so}(F) = \{A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_1^* \end{bmatrix} : \ A_1 \in \mathfrak{gl}_m(F), \ A_2, A_3 \in \mathfrak{m}(F)\}.$$

Define the *half-trace* $\tau : \mathfrak{so}(F) \longrightarrow F$ by $\tau(A) = \mathrm{tr}(A_1)$, and set

$$\mathfrak{so}'(F) := \ker \tau = \{A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_1^* \end{bmatrix} : \ A_1 \in \mathfrak{sl}_m(F), \ A_2, A_3 \in \mathfrak{m}(F)\}.$$

10.1 Lemma. *$\mathfrak{so}'(F)$ is the commutator subalgebra of $\mathfrak{so}(F)$.*

Proof. If S, S' are symmetric matrices over F, and all diagonal entries of S are zero, then $\mathrm{tr}(SS') = 0$. If $M, N \in \mathfrak{m}(F)$, we can take $S = QM, S' = NQ^{-1}$, so that $\mathrm{tr}(MN) = \mathrm{tr}(QMNQ^{-1}) = 0$. For $A, B \in \mathfrak{so}(F)$, we find

$$\tau([A, B]) = \mathrm{tr}([A_1, B_1] + A_2B_3 + A_3B_2) = \mathrm{tr}(A_2B_3) + \mathrm{tr}(A_3B_2) = 0.$$

Conversely, since $\mathfrak{sl}_m(F)$ is the commutator subalgebra of $\mathfrak{gl}_m(F)$, it suffices to prove that $\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \in [\mathfrak{so}(F), \mathfrak{so}(F)]$ for all $M \in \mathfrak{m}(F)$. Since

$$[\begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix}, \begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}] = \begin{bmatrix} 0 & 0 \\ YX + X^*Y & 0 \end{bmatrix},$$

it suffices to find $Y \in \mathfrak{m}(F)$ making the map

$$\phi_Y : \mathfrak{gl}_m(F) \longrightarrow \mathfrak{m}(F), \quad \phi_Y(X) = YX + X^*Y$$

surjective. Now $X \in \ker \phi_Y$ iff $(QY)X + (X^t)(QY) = 0$.

If $m = 2\ell$ is even, we can choose $Y \in \mathfrak{m}(F)$ of rank m, so that QY is the matrix of a nondegenerate symplectic form on $F^{2\ell}$. Then $\ker \phi_Y = \mathfrak{sp}_{2\ell}(F)$ has dimension $2\ell^2 + \ell$. Since $\dim \mathfrak{m}(F) = (m/2)(m-1)$, this shows that ϕ_Y surjective.

If $m = 2\ell + 1$, we can take Y of rank $m - 1$. Then $\ker \phi_Y \simeq \mathfrak{sp}_{2\ell}(F) \oplus F^m$, again implying that ϕ_Y is surjective. $\square$

10.2 Lemma. *If F is a subfield of an algebraic closure $\bar{\mathbf{F}}_2$, then $F \cdot I$ and $\mathfrak{so}'(F)$ are the only proper $O(F)$-invariant subspaces of $\mathfrak{so}(F)$. Hence, if m is odd, we have the irreducible $O(F)$-decomposition $\mathfrak{so}(F) = F \cdot I \oplus \mathfrak{so}'(F)$, and if m is even, we have $F \cdot I \subset \mathfrak{so}'(F) \subset \mathfrak{so}(F)$ indecomposable.*

Proof. It is clear from 10.1 that $\mathfrak{so}'(F)$ is indeed an $O(F)$ invariant subspace of $\mathfrak{so}(F)$. For the moment, let $F = \bar{\mathbf{F}}_2$. Then we can choose a basis of V so that $q(x) = \sum x_i x_{m+i}$. The diagonal matrices in $O(F)$ have the form $t = \mathrm{diag}(t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1})$ and comprise a maximal torus T, which acts diagonalizably on any $O(F)$-invariant subspace $U \subseteq \mathfrak{so}(F)$. Since n is even and ≥ 6, the roots of T in $\mathfrak{so}(F)$ form a single orbit under the Weyl group of T. Hence if U does not consist of diagonal matrices, then all roots must appear in U. The calculation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

shows that U contains all matrices of the form $\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$, where $D \in \mathfrak{gl}_m(F)$ is a diagonal matrix of trace zero. Hence $\mathfrak{so}'(F) \subset U$.

