Epipelagic representations and invariant theory

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Abstract

We introduce a new approach to the representation theory of reductive p-adic groups G, based on the Geometric Invariant Theory (GIT) of Moy-Prasad quotients. Stable functionals on these quotients are used to give a new construction of supercuspidal representations of G having small positive depth, called *epipelagic*. With some restrictions on p, we classify the stable and semistable functionals on Moy-Prasad quotients. The latter classification determines the nondegenerate K-types for G as well as the depths of irreducible representations of G. The main step is an equivalence between Moy-Prasad theory and the theory of graded Lie algebras, whose GIT was analyzed by Vinberg and Levy. Our classification shows that stable functionals arise from $\mathbb{Z}$-regular elliptic automorphisms of the absolute root system of G. These automorphisms also appear in the Langlands parameters of epipelagic representations, in accordance with the conjectural local Langlands correspondence.

1 Introduction

Let k be a locally compact field which is complete with respect to a discrete valuation, with ring of integers A and residue field $\mathfrak{f}$ of characteristic p. Let K be a maximal unramified extension of k, with residue field $\mathfrak{F}$, an algebraic closure of $\mathfrak{f}$. Let G be a connected reductive algebraic group over k. We will assume throughout this paper that G splits over a tamely ramified extension of k. In this introduction we in fact assume G is split over K and that G is semisimple and simply-connected.

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In the paper we introduce a new approach to the representation theory of $G(k)$, based on the Geometric Invariant Theory (GIT) of Moy-Prasad filtrations of $G(K)$.

Recall that the group $G(K)$ acts on its Bruhat-Tits building $\mathcal{B}(G,K)$ and for each $x \in \mathcal{B}(G,K)$ the stabilizer $G(K)_x$ is a pro-algebraic group. In their foundational paper [27], Moy and Prasad constructed, for each $x \in \mathcal{B}(G,K)$, a filtration by normal subgroups of $G(K)_x$:

$$G(K)_x > G(K)_{x,r_1} > G(K)_{x,r_2} > \cdots ,$$

indexed by an increasing discrete sequence $\mathbf{r}(x) = (r_1, r_2, \dots)$ of positive real numbers which depends on x. The quotient $\mathsf{G}_x := G(K)_x / G(K)_{x,r_1}$ is a connected reductive group defined over $\mathfrak{f}$, and for $r = r_i \in \mathbf{r}(x)$ the quotient $\mathsf{V}_{x,r} := G(K)_{x,r}/G(K)_{x,r+}$, where $r+ = r_{i+1}$, affords an algebraic representation of G_x, induced by the conjugation action of $G(K)_x$ on $G(K)_{x,r}$.

We are interested in the GIT of G_x acting on the dual representation $\check{\mathsf{V}}_{x,r}$, for each $r \in \mathbf{r}(x)$. In this context, recall that a linear functional $\lambda \in \check{\mathsf{V}}_{x,r}$ is **semistable** if its orbit under G_x does not contain zero in its closure under the Zariski topology on $\check{\mathsf{V}}_{x,r}$, and λ is **stable** if its orbit is closed, and its stabilizer in G_x is finite.

Let π be an irreducible admissible representation of $G(k)$ of positive depth. That is, π has no nonzero vectors invariant under $G(k)_{y,s_1}$ for any $y \in \mathcal{B}(G,k)$, where s_1 is the smallest element of $\mathbf{r}(y)$. For such representations Moy and Prasad show that there is a point $x \in \mathcal{B}(G,k)$ and $r \in \mathbf{r}(x)$, such that the space of vectors in π fixed by $G(k)_{x,r+}$, regarded as a representation of the abelian group $\mathsf{V}_{x,r}$, contains a character of $\mathsf{V}_{x,r}$ arising from a semistable functional $\lambda \in \check{\mathsf{V}}_{x,r}$. This led them to the notion of the "depth" of π, and thence to a rough classification of the irreducible representations of $G(k)$. To sharpen this classification it is essential to answer the following.

Question 1: For which pairs (x,r) does $\check{\mathsf{V}}_{x,r}$ contain semistable vectors?

At the time of [27], this appeared to be a complicated problem in linear programming, and the problem has remained open until now.

The starting point of our work in this paper was the observation that *stable* vectors are also important for the representation theory of $G(k)$, as they can be used to give a new construction of certain supercuspidal representations of small positive depth. We will discuss this construction shortly. For the moment, suffice it to say that it became essential for us to answer the following.

Question 2: For which pairs (x,r) does $\check{\mathsf{V}}_{x,r}$ contain stable vectors?

With some restrictions on p, we answer Questions 1 and 2 by showing that the representations $\mathsf{G}_x \to \mathrm{GL}(\check{\mathsf{V}}_{x,r})$ are exactly the representations arising from gradings on semisimple Lie algebras over $\mathfrak{F}$. To explain this, we may assume, by passing to a smaller group, that $\mathbf{r}(x) =$

$(1/m,\ 2/m,\ \dots)$, for some positive integer m. Then the G_x-modules $\mathsf{V}_{x,i/m}$ depend only on i modulo m. On the other hand, from the Kac-Vinberg theory of periodic gradings on Lie algebras, the point x also determines a $\mathbb{Z}/m$-grading $\oplus\mathfrak{L}_i$ on the $\mathfrak{F}$-Lie algebra $\mathfrak{L}$ whose root system is that of G. Assuming p does not divide m, we show (see Thm. 4.1) that this grading coincides with the associated grading of the Moy-Prasad filtration $G(K)_x > G(K)_{x,1/m} > \cdots > G(K)_{x,1}$. In particular $\mathfrak{L}_0$ is the Lie algebra of G_x, which therefore acts on the summands $\mathfrak{L}_i$ and we have isomorphisms of representation pairs:

$$(\mathsf{G}_x, \check{\mathsf{V}}_{x,i/m}) \simeq (\mathsf{G}_x, \mathfrak{L}_i). \tag{1}$$

The point of (1) is that the GIT of the representations $(\mathsf{G}_x, \mathfrak{L}_i)$ is very simple. This was discovered by Vinberg (over $\mathbb{C}$) [38] and extended to fields of good odd positive characteristic by Levy [25]. To describe their results we may assume $i = 1$. Vinberg and Levy show that the GIT of $(\mathsf{G}_x, \mathfrak{L}_1)$ is similar to that of the adjoint representation of a reductive group. In particular, there is a *Cartan subspace* $\mathfrak{c} \subset \mathfrak{L}_1$, unique up to G_x-conjugacy, and a *little Weyl group* $W_\mathfrak{c}$ such that the restriction map on invariant polynomial rings is an isomorphism

$$\mathfrak{F}[\mathfrak{L}_1]^{\mathsf{G}_x} \xrightarrow{\sim} \mathfrak{F}[\mathfrak{c}]^{W_\mathfrak{c}},$$

and these rings are themselves polynomial in $\dim\mathfrak{c}$ homogeneneous generators. All closed G_x-orbits in $\mathfrak{L}_1$ meet $\mathfrak{c}$, so $\mathfrak{L}_1$ has semistable vectors exactly when $\mathfrak{c} \neq 0$.

As preparation for this paper we have completed, in [33], the classification of stable and semistable gradings in the Vinberg-Levy theory. Via (1), this transfers back to the Moy-Prasad setting, resulting in the answers to Questions 1 and 2.

Both answers involve interesting aspects of the Weyl group of G (and require mild restrictions on p). We refer to Thm. 8.3 for the answer to Question 1, and confine ourselves here to Question 2. Let $\mathcal{A}$ be the apartment of a maximal K-split torus $T \subset G$. Say that $x \in \mathcal{A}$ is a **stability point** if $\check{\mathsf{V}}_{x,r}$ contains stable vectors, for some $r \in \mathbf{r}(x)$. Again by passing to a smaller group we may assume that $\mathbf{r}(x) = (1/m,\ 2/m, \dots)$, for some positive integer m, and that $r = 1/m$. We show (see Cor. 5.1) that x is a stability point if and only if m is the order of a regular elliptic element σ in the Weyl group of T and x is conjugate under the affine Weyl group to the point

$$x_m := x_0 + \tfrac{1}{m}\check{\rho},$$

where x_0 is a hyperspecial point in $\mathcal{A}$ and $\check{\rho}$ is the half-sum of a set of positive co-roots of T. The Vinberg grading on $\mathfrak{L}$ corresponding to x_m is given by the eigenspaces of a lift of σ to $\mathrm{Aut}(\mathfrak{L})$.

Stability points are barycenters of certain facets in $\mathcal{A}$, and the above condition on m shows that they are quite rare. Examples include the barycenter of an alcove (an open facet in $\mathcal{A}$). In type

A_n these are the only stability points, but in every other type there are stability points which are barycenters of facets in the boundary of an alcove. For example in E_8 there are 12 conjugacy classes of stability points $x \in \mathcal{A}$ (see section 5.1) corresponding to the 12 elliptic regular classes in $W(E_8)$.

We turn now to supercuspidal representations. Assume that $x \in \mathcal{A} \cap \mathcal{B}(G,k)$ and that we are given a stable functional $\lambda \in \check{\mathsf{V}}_{x,r_1}(\mathfrak{f})$, with isotropy group $\mathsf{A}_\lambda \subset \mathsf{G}_x$. With this data we construct (see section 2.5) a family of irreducible supercuspidal representations $\pi_x(\lambda, \rho)$ of $G(k)$, parameterized by irreducible representations ρ of a certain finite-dimensional semisimple algebra $\mathcal{H}_\lambda$ (which in most cases is the group algebra of $\mathsf{A}_\lambda(\mathfrak{f})$). These are the "epipelagic representations" in the title. [1] They have depth r_1, the smallest element of $\mathbf{r}(x)$.

If x is the barycenter of an alcove then the $\pi_x(\lambda, \rho)$ are the "simple supercuspidal representations" of [14], of depth $1/h$, where h is the Coxeter number of G. (For G of type A_n, such representations were constructed by Carayol [10].) In this case the group G_x is a torus and stable functionals are just those with nonzero component in each weight space. When x is contained in the boundary of an alcove, as in the E_8 examples mentioned above, the group G_x is no longer a torus, and the GIT of its action on $\check{\mathsf{V}}_{x,r_1}$ becomes the essential point in the construction. For such points x the epipelagic representations in this paper have no analogue in type A_n.

If $p > h$ then one can also construct our epipelagic representations following Adler [1]. (The generalization of Adler's approach in [39] gives all supercuspidal representations of $G(k)$ when p is large [21].) To implement Adler's method for epipelagic representations one must construct a certain tamely ramified anisotropic torus in $G(k)$ having x as its unique fixed-point in $\mathcal{B}(G,k)$. This was carried out by Kaletha [19] in work based on an earlier version of this paper. In contrast, the GIT approach used here does not require any restrictions on p; it works even when the above tamely ramified torus does not exist. Instead, the required input is a stable rational vector in $\check{\mathsf{V}}_{x,r_1}$.

Epipelagic representations have already found other applications to automorphic forms [13], [16], [22], the explicit Langlands correspondence [9] and inverse Galois theory [20]. Given a stable point in $\check{\mathsf{V}}_{x,r_1}$, the construction of the corresponding epipelagic representations is very simple and does not require the full structure theory of G. For this reason we describe our construction early in the paper, with a minimum of technicalities, in order to make the epipelagic construction more accessible to non-specialists.

Finally, it is natural to ask about the Langlands parameters corresponding to our epipelagic representations $\pi_x(\lambda, \rho)$. If $p > h$, a stable functional $\lambda \in \check{\mathsf{V}}_{x,1}$ leads naturally to a discrete Langlands

[1] In Oceanography, the *Epipelagic zone* is the uppermost layer of the open sea where photosynthesis can occur.

parameter

$$\varphi_\lambda : \mathcal{W}_k \longrightarrow {}^L G,$$

from the Weil group of k to the L-group of G, whose image of Frobenius is determined by the $\mathfrak{f}$-structure on a Cartan subspace of $\check{\mathsf{V}}_{x,r_1}$ containing λ. The inertia subgroup $\mathcal{I}_k \leq \mathcal{W}_k$ maps to the normalizer of a maximal torus in ${}^L G$ and the projection of $\varphi_\lambda(\mathcal{I}_k)$ to the Weyl group of ${}^L G$ is generated by the elliptic $\mathbb{Z}$-regular element σ corresponding to x, as above.