If U consists of diagonal matrices, then the calculation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & s+t \\ 0 & t \end{bmatrix}$$

implies that $U = F \cdot I$. This proves the lemma for $F = \bar{\mathbf{F}}_2$, and shows that the highest weights of the composition factors of $\mathfrak{so}(\bar{\mathbf{F}}_2)$ take values $0, 1$ on the simple co-roots of $O(\bar{\mathbf{F}}_2)$. By Steinberg's theorem [St, 1.3] each composition factor of $\mathfrak{so}(\bar{\mathbf{F}}_2)$ remains irreducible under $O(F)$ for any subfield $F \subset \bar{\mathbf{F}}_2$. Since any $O(F)$-invariant subspace of $\mathfrak{so}(F)$ remains invariant after extending scalars, the lemma is proved. □

11. Higher Levels — In this section we assume L has even rank $n = 2m$ and q is nonsingular on $\mathbf{F}_2 \otimes L$. As a quadratic space over $\mathbf{Z}_2$, we have (see section 2)

$$\mathbf{Z}_2 \otimes L \simeq (m-d) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \perp d \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{11a}$$

where $d \in \{0, 1\}$ is the defect of q.

Let $U_k = \ker[SO(\mathbf{Z}_2) \longrightarrow SO(\mathbf{Z}/2^k\mathbf{Z})]$. The map $\partial_k(u) = 2^{-k}(u - I)$ (see section 8) is an injection

$$\partial_k : U_k/U_{k+1} \hookrightarrow \mathfrak{so}(\mathbf{F}_2) \tag{11b}$$

whose image is an $O(\mathbf{F}_2)$-invariant subspace of $\mathfrak{so}(\mathbf{F}_2)$.

11.1 Lemma. *The map (11b) is surjective for all $k \geq 1$.*

Proof. First suppose $m = 1$, so that $\mathfrak{so}(\mathbf{F}_2) = \{0, I\}$. If $d = 0$, let

$$u = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}, \quad s = 1 + 2^k z, \quad z \in \mathbf{Z}_2^\times.$$

Then $u \in U_k$ and $\partial_k(u) = I$. If $d = 1$, let

$$u = \frac{1}{s^2 - s + 1}\begin{bmatrix} 1 - s^2 & 2s - s^2 \\ s^2 - 2s & 1 - 2s \end{bmatrix}, \quad s = 2^k z,\ z \in \mathbf{Z}_2^\times.$$

Then $u \in U_k$ and $\partial_k(u) = I$.

For $m > 1$, the splitting (11a) shows that the image of ∂_k contains non-scalar diagonal matrices in $\mathfrak{so}(\mathbf{F}_2) \setminus \mathfrak{so}'(\mathbf{F}_2)$. The lemma now follows from 10.2. □

11.2 Lemma. *The spinor norm $\delta_{\mathbf{Q}_2}$ is trivial on U_3.*

Proof. If $k \geq 2$ and $u \in U_k$, then $u^2 \in U_{k+1}$, and from (8a) we have

$$\partial_{k+1}(u^2) = \partial_k(u). \tag{11c}$$

From 11.1 it follows that the squaring map

$$U_k/U_{k+1} \longrightarrow U_{k+1}/U_{k+2}$$

is surjective for $k \geq 2$. Hence, given $u \in U_3$, there are elements $u_k \in U_k$, $k \geq 2$, such that

$$u = u_2^2 u_4 = u_2^2 u_3^2 u_5 = \cdots .$$

But $\delta_{\mathbf{Q}_2}$ takes values in the finite group $\mathbf{Q}_2^\times/\mathbf{Q}_2^{\times 2}$, and is 2-adically continuous [Ki,1.6.5], so $\delta_{\mathbf{Q}_2}$ is trivial on squares and on U_k for some k. Hence $\delta_{\mathbf{Q}_2}(u) = 1$. □