This parameter φ_λ satisfies properties expected of the local Langlands conjecture (see [14], for example). First, φ_λ is itself "epipelagic", in the sense that the upper ramification filtration $\varphi_\lambda(\mathcal{W}_k)$ has a single upper break at r_1, equal to the depth of the representations $\pi_x(\lambda, \rho)$. Second, the adjoint Swan conductor of φ_λ is equal to the dimension of G_x, and the centralizer of φ_λ in the dual group $\hat{G}$ has the same cardinality as the isotropy group $\mathsf{A}_\lambda(\mathfrak{f})$. It follows that the Euler-Poincaré formal degree [14, 7.1] of $\pi_x(\lambda, \rho)$ is given in terms of the adjoint gamma value $\gamma(\varphi_\lambda, \hat{\mathfrak{g}}, 0)$ as proposed in [17] (see also [14]).

If p is small, then (1) does not immediately apply. In many cases however, one can still find stable points in $\check{\mathsf{V}}_{x,r_1}(\mathfrak{f})$, hence epipelagic representations. In sections 7.4 and 7.5 we illustrate the epipelagic construction at these interesting primes p, and we again construct candidates for their Langlands parameters. Here, though the epipelagic representations have a uniform construction, the parameters depend crucially on the particular prime p, and the depth of the representation no longer agrees with the depth of the parameter. This different behavior at small primes was observed already in [14].

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2 Supercuspidal representations

2.1 Compact induction

We will review here some results on compact induction which are well-known to experts. See [8] for more details.

Let G be a locally profinite group with center Z. Let H be an open subgroup of G such that $Z \subset H$ and H/Z is compact, and let τ be a smooth finite-dimensional representation of H on a complex vector space W. The *compactly-induced representation* $\operatorname{ind}_H^G \tau$ is realized on the complex vector space of functions $f : G \to W$ satisfying the two conditions:

- $f(hx) = \tau(h) f(x)$ for all $h \in H$ and $x \in G$;
- f is supported on only finitely many cosets of H in G.

The group G acts on $\operatorname{ind}_H^G \tau$ by right translations: $[g \cdot f](x) = f(xg)$.

For $g \in G$, let $H^g = g^{-1} H g$ and let $\tau^g : H^g \to GL(W)$ be the conjugate representation, given by $\tau^g(h^g) = \tau(h)$. The *intertwining* of τ is the set $I(G, H, \tau) = \{g \in G : \ \tau \simeq \tau^g \ \text{ on } \ H \cap H^g\}$. Note that $I(G, H, \tau)$ is preserved under left-multiplication by H, hence is a union of cosets of H in G.

Now assume that G is the group of k-rational points in a connected reductive algebraic group defined over a field k which is complete and locally compact, with respect to a discrete valuation. A proof of the following basic result can be found in [8, 3.11.4].

Lemma 2.1. *Let H be an open subgroup of G containing the center Z of G with H/Z compact, and let τ be an irreducible smooth representation of H. Then the representation $\operatorname{ind}_H^G \tau$ is irreducible for G if and only if $I(G, H, \tau) = H$.*

If $J < H$ are two open subgroups of G containing and compact modulo Z and τ is a smooth representation of J then $\operatorname{ind}_J^H \tau$ is smooth and we have *transitivity of induction*:

$$\operatorname{ind}_J^G \tau \simeq \operatorname{ind}_H^G \operatorname{ind}_J^H \tau.$$

Suppose in addition that J is normal in H, and let $\chi : J \to \mathbb{C}^\times$ be a smooth character of J. If $h \in H$ then χ^h is again a character of J, possibly equal to χ, and $I(G, J, \chi)$ is a union of cosets of the stabilizer $H_\chi = \{h \in H : \ \chi^h = \chi\}$. We have $J \trianglelefteq H_\chi$ and the quotient $A_\chi := H_\chi / J$ is a finite group.

From Mackey theory for finite groups, we deduce the following. The intertwining algebra

$$\mathcal{H}_\chi := \operatorname{End}_{H_\chi}\left(\operatorname{ind}_J^{H_\chi} \chi\right)$$

has dimension equal to $|A_\chi|$, and we have a bijection, denoted $\rho \mapsto \chi_\rho$, from the set $\operatorname{Irr}(\mathcal{H}_\chi)$ of simple $\mathcal{H}_\chi$-modules (up to equivalence) to the set of irreducible constituents of $\operatorname{ind}_J^{H_\chi} \chi$ such that

$$\operatorname{ind}_J^{H_\chi} \chi = \bigoplus_{\rho \in \operatorname{Irr}(\mathcal{H}_\chi)} (\dim \rho) \cdot \chi_\rho. \tag{2}$$

Finally, each χ_ρ is χ-isotypic when restricted to J, and induces irreducibly to H.

Lemma 2.2. *Assume that $I(G, J, \chi) = H_\chi$. Then the following hold.*

1. *The representation $\operatorname{ind}_J^G \chi$ has a finite direct sum decomposition*

$$\operatorname{ind}_J^G \chi = \bigoplus_{\rho \in \operatorname{Irr}(\mathcal{H}_\chi)} (\dim \rho) \cdot \operatorname{ind}_{H_\chi}^G \chi_\rho. \tag{3}$$

2. *For each $\rho \in \operatorname{Irr}(\mathcal{H}_\chi)$, the compactly induced representation $\operatorname{ind}_{H_\chi}^G \chi_\rho$ is irreducible.*

3. *If ρ, ρ' are inequivalent simple modules for $\mathcal{H}_\chi$, then $\operatorname{ind}_{H_\chi}^G \chi_\rho$ and $\operatorname{ind}_{H_\chi}^G \chi_{\rho'}$ are inequivalent representations of G.*

Proof. The direct sum decomposition (3) follows from (2) and transitivity of induction.

To see that the summands are irreducible, we consider their intertwining. Suppose $g \in I(G, H_\chi, \chi_\rho)$. This means $\chi_\rho = (\chi_\rho)^g$ on $H_\chi \cap H_\chi^g$. Restricting to J, we get $\chi = \chi^g$ on $J \cap J^g$, so that $g \in I(G, J, \chi)$, which is equal to H_χ by assumption. Hence $I(G, H_\chi, \chi_\rho) = H_\chi$, so $\operatorname{ind}_{H_\chi}^G \chi_\rho$ is irreducible, by Lemma 2.1.

A variant of Lemma 2.1 can be used to show inequivalence, or one can argue as follows. From Mackey theory for G [24] and Frobenius reciprocity [8, 1.2.5], it follows that

$$\operatorname{End}_G(\operatorname{ind}_J^G \chi) \simeq \bigoplus_{g \in J \backslash G / J} \operatorname{Hom}_{J \cap J^g}(\chi, \chi^g) = \bigoplus_{g \in A_\chi} \operatorname{Hom}_J(\chi, \chi), \tag{4}$$

again using that $I(G, J, \chi) = H_\chi$. Therefore, $\dim \operatorname{End}_G(\operatorname{ind}_J^G \chi) = |A_\chi| = \dim \mathcal{H}_\chi$. It follows that there can be no nonzero intertwining maps between summands for $\rho \not\simeq \rho'$ in (3). This completes the proof.

□

Remark 1. If χ extends to a character $\tilde{\chi}$ on H_χ then $\mathcal{H}_\chi$ is just the group algebra of A_χ. Each $\rho \in \operatorname{Irr}(A_\chi)$ may be viewed as a representation of H_χ via the projection $H_\chi \to A_\chi$, and we may take $\chi_\rho = \tilde{\chi} \otimes \rho$.

In our applications of Lemma 2.2, the group J will always be a pro-p group. In most cases, A_χ will have order prime to p, so the quotient $H_\chi \to A_\chi$ will split and χ will extend to H_χ. We can also extend χ whenever $H^2(A_\chi, \mathbb{C}^\times) = 1$, for example if A_χ is cyclic or symmetric on three letters.

Remark 2. The formal degree of $\operatorname{ind}_{H_\chi}^G \chi_\rho$ with respect to a Haar measure μ on G is given by

$$\deg_\mu \left(\operatorname{ind}_{H_\chi}^G \chi_\rho \right) = \frac{\dim \chi_\rho}{\mu(H_\chi)} = \frac{\dim \chi_\rho}{|A_\chi|} \cdot \frac{1}{\mu(J)}. \tag{5}$$

2.2 Non-archimedean local fields

Let k be a locally compact field which is complete with respect to a discrete valuation v, with value group $\mathsf{v}(k^\times) = \mathbb{Z}$. Let $A = \{a \in k : \mathsf{v}(a) \geq 0\}$ denote the ring of integers of k, and let p be the characteristic of the residue field $\mathfrak{f}$ of k.

We fix a separable closure $\bar{k}$ of k and let K be the maximal unramified extension of k in $\bar{k}$. The valuation v extends to K, again taking values in $\mathbb{Z}$. The residue field $\mathfrak{F}$ of K is an algebraic closure of $\mathfrak{f}$.

2.3 Semisimple groups

For the rest of the paper, G denotes a connected semisimple algebraic group defined over k. Let S be a maximal k-split torus in G and let T be a maximal K-split torus of G containing S. We may and shall choose T to be defined over k [7, 5.1.12]. Let $F : G(K) \to G(K)$ be the Frobenius action, arising from the given k-structure on G; the constructions in section 2.1 will be applied to the locally profinite group of k-rational points: $G(k) = G(K)^F$.

Let $\mathcal{A}(T, K)$ be the apartment of T in the Bruhat-Tits building $\mathcal{B}(G, K)$ of $G(K)$ and let Ψ_K denote the set of affine K- roots of T in G. Each $\psi \in \Psi_K$ is an affine function on $\mathcal{A}(T, K)$ whose gradient $\dot{\psi}$ is a K-root of T in G. We have $\dot{\psi}_1 = \dot{\psi}_2$ if and only if $\psi_1 - \psi_2$ is a constant function.

The Frobenius map F extends to a mapping on $\mathcal{B}(G, K)$ which preserves and acts affinely on $\mathcal{A}(T, K)$. We may identify

$$\mathcal{B}(G, k) = \mathcal{B}(G, K)^F, \qquad \text{and} \qquad \mathcal{A}(S, k) = \mathcal{A}(T, K)^F,$$

where $\mathcal{B}(G, k)$ is the building of $G(k)$ and $\mathcal{A}(S, k)$ is the apartment in $\mathcal{B}(G, k)$ corresponding to S [37, 2.6.1].

Each affine K-root in Ψ_K restricts to an affine function (possibly the zero function) on $\mathcal{A}(S, k)$. We let $\mathcal{A}(T, K)_{\mathbb{Q}}$ and $\mathcal{A}(S, k)_{\mathbb{Q}}$ be the subsets of the respective apartments where all roots in Ψ_K take rational values.

For $x \in \mathcal{A}(T, K)$ and $r \geq 0$, let

$$T(K)_r, \quad T(K)_{r+}, \quad G(K)_{x,r}, \quad G(K)_{x,r+}$$

denote the filtration groups defined by Moy and Prasad [27]. For example, $G(K)_{x,0}$ is the parahoric subgroup of $G(K)$ attached to x by Bruhat-Tits theory and $G(K)_{x,0+}$ is the pro-unipotent radical of $G(K)_{x,0}$. The top quotient

$$\mathsf{G}_x := G(K)_{x,0}/G(K)_{x,0+}$$

is a connected reductive group over the residue field $\mathfrak{F}$. For $r > 0$ the quotient

$$\mathsf{V}_{x,r} := G(K)_{x,r}/G(K)_{x,r+}$$

is abelian, and is in fact a finite-dimensional vector space over $\mathfrak{F}$. The conjugation action of $G(K)_{x,0}$ on $G(K)_{x,r}$ induces an algebraic representation of G_x on $\mathsf{V}_{x,r}$. If $x \in \mathcal{A}(S,k)$, then both G_x and its representation $\mathsf{V}_{x,r}$ are defined over $\mathfrak{f}$, with Frobenius action induced by F.

The quotient

$$\mathsf{T} := T(K)_0/T(K)_{0+}$$

is a maximal torus in G_x, whose character group may be identified with that of T. The nonzero weights of T in $\mathsf{V}_{x,r}$ are the gradients $\dot{\psi}$ where $\psi(x) = r$.