The values of $\delta_{\mathbf{Q}_2}$ on $U_{1,2}$ can be expressed in terms of the half-trace $\tau : \mathfrak{so}(\mathbf{F}_2) \longrightarrow \mathbf{F}_2$ (see section 10) and the isomorphism

$$\epsilon_1 \times \epsilon_2 : \mathbf{Z}_2^\times/\mathbf{Z}_2^{\times 2} \longrightarrow \mathbf{F}_2 \times \mathbf{F}_2, \quad \epsilon_1(x) \equiv \frac{x-1}{2} \mod 2, \quad \epsilon_2(x) \equiv \frac{x^2-1}{8} \mod 2.$$

11.3 Lemma. *For $k = 1, 2$ we have $\epsilon_k \circ \delta_{\mathbf{Q}_2} = \tau \circ \partial_k$ on U_k.*

Proof. Again start with $m = 1$. For $d = 0$, we have

$$SO(\mathbf{Z}_2) = \{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbf{Z}_2^\times\}, \quad \delta_{\mathbf{Q}_2}(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}) = a,$$

and the claim is immediate. For $d = 1$, one checks that

$$SO(\mathbf{Z}_2) = \{\begin{bmatrix} a & -b \\ b & a+b \end{bmatrix} : a, b \in \mathbf{Z}_2,\ a^2 + ab + b^2 = 1\}, \quad \delta_{\mathbf{Q}_2}(\begin{bmatrix} a & -b \\ b & a+b \end{bmatrix}) = 2 - a - 2b.$$

Note that if $\begin{bmatrix} a & -b \\ b & a+b \end{bmatrix} \in U_k$, then $b \in 2^{k+1}\mathbf{Z}_2$. The claim now follows from straightforward calculations.

Now let $m > 1$. The group U_k has the triangular decomposition

$$U_k = U_k^- T_k U_k^+,$$

where the subgroups $U_k^\pm$ are generated by subgroups of root groups, or products thereof. These root groups lift to $\mathrm{Spin}(\mathbf{Q}_2)$, hence $U_k^\pm \subset SO'(\mathbf{Z}_2)$. It remains to verify 11.3 on T_k. But T_k is a product of orthogonal groups on two-dimensional spaces, so the result follows from the case $m = 1$. □

Recall that $U_k' = U_k \cap \ker \delta_{\mathbf{Q}_2}$.

11.4 Lemma. *We have*

$$\partial_k(U'_k/U'_{k+1}) = \begin{cases} \mathfrak{so}'(\mathbf{F}_2) & \text{if} \quad 1 \le k \le 2 \\ \mathfrak{so}(\mathbf{F}_2) & \text{if} \quad k \ge 3. \end{cases}$$

Proof. If $k \ge 3$ this is immediate from 11.1,2. Assume $k = 1, 2$. Then

$$\partial_k(U'_k/U'_{k+1}) \subseteq \mathfrak{so}'(\mathbf{F}_2)$$

by 11.3. If $m = 1$ the latter space is zero, so assume $m > 1$.

From 11.1, we have

$$[\mathfrak{so}'(\mathbf{F}_2) : \partial_k(U'_k)] = \frac{1}{2}[\partial_k(U_k) : \partial_k(U'_k)].$$

The map ∂_k induces an exact sequence

$$1 \longrightarrow U_{k+1}/U'_{k+1} \longrightarrow U_k/U'_k \longrightarrow \partial_k(U_k)/\partial_k(U'_k) \longrightarrow 1,$$

so

$$[\mathfrak{so}'(\mathbf{F}_2) : \partial_k(U'_k)] = \frac{[U_k : U'_k]}{2[U_{k+1} : U'_{k+1}]}.$$