2.4 Stable vectors

Let $\check{\mathsf{V}}_{x,r} = \operatorname{Hom}(\mathsf{V}_{x,r}, \mathfrak{F})$ be the representation of G_x dual to $\mathsf{V}_{x,r}$. Following [29], we say that a functional $\lambda \in \check{\mathsf{V}}_{x,r}$ is **stable** if the following two conditions hold:

- the orbit $\mathsf{G}_x \cdot \lambda$ is Zariski-closed in $\check{\mathsf{V}}_{x,r}$ and
- the stabilizer of λ in G_x is finite, as an algebraic group.

Given two points $x, y \in \mathcal{A}(T,K)_{\mathbb{Q}}$, and a rational number $r > 0$, the image $\mathsf{V}_{x,y,r}$ of $G(K)_{y,r+} \cap G(K)_{x,r}$ in $\mathsf{V}_{x,r}$ is a T-stable subspace of $V_{x,r}$ with weight decomposition

$$\mathsf{V}_{x,y,r} := \bigoplus_{\substack{\psi \in \Psi_{x,r} \\ \psi(y) > r}} \mathsf{V}_{x,r}(\dot{\psi}).$$

The key step for constructing epipelagic representations is the following.

Lemma 2.3. *Let $\lambda \in \check{\mathsf{V}}_{x,r}$ be a stable functional. If λ vanishes identically on $\mathsf{V}_{x,y,r}$ then $y = x$.*

Proof. Let $X_*(T)$ be the co-character group of T. The vector space $\mathbb{Q} \otimes X_*(T)$ acts transitively on $\mathcal{A}(T,K)_{\mathbb{Q}}$ by translations, so the difference $v = y - x$ lies in $\mathbb{Q} \otimes X_*(T)$. Assume that $v \neq 0$. Then we can write $v = s\gamma$, where $\gamma \in X_*(T)$, and s is a positive rational number. For all $\psi \in \Psi_{x,r}$, we have

$$\psi(y) = \psi(x + s\gamma) = \psi(x) + s\langle\dot{\psi}, \gamma\rangle = r + s\langle\dot{\psi}, \gamma\rangle.$$

Since $s > 0$, the weight decomposition of $\mathsf{V}_{x,y,r}$ is

$$\mathsf{V}_{x,y,r} = \bigoplus_{\substack{\psi \in \Psi_{x,r} \\ \langle \dot{\psi}, \gamma \rangle > 0}} \mathsf{V}_{x,r}(\dot{\psi}).$$

The weight decomposition of λ is of the form

$$\lambda = \sum_{\psi \in \Psi_{x,r}} \lambda_{-\dot{\psi}},$$

where $\lambda_{-\dot{\psi}}$ has T-weight $-\dot{\psi}$. If λ vanishes identically on $\mathsf{V}_{x,y,r}$ then $\lambda_{-\dot{\psi}} = 0$ if $\langle \dot{\psi}, \gamma \rangle > 0$, so we have

$$\lambda = \mu + \sum_{\substack{\psi \in \Psi_{x,r} \\ \langle \dot{\psi}, \gamma \rangle < 0}} \lambda_{-\dot{\psi}} \ ,$$

where

$$\mu = \sum_{\substack{\psi \in \Psi_{x,r} \\ \langle \dot{\psi}, \gamma \rangle = 0}} \lambda_{-\dot{\psi}}.$$

Since

$$\gamma(t) \cdot \lambda = \mu + \sum_{\substack{\psi \in \Psi_{x,r} \\ \langle \dot{\psi}, \gamma \rangle < 0}} t^{-\langle \dot{\psi}, \gamma \rangle} \lambda_{-\dot{\psi}},$$

it follows that

$$\lim_{t \to 0} \gamma(t) \cdot \lambda = \mu.$$

Since the G_x-orbit of λ is closed, we have $\mu \in \mathsf{G}_x \cdot \lambda$, so μ is also a stable vector in $\check{\mathsf{V}}_{x,r}$. But γ takes values in the stabilizer of μ in G_x, which is finite, so $\gamma = 0$, contradicting the assumption $y \neq x$. $\square$

Remark: If we assume $x, y \in \mathcal{A}(S, k)_{\mathbb{Q}}$, then the proof of Lemma 2.3 requires only the weaker assumption that the stabilizer of λ in G_x is anisotropic over $\mathfrak{f}$ (cf. [28, Cor. 4.9]). We will not use this variant of the result.

2.5 Epipelagic supercuspidal representations

For a given point $x \in \mathcal{A}(S, k)$, let $r(x)$ be the smallest positive value in the set $\{\psi(x) : \ \psi \in \Psi_K\}$ of values of all affine K-roots. Thus, we have $G(K)_{x,0+} = G(K)_{x,r(x)}$. We say that an irreducible representation π of $G(k)$ is **epipelagic** if it has a nonzero vector fixed under the subgroup

$G(k)_{x,r(x)+}$ and π has depth $r(x)$. In this section we show how stable points in $\check{V}_{x,r(x)}(\mathfrak{f})$ give rise to epipelagic supercuspidal representations of $G(k)$.

We will abbreviate

$$J_x := G(k)_{x,r(x)}, \quad J_x^+ := G(k)_{x,r(x)+}, \quad \mathsf{V}_x := \mathsf{V}_{x,r(x)}, \quad \check{\mathsf{V}}_x := \check{\mathsf{V}}_{x,r(x)}, \quad \mathsf{V}_{x,y} = \mathsf{V}_{x,y,r(x)}.$$

The isotropy group $G(K)_x := \mathrm{Stab}_{G(K)}(x)$ contains $G(K)_{x,0}$ with finite index. The actions of $G(K)_{x,0}$ on V_x and $\check{\mathsf{V}}_x$ extend to $G(K)_x$ which preserves the set of G_x-stable functionals in $\check{\mathsf{V}}_x$.

Let $\lambda \in \check{\mathsf{V}}_x(\mathfrak{f})$ be an $\mathfrak{f}$-rational stable functional with stabilizer $\mathrm{Stab}_{G(K)_x}(\lambda)$ in $G(K)_x$, and set

$$H_x := G(K)_x^F = G(k)_x, \qquad H_{x,\lambda} := \mathrm{Stab}_{G(K)_x}(\lambda)^F = \mathrm{Stab}_{H_x}(\lambda).$$

Since λ has finite stabilizer in G_x, it follows that the quotient group $\mathrm{Stab}_{G(K)_x}(\lambda)/G(K)_{x,r(x)}$ is finite. We set

$$A_{x,\lambda} := H_{x,\lambda}/J_x = \left[\mathrm{Stab}_{G(K)_x}(\lambda)/G(K)_{x,r(x)}\right]^F, \tag{6}$$

where the latter equality follows from the fact that $G(K)_{x,r(x)}$ has trivial $\mathrm{Gal}(K/k)$-cohomology, by the pro-finite version of the Lang-Steinberg theorem.

Fix a nontrivial character $\chi : \mathfrak{f}^+ \to \mathbb{C}^\times$ of the additive group of $\mathfrak{f}$. The composition

$$\chi_\lambda := \chi \circ \lambda : \mathsf{V}_x(\mathfrak{f}) \to \mathbb{C}^\times$$

is a character of J_x which is trivial on J_x^+, and $H_{x,\lambda}$ is the stabilizer of χ_λ in H_x. We consider the compactly-induced representation

$$\pi_x(\lambda) := \mathrm{ind}_{J_x}^{G(k)} \chi_\lambda,$$

and the intertwining algebra

$$\mathcal{H}_{x,\lambda} := \mathrm{End}_{H_{x,\lambda}} \left(\mathrm{ind}_{J_x}^{H_{x,\lambda}} \chi_\lambda\right).$$

If ρ is a simple $\mathcal{H}_{x,\lambda}$-module, let $\chi_{\lambda,\rho}$ denote the corresponding irreducible constitutent for $H_{x,\lambda}$ in $\mathrm{ind}_{J_x}^{H_{x,\lambda}} \chi_\lambda$, as in (2).

Proposition 2.4. *Suppose that $\lambda \in \check{\mathsf{V}}_x(\mathfrak{f})$ is an $\mathfrak{f}$-rational stable functional for the action of G_x on $\check{\mathsf{V}}_x$. Then the following hold.*

1. *The representation $\pi_x(\lambda)$ has a finite direct sum decomposition*

$$\pi_x(\lambda) = \bigoplus_{\rho \in \mathrm{Irr}(\mathcal{H}_{x,\lambda})} \dim \rho \cdot \pi_x(\lambda, \rho),$$

   where $\pi_x(\lambda, \rho) := \mathrm{ind}_{H_{x,\lambda}}^{G(k)} \chi_{\lambda,\rho}$ is an irreducible supercuspidal representation of $G(k)$, for each $\rho \in \mathrm{Irr}(\mathcal{H}_{x,\lambda})$.

2. *If ρ and ρ' are inequivalent simple modules for $\mathcal{H}_{x,\lambda}$ then $\pi_x(\lambda, \rho)$ and $\pi_x(\lambda, \rho')$ are inequivalent representations of $G(k)$.*

3. *The formal degree of $\pi_x(\lambda, \rho)$ with respect to a Haar measure μ on $G(k)$ is given by*

$$\deg_\mu(\pi_x(\lambda, \rho)) = \frac{\dim \chi_{\lambda,\rho}}{|A_{x,\lambda}|} \cdot \frac{1}{\mu(J_x)}.$$

Proof. By Lemma 2.2 (taking $J = J_x$ and $H = H_x$, so that $H_{\chi_\lambda} = H_{x,\lambda}$), it suffices to show that if $g \in G(k)$ and

$$\chi_\lambda = \chi_\lambda^g \quad \text{on} \quad J_x \cap J_x^g, \tag{7}$$

then $g \in H_x$. Since $H_x \supset G(k)_{x,0}$ and the latter contains an Iwahori subgroup of $G(k)$ we may write $g = anb$, where $n \in G(k)$ normalizes S and $a, b \in H_x$ [37, 3.3.1]. Since J_x is normal in H_x, condition (7) is equivalent to

$${}^b\chi_\lambda = (\chi_\lambda^a)^n \quad \text{on} \quad J_x \cap J_x^n, \tag{8}$$

so we certainly have

$${}^b\chi_\lambda = (\chi_\lambda^a)^n \quad \text{on} \quad J_x \cap (J_x^+)^n. \tag{9}$$

But J_x^+ is also normal in H_x and χ_λ is trivial on J_x^+, so $(\chi_\lambda^a)^n$ is trivial on $(J_x^+)^n$. Thus, (9) implies that ${}^b\chi_\lambda$ is trivial on $J_x \cap (J_x^+)^n$.

Now $\mathsf{V}_x(\mathfrak{f}) = J_x/J_x^+$ and the image of $J_x \cap (J_x^+)^n$ in $\mathsf{V}_x(\mathfrak{f})$ is the subspace $\mathsf{V}_{x,y}(\mathfrak{f})$, where $y = n^{-1} \cdot x$. Moreover ${}^b\chi_\lambda = \chi_{b\cdot\lambda}$, and $b \cdot \lambda$ is a stable vector in $\check{\mathsf{V}}_x(\mathfrak{f})$ which vanishes on $\mathsf{V}_{x,y}(\mathfrak{f})$. Since $\mathsf{V}_{x,y}(\mathfrak{f})$ contains an $\mathfrak{F}$- basis of $\mathsf{V}_{x,y}$ it follows that $b \cdot \lambda$ vanishes on $\mathsf{V}_{x,y}$. Now Lemma 2.3 implies that $y = x$, so that $n \in H_x$. It follows that $g = anb \in H_x$, as we wished to show. □

2.6 Example: simple supercuspidal representations

The above construction of supercuspidal representations requires the existence of stable orbits. For certain points $x \in \mathcal{A}(S, k)$, such orbits do indeed exist, and can be found in a uniform way for all tamely ramified groups. Here we assume for convenience that G is absolutely simple. Let E_0 be the smallest tamely ramified extension of K splitting G. The Galois group $\Gamma_0 = \mathrm{Gal}(E_0/K)$ is cyclic of order $e := [E_0 : K]$. Since G is quasi-split over K, a generator of $\mathrm{Gal}(E_0/K)$ acts on the based root datum $(X, \Delta, \check{R}, \check{\Delta})$ of G via an automorphism $\vartheta \in \mathrm{Aut}(R, \Delta)$ of order e. (See section 3.1 for more details on the root datum and Galois action.)

An *alcove* in $\mathcal{A}(T, K)$ is a connected component of the set of points in $\mathcal{A}(T, K)$ on which no affine K-root vanishes. There exists and we choose an alcove $\mathcal{C}$ in $\mathcal{A}(T, K)$ for which $F(\mathcal{C}) = \mathcal{C}$ [37, 3.4.3]. Each hyperplane bounding $\mathcal{C}$ is the zero locus of a unique affine K-root which is positive on $\mathcal{C}$. Let $\Pi_{\mathcal{C}} = \{\psi_0, \psi_1, \dots, \psi_{\ell_\vartheta}\}$ be the set of these affine K-roots, where $\ell_\vartheta = \dim \mathcal{A}(T, K)$ is the K-rank of G. The affine roots in $\Pi_{\mathcal{C}}$ satisfy a single linear relation

$$\sum_{i=0}^{\ell_\vartheta} b_i \psi_i = \frac{1}{e}, \tag{10}$$

where the numbering is chosen so that $b_0 = 1$ (see also section 3.3). All of the b_i are positive integers and their sum is

$$\sum_{i=0}^{\ell_\vartheta} b_i = \frac{h_\vartheta}{e}, \tag{11}$$

where h_ϑ is the twisted Coxeter number of (R, ϑ) (cf. [31]). If $\vartheta = 1$ (i.e. if G is K-split) then h_ϑ is the usual Coxeter number of G.

The **barycenter** of $\mathcal{C}$ is the unique point $x \in \mathcal{C}$ at which all $\psi_i \in \Pi_{\mathcal{C}}$ take the same value. This common value must be $1/h_\vartheta$, by equation (11). From the uniqueness of x we have $F(x) = x$, so in fact $x \in \mathcal{A}(S, k)$.

It follows that the minimal positive value $r(x)$ (see section 2.5) is given by

$$r(x) = \frac{1}{h_\vartheta}.$$

The subgroup H_x is the normalizer of the Iwahori subgroup of $G(k)$ attached to $\mathcal{C}$ and J_x is the pro-unipotent radical of H_x. The group G_x is the torus T whose character group $X^*(\mathsf{T}) = X_\vartheta$ is the coinvariant group $X/(1-\vartheta)X$ modulo torsion, and $\check{\mathsf{V}}_x$ decomposes under T into a direct sum of lines

$$\check{\mathsf{V}}_x = \bigoplus_{i=0}^{\ell_\vartheta} \check{\mathsf{V}}_x(-\dot{\psi}_i),$$

where $\check{\mathsf{V}}_x(-\dot{\psi}_i)$ affords the character $-\dot{\psi}_i$. A functional $\lambda \in \check{\mathsf{V}}_x$ is stable for T precisely when each of its components $\lambda_i \in \check{\mathsf{V}}_x(-\dot{\psi}_i)$ is nonzero. In this case the stabilizer of λ in T is the intersection of root kernels

$$\mathsf{T}_\lambda = \bigcap_{i=0}^{\ell_\vartheta} \ker(\dot{\psi}_i).$$

Thus, G has epipelagic representations of depth $1/h_\vartheta$, associated to the barycenter x of an alcove in $\mathcal{A}(T, K)$.

If G is simply-connected then $H_{x,\lambda} = Z(k) \cdot J_x$, where $Z(k)$ is the center of $G(k)$, so there are $|Z(k)|$ of these representations, one with each central character. If G is also split over k these are the simple supercuspidal representations constructed in [14].

3 Semistable points and barycenters

We have just seen that if x is the barycenter of an alcove in $\mathcal{A}(T, K)$ then G_x has stable orbits in $\check{\mathsf{V}}_{x,1/h_\vartheta}$. In this section we give a necessary condition on a rational point $x \in \mathcal{A}(T, K)$ for G_x to have semistable orbits in $\check{\mathsf{V}}_{x,r(x)}$; we will show this can hold only if x is the barycenter of some facet in $\mathcal{A}(T, K)$.

There are still no restrictions on the residue characteristic p other than G being split over a tamely ramified extension of k. For convenience we assume in this section that G is absolutely simple.

3.1 Root data

We now require a more detailed account of the structure of G. Recall that $\bar{k}$ is a fixed separable closure of k, and that $E_0 \subset \bar{k}$ is the minimal tame extension of K which splits G. Let e be the degree of E_0/K and choose a generator σ_0 of the Galois group $\mathrm{Gal}(E_0/K)$.

Recall that S is a maximal k-split torus in G, contained in a maximal K-split k-torus of G defined over k. Since G is quasi-split over K, the centralizer $M = C_G(T)$ is a maximal torus of G contained in a K-rational Borel subgroup B of G. Let $N = N_G(M)$ be the normalizer of M in G, and let $W = N/M$ be the absolute Weyl group of G.

The based root datum of the split group $G \times_k E_0$ is a quadruple $\Phi := (X, \Delta, \check{X}, \check{\Delta})$, where $X = X^*(M)$ and $\check{X} = X_*(M)$ are the character and co-character groups of M, and Δ and $\check{\Delta}$ are the simple roots and co-roots of M in B. The latter are contained in the sets R and $\check{R}$ of all roots and co-roots of M in G. Define $\check{\rho} \in \frac{1}{2}\check{X}$ to be one-half the sum of the co-roots which are positive with respect to $\check{\Delta}$.

Let $\mathrm{Aut}(R)$ be the subgroup of $\mathrm{Aut}(X)$ preserving R and let $\mathrm{Aut}(R,\Delta)$ be the subgroup of $\mathrm{Aut}(R)$ preserving Δ. Via the action of W on R we may identify

$$\mathrm{Aut}(R) = W \rtimes \mathrm{Aut}(R,\Delta).$$

The action of $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ on $G(\bar{k})$ gives a Galois action on X, via the homomorphism $\varrho : \Gamma_k \to \mathrm{Aut}(R)$ given by $\varrho_\gamma(\chi)(s) = \gamma(\chi(s^\gamma))$, for $\gamma \in \Gamma_k$, $\chi \in X$ and $s \in M$. This map factors through $\mathrm{Gal}(E_0/k)$ and the image of the cyclic subgroup $\mathrm{Gal}(E_0/K) = \langle \sigma_0 \rangle$ is generated by the automorphism

$$\vartheta := \varrho(\sigma_0) \in \mathrm{Aut}(R,\Delta)$$

of order e. The full image $\varrho(\Gamma_k)$ is generated by ϑ and the image $\phi = \varrho(F)$ of Frobenius, with the relation

$$\phi\vartheta\phi^{-1} = \vartheta^q.$$

This Frobenius element ϕ need not preserve Δ (it does so if G is quasi-split over k and B is chosen to be defined over k); in general, we have a unique decomposition

$$\phi = w \cdot \phi_0, \tag{12}$$

where $w \in W$ and $\phi_0 \in \mathrm{Aut}(R,\Delta)$.

Since G splits over E_0 we may choose a pinning on $G \times_k E_0$. This consists of root group homomorphisms u_α, for each $\alpha \in R$, from the additive group to B such that u_α is defined over E_0, and we have

(i) $tu_\alpha(b)t^{-1} = u_\alpha(\alpha(t)b)$ for all $t \in M(E_0)$ and

(ii) $\sigma_0\left(u_\alpha(b)\right) = u_{\vartheta\alpha}(c_\alpha\sigma_0(b))$, where $c_\alpha = \pm 1$, $c_\alpha c_{-\alpha} = 1$ and if $\alpha \in \pm\Delta$ then $c_\alpha = 1$.

The elements

$$n_\alpha := u_\alpha(-1) \cdot u_{-\alpha}(1) \cdot u_\alpha(-1),$$

for $\alpha \in R$, belong to $N(E_0)$ and generate a finite subgroup N_0 of $N(E_0)$. Under the projection $N \to W$, each element n_α maps to the corresponding reflection in W. It follows that $N(E_0) = N_0 \cdot M(E_0)$. From (ii) above it follows that $\sigma_0(n_\alpha) = n_{\vartheta\alpha}^{\pm 1}$ for all $\alpha \in R$, so we have

$$\sigma_0(N_0) = N_0. \tag{13}$$

3.2 Apartments and affine root systems

Let $\mathcal{A}(S,k)$, $\mathcal{A}(T,K)$, $\mathcal{A}(M,E_0)$ be the respective apartments of the tori S,T,M in the Bruhat-Tits buildings of $G(k)$, $G(K)$ and $G(E_0)$. Recall that $\mathcal{A}(M,E_0)$ is an affine space under the vector space $V := \mathbb{R}\otimes\check{X}$, on which the Galois group $\mathrm{Gal}(E_0/k)$ acts by affine transformations, and we have

$$\mathcal{A}(T,K) = \mathcal{A}(M,E_0)^{\mathrm{Gal}(E_0/K)}, \qquad \mathcal{A}(S,k) = \mathcal{A}(M,E_0)^{\mathrm{Gal}(E_0/k)}.$$

The group $N(E_0)$ also acts on $\mathcal{A}(M,E_0)$ via affine transformations. Since $N(E_0) = N_0\cdot M(E_0)$ and N_0 is finite, it follows that N_0 has a unique fixed-point $x_0 \in \mathcal{A}(M,E_0)$. Any linear functional λ on V then gives an affine linear function $\tilde{\lambda} : \mathcal{A}(M,E_0)\to\mathbb{R}$, defined by $\tilde{\lambda}(x) = \lambda(x-x_0)$, and the gradient of $\tilde{\lambda}$ is λ. In particular, each root $\alpha\in R$ gives an affine function $\tilde{\alpha}$ on $\mathcal{A}(M,E_0)$.

From (13) it follows that $x_0\in\mathcal{A}(T,K)$. The system Ψ_K of affine K-roots of $G(K)$ (with respect to T) may be constructed from the data (R,ϑ,x_0) as in [33]: Let R/ϑ be the set of orbits in R under $\langle\vartheta\rangle$, and for each orbit $a\in R/\vartheta$, let β_a denote the restriction to $\check{V}^\vartheta$ of any $\alpha\in a$. The collection

$$R_\vartheta := \{\beta_a :\ a\in R/\vartheta\}$$

is a root system (possibly non-reduced) of linear functionals on $\check{V}^\vartheta$, with basis

$$\Delta_\vartheta := \{\beta_a :\ a\in\Delta/\vartheta\}.$$

It will be convenient to choose a numbering:

$$\Delta_\vartheta = \{\beta_1,\dots,\beta_{\ell_\vartheta}\},$$

where $\ell_\vartheta = |\Delta/\vartheta|$ is the number of $\langle\vartheta\rangle$-orbits in Δ.

There is a unique alcove $\mathcal{C}\subset\mathcal{A}(T,K)$ containing x_0 in its closure, on which $\tilde{\beta}_i > 0$ for all $i=1,\dots,\ell_\vartheta$. We define a certain negative root $\beta_0\in R_\vartheta$ as follows (see [31, 3.11]). If $\vartheta=1$ then β_0 is the lowest root of $R=R_\vartheta$. If $\vartheta\neq 1$ then β_0 is the lowest short root of R_ϑ except if (R,ϑ) has type ${}^2A_{2n}$, in which case β_0 is twice the lowest short root of R_ϑ. The integers $b_0(=1), b_1,\dots,b_{\ell_\vartheta}$ mentioned earlier in (11) are uniquely defined by the relation $\sum_{i=0}^{\ell_\vartheta} b_i\beta_i = 0$, and the affine roots ψ_i defining the walls of $\mathcal{C}$ are given by $\psi_0 = \frac{1}{e}+\tilde{\beta}_0$ and $\psi_i = \tilde{\beta}_i$ for $i=1,\dots,\ell_\vartheta$. Finally, we have

$$\Psi_K = \bigsqcup_{a\in R/\vartheta}\Psi_K(a),$$

where

$$\Psi_K(a) = \begin{cases} \left\{\widetilde{\beta}_a + \frac{n}{|a|} :\ n\in\mathbb{Z}\right\} & \text{if}\quad \beta_a\notin 2R_\vartheta,\\[2ex] \left\{\widetilde{\beta}_a + \frac{1}{2} + n :\ n\in\mathbb{Z}\right\} & \text{if}\quad \beta_a\in 2R_\vartheta.\end{cases}$$

Note the second case only occurs in type ${}^2A_{2n}$, where $a = \{\alpha + \vartheta\alpha\}$, and $\{\alpha, \vartheta\alpha\}$ generate a subsystem of type A_2.