From 11.1,3, we find that

$$[U_k : U'_k] = \begin{cases} 4 & \text{if} \ \ k = 1 \\ 2 & \text{if} \ \ k = 2 \ . \\ 1 & \text{if} \ \ k \ge 3 \end{cases}$$

The result follows. □

Recall that $O'(\mathbf{Z}) = O(\mathbf{Z}) \cap \ker \delta_{\mathbf{Q}}$. From (9c), we have

$$O'(\mathbf{Z}_2) = O'(\mathbf{Z})U'_k, \qquad \text{for all} \quad k \ge 1,$$

as long as $1 \in q(L)$. We can now prove the following density criterion for subgroups of $O'(\mathbf{Z})$.

11.5 Proposition. *Assume $1 \in q(L)$, and $n \ge 6$. Let H be a subgroup of $O'(\mathbf{Z})$, and set $H_k = H \cap U'_k$. Then H is dense in $O'(\mathbf{Z}_2)$ (in the 2-adic topology) if and only if the following three conditions hold.*

1. *The composition $H \hookrightarrow O(\mathbf{Z}) \longrightarrow O(\mathbf{F}_2)$ is surjective.*
2. *The image $\partial_k(H_k) \subseteq \mathfrak{so}(\mathbf{F}_2)$ contains a nonscalar matrix for $1 \le k \le 3$.*
3. *The homomorphism $\tau \circ \partial_3 : U_3 \longrightarrow \mathbf{F}_2$ is nontrivial on H_3.*

Proof. The necessity of (1-3) is clear. We assume that (1-3) hold, and must show that $O'(\mathbf{Z}_2) = HU'_k$ for all $k \ge 1$. The latter is true for $k = 1$, by (1), so assume $k \ge 2$.

In view of (1), we have a containment of $O(\mathbf{F}_2)$-invariant subspaces

$$\partial_k(H_k) \subseteq \partial_k(U'_k). \tag{11d}$$

It suffices to show equality in (11d) for all $k \geq 2$.

For $2 \leq k \leq 3$, condition (2) and 10.2 imply that $\mathfrak{so}'(\mathbf{F}_2) \subset \partial_k(H_k)$. Now 11.4 implies equality in (11d), where for $k = 3$ we also invoke condition (3).

From (11c) we have $\partial_k(H_k) \subseteq \partial_{k+1}(H_{k+1})$ for $k \geq 2$. We have already proved that $\partial_3(H_3) = \mathfrak{so}(\mathbf{F}_2)$, so we have equality in (11d) for all $k \geq 4$. □

12. Density for W — Let X be a nonsingular even tree with $n = 2m$ vertices, and let L and (W, S) the associated quadratic lattice and Coxeter group, as in section 1. Then $1 \in q(L)$, since, for example, $q(\alpha_1) = 1$. Let $J \subset S$ be an even subset, with corresponding sublattice L_J, such that $[J]$ is connected and nonsingular. We can label $S = \{1, \ldots, n\}$ so that no edge is contained in $\{1, \ldots, m\}$ or $\{m+1, \ldots, n\}$, and so that $J = \{j, \ldots, m, m+1, \ldots, 2m-j+1\}$, for some $1 \leq j \leq m$. With respect to the basis $\{\alpha_1, \ldots, \alpha_n\}$, the matrix of $\langle\ ,\ \rangle$ on L has the form

$$2I_n - \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & Q_J & * \\ * & Q_J & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} 0 & Q_J \\ Q_J & 0 \end{bmatrix}$ is the adjacency matrix of $[J]$. Any $w \in W(J)$ has matrix on L of the form

$$w = \begin{bmatrix} I & 0 & 0 \\ * & w_1 & * \\ 0 & 0 & I \end{bmatrix},$$

where w_1 is the matrix of w on L_J. If $w \in W_k(J)$ for $k \geq 1$, then $\partial_k(w)$ has the form

$$\partial_k(w) = \begin{bmatrix} 0 & 0 & 0 \\ * & \partial_k(w_1) & * \\ 0 & 0 & 0 \end{bmatrix},$$

and $\partial_k(w_1)$ belongs to the Lie algebra $\mathfrak{so}_J(\mathbf{F}_2)$ with respect to $\begin{bmatrix} 0 & Q_J \\ Q_J & 0 \end{bmatrix}$. Therefore