Let $W_{\mathsf{aff}}(R, \vartheta)$ be the affine Weyl group of the affine root system Ψ_K. This is the group of affine transformations of $\mathcal{A}(T, K)$ generated by the reflections in the hyperplanes $\psi^{-1}(0)$ for $\psi \in \Psi_K$.

3.3 Kac coordinates and barycenters

A point $x \in \mathcal{A}(T, K)$ is **rational** if $\psi(x) \in \mathbb{Q}$ for all affine K-roots $\psi \in \Psi_K$. When this holds, we define the **order** of x to be the smallest positive integer m such that $\psi(x) \in \frac{1}{m}\mathbb{Z}$ for all $\psi \in \Psi_K$.

The order is invariant under the affine Weyl group $W_{\mathsf{aff}}(R, \vartheta)$. If x is a rational point of order m and $x' \in W_{\mathsf{aff}}(R, \vartheta) \cdot x$ is contained in the closure of the alcove $\mathcal{C}$ then there are relatively prime non-negative integers s_i, the **Kac coordinates** of x, such that $\psi_i(x') = s_i/m$ for $i \in [0, \ell_\vartheta]$. The Kac-coordinates and the order are related by:

$$e \cdot \sum_{i=0}^{\ell_\vartheta} b_i s_i = m.$$

In particular, the order m is divisible by e.

A point $x \in \mathcal{A}(T, K)$ is a **barycenter** if x is a rational point with all Kac-coordinates $s_i \in \{0, 1\}$. The order m of a barycenter x is given by

$$m = e \cdot \sum_{s_i=1} b_i.$$

In particular, the order of a barycenter is bounded by the twisted Coxeter number:

$$m \leq e \cdot (b_0 + b_1 + \cdots + b_{\ell_\vartheta}) = h_\vartheta. \tag{14}$$

Each facet $\mathcal{F}$ in the closure of $\mathcal{C}$ contains a unique barycenter, namely the point $x \in \mathcal{F}$ where all ψ_i not vanishing on $\mathcal{F}$ take the same value. This value is $1/m$, where m is the order of x.

Given a point $x \in \mathcal{A}(T, K)$ and a real number $r \geq 0$, let

$$R_{x,r} = \{\dot{\psi} : \ \psi \in \Psi_K \text{ and } \psi(x) = r\}.$$

The roots in $R_{x,r}$ are the nonzero weights of $\mathsf{V}_{x,r}$ under the torus $\mathsf{T} := T(K)_0/T(K)_{0+}$. One checks that for $w \in W_{\mathsf{aff}}(R, \vartheta)$ we have

$$R_{wx,r} = \dot{w}(R_{x,r}). \tag{15}$$

Lemma 3.1. *Suppose $x \in \mathcal{A}(T,K)$ is a rational point of order m. If x is not a barycenter then there exists a co-weight $\check{\lambda} \in \check{X}^\vartheta$ such that $\langle \beta, \check{\lambda}\rangle > 0$ for all $\beta \in R_{x,1/m}$.*

Proof. By (15) we may assume $x \in \overline{\mathcal{C}}$. If $R_{x,1/m}$ contains only positive or only negative roots with respect to Δ_ϑ, then one of $\pm 2\check{\rho}$ satisfies the conclusion of the lemma. Hence we may also assume that $R_{x,1/m}$ contains a positive root as well as a negative root.

Let $(s_0, \dots, s_{\ell_\vartheta})$ be the Kac coordinates of x, and define a function $H : R_\vartheta \to \mathbb{Z}$ by

$$H(\beta) := m \cdot \tilde{\beta}(x) = \sum_{i=1}^{\ell_\vartheta} c_i s_i,$$

where c_i is the coefficient of the simple root β_i in β.

Recall that $\beta = \beta_a$ for some $a \in R/\vartheta$ and that $|a|$ divides e which divides m. We define positive integers e', m' as follows.

If $\beta \notin 2R_\vartheta$, we set

$$e' = \frac{e}{|a|}, \qquad m' = \frac{m}{|a|},$$

so that $e' = 1$ if β is a short root in R_ϑ and $e' = e$ if β is long. If $\beta \in 2R_\vartheta$, we set

$$e' = \frac{e}{2} = 1, \qquad m' = \frac{m}{2}.$$

If $m' = 1$ then $e = m$ so x is a barycenter, by the linear relation (10), so we assume $m' > 1$.

The coefficients c_i of β satisfy

$$-e'b_i \le c_i \le e'b_i \tag{16}$$

for $i = 1, \dots, \ell_\vartheta$. And since x has order m it follows that

$$m' = e'(s_0 - H(\beta_0)),$$

so we have the inequalities

$$e's_0 - m' = e'H(\beta_0) \le H(\beta) \le -e'H(\beta_0) = m' - e's_0. \tag{17}$$

Let $\psi \in \Psi_K$ be an affine root with $\dot{\psi} = \beta$ and $\psi(x) = 1/m$. There is an integer n such that

$$\psi = \begin{cases} \tilde{\beta} + \frac{n}{|a|} & \text{if} \quad \beta = \beta_a \notin 2R \\ \tilde{\beta} + \frac{n}{2} & \text{if} \quad \beta \in 2R. \end{cases}$$

The equation $\psi(x) = 1/m$ is equivalent to the equation

$$H(\beta) = 1 - nm', \tag{18}$$

so that the inequalities (17) become

$$e's_0 - m' \le 1 - nm' \le m' - e's_0. \tag{19}$$

Assume that β is a negative root with respect to Δ_ϑ. Then $n \ge 1$, and the left-hand inequality of (19) yields

$$0 \le (n-1)m' \le 1 - e's_0 \le 1.$$

But $m' > 1$ so we must have $n = 1$. It also follows (since we have assumed R_ϑ contains negative roots) that $s_0 \in \{0, 1\}$. We now have

$$H(\beta) = 1 - m' = 1 - e'(s_0 - H(\beta_0)),$$

which is equivalent to

$$\sum_{i=1}^{\ell_\vartheta} (c_i + e'b_i)s_i = 1 - e's_0.$$

But all $c_i + e'b_i \ge 0$, by (16), so we must have $c_i + e'b_i = 0$ whenever $s_i \ge 2$. It follows that

$$\beta = e'\beta_0 + \sum_{i \in J} (c_i + e'b_i)\beta_i,$$

where $J = \{i \in [1, \ell_\vartheta] : \ s_i \le 1\}$.

Assume that β is a positive root. Since $H(\beta) \ge 0$ we must have $n = 0$ in equation (18), so $H(\beta) = 1$. This means that $c_i = 0$ whenever $s_i \ge 2$ and we have

$$\beta = \sum_{i \in J} c_i \beta_i.$$

We have shown that every root in $R_{x,1/m}$ is a linear combination, with *non-negative* coefficients, of roots in the set $S := \{\beta_j : \ j \in J \cup \{0\}\}$.

The assumption of the lemma is that $J \cup \{0\} \ne [0, \ell_\vartheta]$. Hence S is a basis of a root subsystem R' of R_ϑ. The sum $\check{\lambda}$ of the co-roots of R' which are positive with respect to S satisfies the conclusion of the lemma. □

Lemma 3.1 implies that if x is not a barycenter then $\mathsf{V}_{x,1/m}$ and its dual space $\check{\mathsf{V}}_{x,1/m}$ have no nonzero closed orbits under G_x. In view of (14), this has the following implication.

Proposition 3.2. *If $x \in \mathcal{A}(T,K)$ is a rational point of order m and $\check{\mathsf{V}}_{x,1/m}$ contains a nonzero closed orbit under G_x then x is a barycenter and $m \leq h_\vartheta$, with equality if and only if x is a barycenter of an alcove.*

4 Filtration groups and graded Lie algebras

In this section we show that the representations $(\mathsf{G}_x, \check{\mathsf{V}}_{x,r})$ arising from Moy-Prasad filtrations are the same representations arising in the Vinberg-Levy theory of graded Lie algebras. This will lead to the classification of the points x for which G_x has stable orbits in $\check{\mathsf{V}}_{x,1/m}$. We will show that such points x are those barycenters which arise from W-conjugacy classes of elliptic $\mathbb{Z}$-regular automorphisms of the absolute root system R. These results require additional assumptions on p.

4.1 Moy-Prasad filtrations

Fix a positive integer m which is divisible by e and not divisible by p. Let E/K be the unique extension of K (in $\bar{k}$) of degree $m = [E : K]$. Then E/K is totally and tamely ramified, E contains the splitting field E_0 of G over K, and E is Galois over k. Extend the valuation v to E, so that $\mathsf{v}(E^\times) = \frac{1}{m}\mathbb{Z}$. Let $A_E = \{a \in E :\ \mathsf{v}(a) \geq 0\}$ and $P_E = \{a \in E :\ \mathsf{v}(a) > 0\}$ denote the ring of integers and prime ideal of E, respectively. Since E/K is totally ramified we may and shall identify $\mathfrak{F}$ with the residue field of E, via the inclusion $A_K \hookrightarrow A_E$.

Fix a generator $\sigma \in \mathrm{Gal}(E/K)$ (of order m) such that $\sigma = \sigma_0$ on E_0. We may and shall choose a uniformizer $\pi \in E$ such that $\pi^m = \varpi$ is a uniformizer in k. We have $\sigma(\pi) = \zeta\pi$, where $\zeta \in A_K^\times$ is a root of unity of order m. Let F now denote the unique Frobenius element of $\mathrm{Gal}(E/k)$ such that $F(\pi) = \pi$. Let $\vartheta = \varrho(\sigma) \in \mathrm{Aut}(R, \Delta)$ be the image of σ under the Galois action $\varrho : \mathrm{Gal}(E/K) \to \mathrm{Aut}(R, \Delta)$ arising from the K-structure on $G \times_k K$.

We have $\mathcal{A}(M, E) = \mathcal{A}(M, E_0)$ and the set of affine functions

$$\Psi_E := \left\{\tilde{\alpha} + \frac{n}{m} :\ \alpha \in R,\ n \in \mathbb{Z}\right\}$$

is the system of **affine E-roots** of $G(E)$ (with respect to $M(E)$), and x_0 is a hyperspecial point in $\mathcal{A}(M, E)$.

Each affine E-root $\psi \in \Psi_E$ with gradient $\dot{\psi}$ determines an affine root group $U_\psi \subset G(E)$, which may be described explicitly in terms of the basepoint x_0 and the root groups $\{u_\alpha :\ \alpha \in R\}$ as

$$U_\psi = \{u_{\dot{\psi}}(b) :\ b \in E,\ \mathsf{v}(b) \geq \psi(x_0)\},$$

and for each $x \in \mathcal{A}(M,E)$ and $r \geq 0$ the Moy-Prasad filtration subgroup of $G(E)$ is given by

$$G(E)_{x,r} = \langle U_\psi, M(E)_r : \ \psi \in \Psi_E, \ \psi(x) \geq r \rangle,$$

where $M(E)_0 = \check{X} \otimes A_E^\times$ is the parahoric subgroup of $M(E)$ and

$$M(E)_r = \{t \in M(E)_0 : \ \mathsf{v}(\chi(t) - 1) \geq r \quad \forall \chi \in X\},$$

for $r > 0$. As before, we set

$$G(E)_{x,r+} = \bigcup_{s>r} G(E)_{x,s} \qquad \overline{G(E)}_{x,r} = G(E)_{x,r}/G(E)_{x,r+}.$$

The group $G(E)_{x,0}$ is the parahoric subgroup of $G(E)$ attached to the point x via Bruhat-Tits theory; it is contained with finite index in the full stabilizer $G(E)_x$ of x in $G(E)$. The quotient $\overline{G(E)}_{x,0}$ is a connected reductive group over the residue field $\mathfrak{F}$ of E, and for $r > 0$ the quotient $\overline{G(E)}_{x,r}$ is a finite-dimensional $\mathfrak{F}$-vector space, affording a rational representation of $\overline{G(E)}_{x,0}$, induced by conjugation.

If $x = x_0$ and $r = 0$, we have

$$\begin{aligned} G(E)_{x_0,0} &= \langle u_\alpha(A_E), \ \check{\lambda}(A_E^\times) : \ \alpha \in R, \ \check{\lambda} \in \check{X} \rangle, \\ G(E)_{x_0,0+} &= \langle u_\alpha(P_E), \ \check{\lambda}(1 + P_E) : \ \alpha \in R, \ \check{\lambda} \in \check{X} \rangle. \end{aligned}$$

The quotient

$$\mathsf{G} := \overline{G(E)}_{x_0,0} \tag{20}$$

is a connected reductive $\mathfrak{f}$-group with the same based root datum $(X, \Delta, \check{X}, \check{\Delta})$ as that of G and $\mathfrak{f}$-structure given by the Frobenius element ϕ_0 (see (12)).

The Galois action of $\mathrm{Gal}(E/K) = \langle \sigma \rangle$ on $G(E)$ imparts additional structure to G. For each $\alpha \in R$ let $\mathsf{u}_\alpha : \mathfrak{F}^+ \to \mathsf{G}$ be the root group fitting into the commutative diagram

$$\begin{array}{ccc} A_E & \xrightarrow{u_\alpha} & G(E)_{x_0,0} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{F} & \xrightarrow{\mathsf{u}_\alpha} & \mathsf{G}, \end{array}$$

where the vertical maps are the obvious projections. There is a unique algebraic automorphism of G, again denoted by ϑ, such that $\vartheta(\mathsf{u}_\alpha(t)) = \mathsf{u}_{\vartheta(\alpha)}(c_\alpha t)$ for each $\alpha \in R$ and $t \in \mathfrak{F}$, where $c_\alpha = \pm 1$ and $c_\alpha = +1$ if $\alpha \in \pm\Delta$. Since E/K is totally ramified, it follows that $\mathsf{u}_\alpha(\mathfrak{F})$ is in fact the projection to G of $u_\alpha(A_K)$, which implies that $\sigma(\mathsf{u}_\alpha(t)) = \vartheta(\mathsf{u}_\alpha(t))$ for all $t \in \mathfrak{F}^\times$. Likewise $\check{\lambda}(A_E^\times)$ and $\check{\lambda}(A_K^\times)$ have the same projection to G, for each $\check{\lambda} \in \check{X}$, so $\sigma(\check{\lambda}(t)) = \vartheta(\check{\lambda}(t))$ for $t \in \mathfrak{F}^\times$. Thus, the *Galois* action of σ on $G(E)_{x_0,0}$ induces the *algebraic* action of ϑ on G, and we have

$$(\mathsf{G}^\vartheta)^\circ = \overline{G(K)}_{x_0,0}.$$

Let $\mathfrak{L}$ be the Lie algebra of G. As before, we may identify

$$\mathfrak{L} = \overline{\mathfrak{g}(E)}_{x_0,0}$$

and the Galois action of σ on $\mathfrak{g}(E)_{x_0,0}$ induces the algebraic action of ϑ on $\mathfrak{L}$, so that

$$\mathfrak{L}^\vartheta = \overline{\mathfrak{g}(K)}_{x_0,0}.$$

If $x \in \mathcal{A}(T, K)$ then $\sigma(x) = x$ and it is clear that we have equality of full stabilizers:

$$(G(E)_x)^\sigma = G(K)_x, \tag{21}$$

and since E/K is tame we also have, for $r > 0$,

$$(G(E)_{x,r})^\sigma = G(K)_{x,r}. \tag{22}$$

When $r = 0$ equality (22) need not hold, because the group $\left(\overline{G(E)}_{x,0}\right)^\sigma$ need not be connected; its identity component is $\overline{G(K)}_{x,0}$.

4.2 Graded Lie algebras

We are ready to show that the representation pairs $(\mathsf{G}_x, \mathsf{V}_{x,r})$ are exactly those appearing in the Vinberg-Levy theory of graded Lie algebras.

Let $x \in \mathcal{A}(S, k)$ be a rational point of order m. Then x becomes hyperspecial in $\mathcal{A}(M, E)$. The Lie algebra $\mathfrak{g}(E)$ is filtered by Moy-Prasad lattices $\mathfrak{g}(E)_{x,r}$, with jumps in $\frac{1}{m}\mathbb{Z}$, and we have $\mathfrak{g}(E)_{x,i/m} = \pi^i \mathfrak{g}(E)_{x,0}$ for all $i > 0$, along with canonical isomorphisms on successive quotients

$$\overline{G(E)}_{x,i/m} = G(E)_{x,i/m}/G(E)_{x,(i+1)/m} \simeq \mathfrak{g}(E)_{x,i/m}/\mathfrak{g}(E)_{x,(i+1)/m} = \overline{\mathfrak{g}(E)}_{x,i/m}$$

which are equivariant for the $\mathrm{Gal}(E/k)$-action.

This action of $\mathrm{Gal}(E/k)$ depends on x. To see this explicitly, write

$$x = x_0 - \frac{1}{m}\check{\lambda},$$

where $\check{\lambda} \in \check{X}^{\vartheta}$. Consider the elements in $M(E)$ given by

$$t := \check{\lambda}(\pi), \qquad \text{and} \qquad s = \check{\lambda}(\zeta),$$

which satisfy $\sigma(t) = ts$. Identifying ζ with its image in $\mathfrak{F}^\times$, we may regard s as an element of T, so that $\mathrm{Ad}(s)$ is an inner automorphism of $\mathfrak{L}$. We set

$$\theta := \mathrm{Ad}(s)\vartheta \in \mathrm{Aut}(\mathfrak{L}). \tag{23}$$

Under the translation action of $M(E)$ on $\mathcal{A}(M,E)$ (see [37, 1.2]), we have $t \cdot x_0 = x$, so that the conjugation map $c_t : g \mapsto tgt^{-1}$ sends $G(E)_{x_0,0}$ to $G(E)_{x,0}$, and thereby induces an $\mathfrak{F}$-linear isomorphism

$$f : \mathsf{G} \xrightarrow{\sim} \overline{G(E)}_{x,0}.$$

Since $\sigma(t) = ts$, it follows that such that $c_{ts} \circ \sigma = \sigma \circ c_t$. But we have seen that σ induces ϑ on G. It follows that the isomorphism f has the intertwining property:

$$f \circ \theta = \sigma \circ f, \tag{24}$$

so

$$f(\mathsf{G}^\theta) = \left(\overline{G(E)}_{x,0}\right)^{\sigma}.$$

Letting f_0 denote the restriction of f to $(\mathsf{G}^\theta)^\circ$, we get an isomorphism on identity components:

$$f_0 : (\mathsf{G}^\theta)^\circ \xrightarrow{\sim} \mathsf{G}_x = \overline{G(K)}_{x,0}. \tag{25}$$

Recall we have taken $x \in \mathcal{A}(S,k)$, so that $F(x) = x$ and F preserves $G(K)_{x,0}$, inducing an $\mathfrak{f}$-structure on the algebraic group G_x. We need not have $F(x_0) = x_0$, so F does not act on G. However the isomorphism (25) does descend to $\mathfrak{f}$, provided we give G the $\mathfrak{f}$-structure with Frobenius

$$F_f := f^{-1} \circ F \circ f. \tag{26}$$

Turning to higher depths, for each $1 \leq i < m$, the eigenspace

$$\mathfrak{L}(\theta, \zeta^{-i}) := \{v \in \mathfrak{L} : \ \theta(v) = \zeta^{-i} v\}$$

affords a representation of G^θ. We define an $\mathfrak{F}$-linear isomorphism

$$f_i : \mathfrak{L}(\theta, \zeta^{-i}) \longrightarrow \mathsf{V}_{x,i/m},$$

as follows.

Let $v \in \mathfrak{L}(\theta, \zeta^{-i})$. Recalling that f is induced by c_t, we have from (24) that

$$\sigma(df(v)) = \zeta^{-i} \cdot df(v), \tag{27}$$

where $df : \mathfrak{L} \to \overline{\mathfrak{g}(E)}_{x,0}$ is the differential of f.

Let $\tilde{v} \in \mathfrak{g}(E)_{x_0,0}$ be a lift of v. Then $\operatorname{Ad}(t)\tilde{v} \in \mathfrak{g}(E)_{x,0}$ is a lift of $df(v)$, so equation (27) implies that

$$\sigma(\operatorname{Ad}(t)\tilde{v}) \equiv \zeta^{-i} \operatorname{Ad}(t)\tilde{v} \mod \mathfrak{g}(E)_{x,1/m}. \tag{28}$$

Since $\sigma(\pi^i) = \zeta^i \pi^i$ and $\pi^i \operatorname{Ad}(t)\tilde{v} \in \mathfrak{g}(E)_{x,i/m}$, it follows from (28) that

$$\sigma(\pi^i \operatorname{Ad}(t)\tilde{v}) \equiv \pi^i \operatorname{Ad}(t)\tilde{v} \mod \mathfrak{g}(E)_{x,(i+1)/m}.$$

Therefore, $\pi^i \operatorname{Ad}(t)\tilde{v}$ projects to a σ-fixed element

$$\overline{\pi^i \operatorname{Ad}(t)\tilde{v}} \in \left(\overline{\mathfrak{g}(E)}_{x,i/m}\right)^\sigma = \overline{\mathfrak{g}(K)}_{i/m} = \mathsf{V}_{x,i/m}.$$

One checks that the map

$$f_i : \mathfrak{L}(\theta, \zeta^{-i}) \longrightarrow \mathsf{V}_{x,i/m} \tag{29}$$

given by

$$f_i(v) = \overline{\pi^i \operatorname{Ad}(t)\tilde{v}}$$

is a well-defined isomorphism intertwining the adjoint action of an element $g \in (\mathsf{G}^\theta)^\circ$ on $\mathfrak{L}(\theta, \zeta^{-i})$ with the conjugation action of $f_0(g) \in \mathsf{G}_x$ on $\mathsf{V}_{x,i/m}$.

Assume there is a nondegenerate G-invariant symmetric pairing $\mathfrak{L} \times \mathfrak{L} \to \mathfrak{F}$. (This holds, for example if p is odd and is not a torsion prime for G.) Then $\mathfrak{L}(\theta, \zeta^i)$ and $\mathfrak{L}(\theta, \zeta^{-i})$ become contragredient representations of $(\mathsf{G}^\theta)^\circ$. Thus f_i induces an isomorphism

$$\check{f}_i : \mathfrak{L}(\theta, \zeta^i) \xrightarrow{\sim} \check{V}_{x,i/m}. \tag{30}$$

We summarize what we have shown in this section:

Theorem 4.1. *Assume there is a* G*-invariant nondegenerate symmetric form on* $\mathfrak{L}$ *and let* $x \in \mathcal{A}(T,K)$ *be a rational point of order* m *such that* $p \nmid m$*. Let* $\check{\lambda} \in \check{X}^{\vartheta}$ *be defined by* $x_0 - x = \frac{1}{m}\check{\lambda}$*, and let* $E = K(\pi)$ *be the unique tame extension* K *of degree* m*, where* π *is a uniformizer in* E *such that* π^m *is a uniformizer in* k*. Choose a generator* σ *of* $\mathrm{Gal}(E/K)$*, identify the* m^{th} *root of unity* $\zeta := \sigma(\pi)/\pi$ *with its image in* $\mathfrak{F}^{\times}$*, and define*

$$s = \check{\lambda}(\zeta) \in \mathsf{T}, \qquad \theta = \mathrm{Ad}(s)\vartheta \in \mathrm{Aut}(\mathfrak{L}),$$

where $\vartheta = \varrho(\sigma)$*. Then for* $1 \leq i < m$ *the isomorphisms*

$$f_0 : (\mathsf{G}^{\theta})^{\circ} \xrightarrow{\sim} \overline{G(K)}_{x,0}, \qquad\qquad \check{f}_i : \mathfrak{L}(\theta,\zeta^i) \xrightarrow{\sim} \check{\mathsf{V}}_{x,i/m},$$

defined in (25) *and* (30)*, fit into a commutative diagram*

$$\begin{array}{ccc} (\mathsf{G}^{\theta})^{\circ} & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{L}(\theta,\zeta^i)) \\ {\scriptstyle f_0}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{Ad}(\check{f}_i)}\downarrow{\scriptstyle \simeq} \\ \mathsf{G}_x & \longrightarrow & \mathrm{GL}(\check{\mathsf{V}}_{x,i/m}), \end{array}$$

where the lower horizontal map is the dual of the representation of G_x *on* $\mathsf{V}_{x,i/m}$ *induced by the conjugation action of* $G(K)_{x,0}$ *on* $G(K)_{x,i/m}$ *and* $\mathrm{Ad}(\check{f}_i)$ *is conjugation by* $\check{f}_i$*.*

Remark 1: Assuming only that $p \nmid e$, a point $x \in \mathcal{A}(S,k)$ of order m still gives a $\mathbb{Z}/m$-grading $\mathfrak{L} = \oplus \mathfrak{L}_i$ (cf. [34], [33, 1.3]) such that the representations $\mathsf{V}_{x,i/m}$ and $\mathfrak{L}_i$ have the same decomposition under T. It is likely that a version of Thm. 4.1 can be proved in this more general setting.