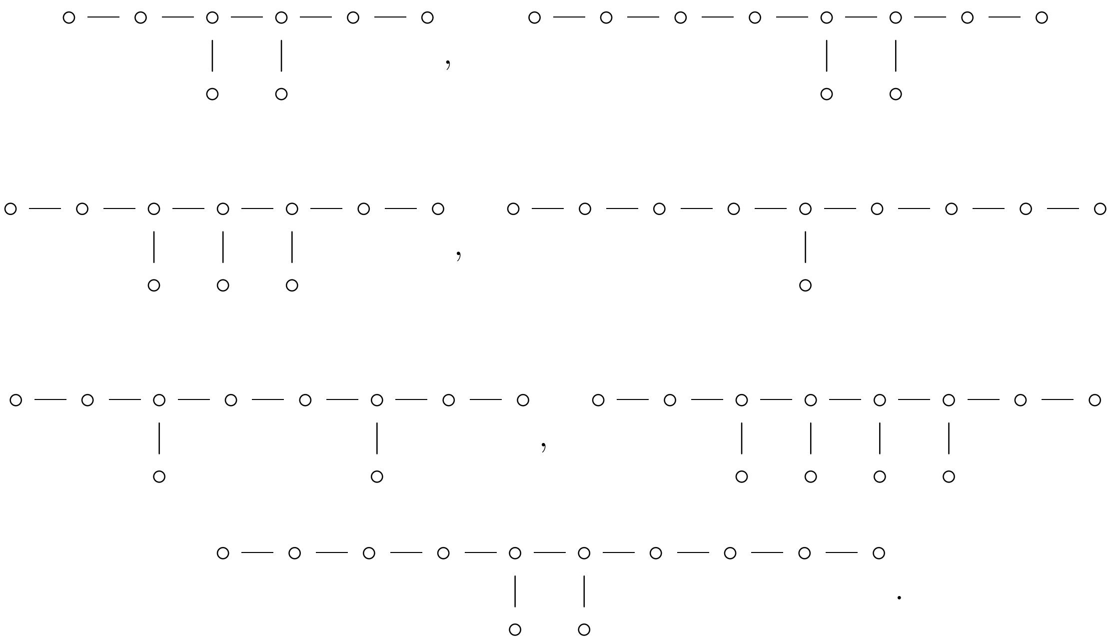
$$\tau(\partial_k(w)) = \tau_J(\partial_k(w_1)),$$

where τ_J is the half-trace on $\mathfrak{so}_J(\mathbf{F}_2)$.

It follows that if conditions (2,3) of 11.5 hold for $H = W(J)$ and $\mathfrak{so}_J(\mathbf{F}_2)$, then they hold for $H = W$ and $\mathfrak{so}(\mathbf{F}_2)$ as well. We know that 11.5(2,3) hold if J has type E_{10} or $T_{3,3,4}$, by 9.4 and 9.5. In 7.3 we verified 11.5(1) for nonsingular X containing a branch node. Thus, we have proved Theorem 2 of the introduction:

12.1 Theorem. *Let X be a nonsingular even tree, containing a subtree of type $T_{3,3,4}$ or E_{10}. Then W is dense in $O'(\mathbf{Z}_2)$.*

The nonsingular even trees to which 12.1 does not apply are few, in the sense that they can be easily listed, by considering all possible sproutings on small trees. The even nonsingular trees which do not contain $T_{3,3,4}$ or E_{10} consist of the family X_{2m} obtained by $m-1$ sproutings at a single vertex in A_2 (so $X_2 = A_2$, $X_4 = A_4$, $X_6 = E_6, \ldots$), and the seven trees



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