Remark 2: Recall that the Frobenius F on $G(E)_{x,0}$ transfers to the twisted Frobenius F_f on $G(E)_{x_0,0}$, defined in (26). On the latter group we have the relation

$$F_f \circ \theta = \theta^q \circ F_f, \tag{31}$$

which follows from (24). This implies that F_f preserves each eigenspace $\mathfrak{L}(\theta,\zeta^i)$, and one checks that the vertical maps in the diagram of Thm. 4.1 interwine the action of F_f on the top row with F on the bottom row.

5 Stable orbits and epipelagic representations

In this section we assume that p is odd and is not a torsion prime for G, and we keep the notation of section 4.2. Let $x \in \mathcal{A}(T,K)$ be a rational point of order m, where $p \nmid m$. By Thm. 4.1, the group G_x has stable orbits in $\check{\mathsf{V}}_{x,1/m}$ if and only if the group $\mathsf{G}_0 := (\mathsf{G}^\theta)^\circ$ has stable orbits in the eigenspace $\mathsf{V} := \mathfrak{L}(\theta,\zeta)$.

The latter eigenspace representations were analyzed by Vinberg [38] (over $\mathbb{C}$) and his results were extended by Levy [25] to algebraically-closed fields of good odd positive characteristic. The main features of this theory are as follows. Define a **Cartan subspace** to be a subspace of V which is abelian under the Lie bracket in $\mathfrak{L}$, consists entirely of semisimple elements of $\mathfrak{L}$, and is maximal with these two properties. Then: 1) Any two Cartan subspaces are conjugate under G_0, every closed G_0-orbit in V meets a fixed Cartan subspace $\mathfrak{c}$, and if two points in $\mathfrak{c}$ are G_0-conjugate then they are conjugate under the normalizer $N_{\mathsf{G}_0}(\mathfrak{c})$. 2) The action of the faithful quotient $W(\mathfrak{c},\theta) = N_{\mathsf{G}_0}(\mathfrak{c})/C_{\mathsf{G}_0}(\mathfrak{c})$ on $\mathfrak{c}$ is generated by reflections. 3) The restriction map gives an isomorphism on rings of polynomial invariants:

$$\mathfrak{F}[\mathsf{V}]^{\mathsf{G}_0} \xrightarrow{\sim} \mathfrak{F}[\mathfrak{c}]^{W(\mathfrak{c},\theta)},$$

and these rings are generated by homogeneous polynomials $f_1,\dots,f_r$, where $r = \dim\mathfrak{c}$ is the **rank** of V, and we have $\prod_{i=1}^r \deg(f_i) = |W(\mathfrak{c},\theta)|$.

If the rank of V is zero (the unstable case) then every orbit of G_0 in V is nilpotent and $\{0\}$ is the only closed orbit. So G_0 has semistable orbits in V exactly when V has positive rank. The pairs $(\mathsf{G}_0,\mathsf{V})$ of positive rank were classified in [38], [25], and [33]; we shall discuss this further in section 8.

The subset of stable representations $(\mathsf{G}_0,\mathsf{V})$ (i.e., those having closed orbits with finite isotropy groups) was also classified in [33]. These are in bijection with certain W-conjugacy classes of automorphisms of R, as we now explain.

We say that an automorphism $\sigma \in \mathrm{Aut}(R)$ is **$\mathbb{Z}$-regular** if the group generated by σ acts freely on R. [2] We say also that σ is **elliptic** if σ fixes no nonzero vector in V. The $\mathbb{Z}$-regular elliptic elements are tabulated for exceptional groups in section 5.1 (see also [33, section 7]).

By [33, Cor. 14], the eigenspace representation $\mathsf{V} = \mathfrak{L}(\theta,\zeta)$ contains a stable orbit under G_0 precisely when θ normalizes a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{L}$ such that the restriction $\theta|_{\mathfrak{s}}$ gives an elliptic $\mathbb{Z}$-regular W-conjugacy class in $\mathrm{Aut}(R)$. In this case, θ is G-conjugate to $\mathrm{Ad}(\check{\rho}(\zeta))\vartheta$, where ϑ is the projection of $\theta|_{\mathfrak{s}}$ to $\mathrm{Aut}(R,\Delta)$. Moreover, this is independent of the choice of ζ, in the sense that if ζ' is another root of unity of order m, then the automorphisms $\mathrm{Ad}(\check{\rho}(\zeta))\vartheta$ and $\mathrm{Ad}(\check{\rho}(\zeta'))\vartheta$ are G-conjugate.

Using Thm. 4.1, with x and θ as related therein, we can transfer this classification to $\check{\mathsf{V}}_{x,1/m}$ and obtain the following.

[2]The $\mathbb{Z}$-regular elements of order m are those which are regular in the sense of Springer [35] and have a regular eigenvalue of order m.

Corollary 5.1. *Let $x \in \mathcal{A}(T,K)$ be a rational point of order m. Then $\check{\mathsf{V}}_{x,1/m}$ contains stable vectors under G_x if and only if m is the order of an elliptic $\mathbb{Z}$-regular element of $w\vartheta \in W\vartheta$ and x is conjugate under $W_{\mathsf{aff}}(R,\vartheta)$ to the point $x_0 + \frac{1}{m}\check{\rho}$.*

In Cor. 5.1, the W-conjugacy class of $w\vartheta$ is determined by m. All stabilizers A_0 in G_0 of stable vectors in $\mathfrak{L}(\theta,\zeta)$ are G_0-conjugate and fit into an exact sequence

$$1 \longrightarrow \mathsf{A}_0 \longrightarrow \mathsf{S}^\theta \longrightarrow \mathsf{Z}_\vartheta \longrightarrow 1. \tag{32}$$

Here $\mathsf{Z}_\vartheta = \mathsf{Z}/(1-\vartheta)\mathsf{Z}$, where Z is the fundamental group of G, and S^θ is the group of θ-invariants in the centralizer S of $\mathfrak{c}$ in G (see [33, section 6.3], where A_0 is denoted by S_0). The group S^θ is finite, of order

$$|\mathsf{S}^\theta| = \det(1 - w\vartheta|_X),$$

and the exponent of S^θ divides m (see [32, 2.1]). Our assumption that $p \nmid m$ therefore implies that A_0 has order prime to p.

The group $A_{x,\lambda}$ of section 2.5 is isomorphic, via the map f_0 of Thm. 4.1, to the group of $\mathfrak{f}$-rational points in A_0, and the group $H_{x,\lambda}$ is the pre-image of $A_{x,\lambda}$ in $G(k)_x$. Hence the finite group $A_{x,\lambda}$ is abelian of order prime to p, and from Remark 1 of section 2.1, it follows that χ_λ extends to $H_{x,\lambda}$ and the intertwining algebra $\mathcal{H}_{x,\lambda}$ of Prop. 2.4 is just the group algebra of $A_{x,\lambda}$. We have shown the following.

Proposition 5.2. *Let m be the order of an elliptic $\mathbb{Z}$-regular automorphism $w\vartheta \in W\vartheta$ and assume p is a non-torsion prime for G, not dividing $2m$. Let $x = x_0 + \frac{1}{m}\check{\rho} \in \mathcal{A}(S,k)$. Then $\check{\mathsf{V}}_{x,1/m}$ contains stable orbits under G_x. If $\lambda \in \check{\mathsf{V}}_{x,1/m}(\mathfrak{f})$ is an $\mathfrak{f}$-rational stable vector then the character χ_λ of $J_x = G(k)_{x,1/m}$ (see 2.5) extends to $H_{x,\lambda}$ and we have the direct sum decomposition*

$$\mathrm{ind}_{J_x}^{G(k)} \chi_\lambda = \bigoplus_{\rho \in \mathrm{Irr}(A_{x,\lambda})} \pi_x(\lambda,\rho),$$

where ρ runs over the irreducible characters of the abelian group $A_{x,\lambda}$ and

$$\pi_x(\lambda,\rho) = \mathrm{ind}_{H_{x,\lambda}}^{G(k)}(\chi_\lambda \otimes \rho)$$

is an irreducible supercuspidal epipelagic representation of $G(k)$.

5.1 Tables for exceptional groups

In this section we tabulate the points $x \in \mathcal{A}(T,K)$ for which G_x has stable orbits in $\check{\mathsf{V}}_{x,1/m}$. For brevity we list just the exceptional groups, by their "Name" in the left-most column of [37, p.61], and we take them to be simply-connected. The latter condition affects only the group A_0 (see (32)). For classical groups, see [33, 7.2].

E_6

m	x	w	A_0	$\dim \mathfrak{c}$
$12 = h_\vartheta$	1 1 1 1 1 1 1	E_6	$\boldsymbol{\mu}_3$	1
9	1 1 0 1 1 1 1	$E_6(a_1)$	$\boldsymbol{\mu}_3$	1
6	1 0 1 0 1 0 1	$E_6(a_2)$	$\boldsymbol{\mu}_3$	2
3	0 0 1 0 0 0 0	$3A_2$	$\boldsymbol{\mu}_3^3$	3

F_4^I

m	x	w	A_0	$\dim \mathfrak{c}$
$18 = h_\vartheta$	$111 \Leftarrow 11$	$-E_6(a_1)$	1	1
12	$110 \Leftarrow 11$	$-E_6$	1	1
6	$100 \Leftarrow 10$	$-(3A_2)$	1	3
4	$000 \Leftarrow 10$	$-D_4(a_1)$	$\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_4$	2
2	$000 \Leftarrow 01$	-1	$\boldsymbol{\mu}_2^6$	6

E_7

m	x	w	A_0	$\dim \mathfrak{c}$
$18 = h_\vartheta$	1 1 1 1 1 1 1 1	E_7	$\boldsymbol{\mu}_2$	1
14	1 1 1 0 1 1 1 1	$E_7(a_1)$	$\boldsymbol{\mu}_2$	1
6	1 0 0 1 0 0 1 0	$E_7(a_4)$	$\boldsymbol{\mu}_2$	3
2	0 0 0 0 0 0 0 1	$7A_1$	$\boldsymbol{\mu}_2^7$	7

E_8

m	x	w	A_0	$\dim \mathfrak{c}$
$30 = h_\vartheta$	1 1 1 1 1 1 1 1 1	E_8	1	1
24	1 1 0 1 1 1 1 1 1	$E_8(a_1)$	1	1
20	1 1 0 1 0 1 1 1 1	$E_8(a_2)$	1	1
15	1 0 1 0 1 0 1 1 0	$E_8(a_5)$	1	1
12	1 0 1 0 0 1 0 1 0	$E_8(a_3)$	1	2
10	0 0 1 0 0 1 0 1 0	$E_8(a_6) = -2A_4$	1	2
8	0 0 1 0 0 0 1 0 0	$D_8(a_3)$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	2
6	0 0 0 1 0 0 0 1 0	$E_8(a_8) = -4A_2$	1	4
5	0 0 0 1 0 0 0 0 0	$2A_4$	$\boldsymbol{\mu}_5 \times \boldsymbol{\mu}_5$	2
4	0 0 0 0 1 0 0 0 0	$2D_4(a_1)$	$\boldsymbol{\mu}_2^4$	4
3	0 0 0 0 0 0 0 0 1	$4A_2$	$\boldsymbol{\mu}_3^4$	4
2	1 0 0 0 0 0 0 0 0	$8A_1 = -1$	$\boldsymbol{\mu}_2^8$	8

F_4

m	x	w	A_0	$\dim \mathfrak{c}$
$12 = h_\vartheta$	$111 \Rightarrow 11$	F_4	1	1
8	$111 \Rightarrow 01$	B_4	$\boldsymbol{\mu}_2$	1
6	$101 \Rightarrow 01$	$F_4(a_1)$	1	2
4	$101 \Rightarrow 00$	$D_4(a_1)$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	2
3	$001 \Rightarrow 00$	$A_2 + \tilde{A}_2$	$\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$	2
2	$010 \Rightarrow 00$	$4A_1$	$\boldsymbol{\mu}_2^4$	4

G_2

m	x	w	A_0	$\dim \mathfrak{c}$
$6 = h_\vartheta$	$1\,1 \Rrightarrow 1$	G_2	1	1
3	$1\,1 \Rrightarrow 0$	A_2	$\boldsymbol{\mu}_3$	1
2	$0\,1 \Rrightarrow 0$	$A_1 + \tilde{A}_1$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	2

G_2^I

m	x	w	A_0	$\dim \mathfrak{c}$
$12 = h_\vartheta$	$1\,1 \Lleftarrow 1$	F_4	1	1
6	$1\,0 \Lleftarrow 1$	$F_4(a_1)$	1	2
3	$0\,0 \Lleftarrow 1$	$A_2 + \tilde{A}_2$	$\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$	2

6 Cartan subspaces

We have so far considered stable orbits over the algebraic closure $\mathfrak{F}$ of the residue field $\mathfrak{f}$ of k. We now examine how these orbits break up when we descend to $\mathfrak{f}$. In section 7.3 we will use this analysis to attach Langlands parameters to epipelagic representations.

Since each stable orbit determines a Cartan subspace, we must first analyze the $\mathfrak{f}$-rational conjugacy classes of Cartan subspaces. As this is purely a question about groups over $\mathfrak{f}$, we restrict ourselves to that context.

6.1 Rational classes of Cartan subspaces

Let G be a connected reductive group over a finite field $\mathfrak{f} \simeq \mathbb{F}_q$, with Lie algebra $\mathfrak{L}$, and assume the characteristic p of $\mathfrak{f}$ is an odd good prime for G.

Let F be the Frobenius map on G and $\mathfrak{L}$, arising from the given $\mathfrak{f}$-structure. (Here F plays the role of the twisted Frobenius F_f in Remark 2 of section 4.2.)

Let $\mathfrak{t} \subset \mathfrak{b}$ be a Cartan subalgebra contained in a Borel subalgebra of $\mathfrak{L}$, such that $F(\mathfrak{t}) = \mathfrak{t}$ and $F(\mathfrak{b}) = \mathfrak{b}$. The action of F on $\mathfrak{t}$ induces an automorphism $\phi \in \mathrm{Aut}(R, \Delta)$, where $(X, \Delta, \check{X}, \check{\Delta})$ is the based root datum of G. We may extend ϕ to a pinned algebraic automorphism of G (denoting the extension again by ϕ) and write

$$F = \phi F_0,$$

where F_0 is the Frobenius map for the split $\mathfrak{f}$-structure on G. Let N and T be the normalizer and centralizer of $\mathfrak{t}$ in G, with Weyl group $W = N/T$.

Let θ be an automorphism of $\mathfrak{L}$ whose order m is nonzero in $\mathfrak{f}$ such that

$$F \circ \theta = \theta^q \circ F \tag{33}$$

as maps $\mathfrak{L} \to \mathfrak{L}$ (cf. (31)). Then $\theta = \check{\lambda}(\zeta)\vartheta$, where $\zeta \in \mathfrak{F}^\times$ is a root of unity of order m and ϑ is a pinned automorphism of $\mathfrak{L}$ arising from an element of $\mathrm{Aut}(R, \Delta)$, which we also denote by ϑ.

Set $\mathsf{G}_0 = (\mathsf{G}^\theta)^\circ$ and let V be the ζ-eigenspace of θ in $\mathfrak{L}$. Condition (31) implies that $F(\mathsf{G}_0) = \mathsf{G}_0$, $F(\mathsf{V}) = \mathsf{V}$, and the representation of G_0 on V is defined over $\mathfrak{f}$. By the Lang-Steinberg theorem for G_0 there exists a Cartan subspace $\mathfrak{c}$ in V for which $F(\mathfrak{c}) = \mathfrak{c}$. The groups

$$N_{\mathfrak{c}} = N_{\mathsf{G}_0}(\mathfrak{c}), \qquad S_{\mathfrak{c}} = C_{\mathsf{G}_0}(\mathfrak{c}), \qquad W(\mathfrak{c}, \theta) = N_{\mathfrak{c}}/S_{\mathfrak{c}}$$

are also preserved by F.

The relation (33) also implies that

$$\phi\vartheta = \vartheta^q\phi \in \mathrm{Aut}(R, \Delta). \tag{34}$$

Assume that G_0 has stable orbits in V and let $\Sigma_m \subset W\vartheta$ be the unique W-conjugacy class of $\mathbb{Z}$-regular elliptic elements of order m (see section 5). We consider the set of pairs

$$M(\vartheta, \phi) = \{(\sigma, \tau) \in \Sigma_m \times W\phi : \ \tau\sigma\tau^{-1} = \sigma^q\},$$

on which W acts by conjugation in both factors. Given $(\sigma, \tau) \in M(\vartheta, \phi)$, we have a twisted Frobenius map

$$F_\tau := \tau \circ F_0$$

on $\mathfrak{t}$, preserving the eigenspace

$$\mathfrak{t}(\sigma,\zeta) := \{v \in \mathfrak{t} : \ \sigma(v) = \zeta v\}.$$

We define a map

$$\{\mathfrak{c} : \ F(\mathfrak{c}) = \mathfrak{c}\}/\mathsf{G}_0^F \longrightarrow M(\vartheta,\phi)/W, \tag{35}$$

from the set of G_0^F-orbits of Cartan subspaces in V preserved by F to the set of W-conjugacy classes in $M(\vartheta,\phi)$, as follows.

Since G_0 has stable orbits in V, it follows [33, Prop. 1] that $\mathfrak{c}$ contains regular elements in $\mathfrak{L}$. The centralizer $\mathfrak{s} = \mathfrak{z}(\mathfrak{c})$ is a Cartan subalgebra of $\mathfrak{L}$ so there exists $g \in \mathsf{G}$ such that $\mathfrak{s} = \mathrm{Ad}(g)\mathfrak{t}$. Now $\mathfrak{s}$ and $\mathfrak{t}$ are both preserved by F, so we have $g^{-1}F(g) \in N$, projecting to an element $u \in W$ and we set $\tau = u\phi \in W\phi$. Then $\mathrm{Ad}(g) : \mathfrak{t} \to \mathfrak{s}$ intertwines the action of F on $\mathfrak{s}$ with the action of F_τ on $\mathfrak{t}$.

The automorphism

$$\theta' := \mathrm{Ad}(g)^{-1} \circ \theta \circ \mathrm{Ad}(g)$$

preserves $\mathfrak{t}$ and by restriction gives an automorphism $\sigma := \theta'|_{\mathfrak{t}} \in \Sigma_m$. The relation (31) implies that $(\sigma,\tau) \in M(\vartheta,\phi)$. A different choice of g such that $\mathfrak{s} = \mathrm{Ad}(g)\mathfrak{t}$ leads to a W-conjugate of (σ,τ). And replacing $\mathfrak{c}$ by a G_0^F-conjugate of $\mathfrak{c}$ leads to the same pair (σ,τ). Thus, sending $\mathfrak{c} \mapsto (\sigma,\tau)$ gives the well-defined map (35). It has the following property.

Lemma 6.1. *If, under the map (35), the G_0^F-orbit of $\mathfrak{c}$ corresponds to the W-conjugacy class of (σ,τ), then there exist isomorphisms*

$$\gamma : W(\mathfrak{c},\theta) \xrightarrow{\sim} C_W(\sigma), \qquad \delta : \mathfrak{c} \xrightarrow{\sim} \mathfrak{t}(\sigma,\zeta),$$

where $C_W(\sigma)$ is the centralizer of σ in W, such that for all $w \in W(\mathfrak{c},\theta)$ the map δ intertwines the action of wF on $\mathfrak{c}$ with the action of $\gamma(w)F_\tau$ on $\mathfrak{t}(\sigma,\tau)$.

Proof. The map $\mathrm{Ad}(g)$ in the definition of (35) intertwines the pairs $(\mathfrak{s},F)$ and $(\mathfrak{t},F_\tau)$, along with the actions of θ on $\mathfrak{s}$ and σ on $\mathfrak{t}$, and induces an isomorphism $C_{W(\mathfrak{s})}(\theta) \xrightarrow{\sim} C_W(\sigma)$. And from [33, Cor. 21] we have $C_{W(\mathfrak{s})}(\theta) = W(\mathfrak{c},\theta)$. Thus, $\mathrm{Ad}(g)$ induces maps γ and δ as in the assertion of the lemma. □

7 Langlands parameters for epipelagic representations

We return to the setting of p-adic groups, and assume that our connected k-group G is absolutely simple and also simply-connected. Recall that the k-structure on the quasi-split k-form of G is determined by a homomorphism

$$\varrho : \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{Aut}(R, \Delta)$$

whose image Θ is generated by elements $\vartheta, \phi \in \mathrm{Aut}(R, \Delta)$ satisfying $\phi\vartheta = \vartheta^q\phi$, where ϑ generates the image of the inertia subgroup of $\mathrm{Gal}(\bar{k}/k)$ and has order prime to p, and ϕ is the image of a Frobenius element ($a \mapsto a^q$) in $\mathrm{Gal}(\bar{k}/k)$.

Let $\hat{G}$ be a connected adjoint Lie group whose root datum is dual to that of G, and let $\hat{\mathfrak{g}}$ be the Lie algebra of $\hat{G}$. Then $\hat{G} = \mathrm{Aut}(\hat{\mathfrak{g}})^\circ$ is the identity component of the automorphism group of the complex simple Lie algebra $\hat{\mathfrak{g}}$. Fix a pinning $(\hat{T}, \hat{B}, \{\hat{e}_\alpha\})$ in $\hat{G}$, consisting of a maximal torus $\hat{T}$ contained in a Borel subgroup $\hat{B}$, along with simple root vectors $\hat{e}_\alpha$ for each $\check{\alpha} \in \check{\Delta}$. We identify $W = \hat{N}/\hat{T}$, where $\hat{N}$ is the normalizer of $\hat{T}$ in $\hat{G}$.

The group $\mathrm{Aut}(R, \Delta)$ acts on $\hat{G}$ via pinned automorphisms and we have $\hat{G} \rtimes \mathrm{Aut}(R, \Delta) = \mathrm{Aut}(\hat{\mathfrak{g}})$. We set

$$^LG := \hat{G} \rtimes \Theta \subset \mathrm{Aut}(\hat{\mathfrak{g}}),$$

and let $\mathsf{pr} : {}^LG \to \Theta$ be the projection along $\hat{G}$.

We will construct discrete Langlands parameters attached to our epipelagic representations $\pi_x(\lambda, \rho)$. These will be continuous homomorphisms

$$\varphi : \mathcal{W}_k \longrightarrow {}^LG$$

from the Weil group of k, for which $\mathsf{pr} \circ \varphi = \varrho$, and which have the expected properties of the adjoint Swan conductor $b(\varphi)$ and centralizer $A_\varphi = C_{\hat{G}}(\varphi)$.

We first show how these properties impose strong conditions on φ. Then we give a template for the construction of parameters φ satisfying these conditions, and we use the template in two situations: First, when p is a non-torsion prime for G not dividing $2m$, we will attach a discrete parameter to each stable $\mathsf{G}_0(\mathfrak{f})$-orbit of stable vectors in $\check{\mathsf{V}}_{x,1/m}(\mathfrak{f})$. Here the image of φ will lie in the normalizer ${}^LN$ of $\hat{T}$ in ${}^LG$. Then we will consider some cases where p divides $2m$ or is a torsion prime for G; here Thm. 4.1 does not apply but stable orbits persist. In these cases the image of φ will lie in the normalizer in ${}^LG$ of a Jordan subgroup of $\hat{G}$.

7.1 Predictions for the Langlands parameter

According to the conjectural local Langlands correspondence, each epipelagic representation $\pi_x(\lambda, \rho)$ should correspond to a Langlands parameter (see [14, section 7])

$$\varphi : \mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG$$

whose image has finite centralizer $A_\varphi = C_{\hat{G}}(\varphi)$, along with an irreducible projective representation ξ of A_φ, such that certain numerical invariants of the representation $\pi_x(\lambda, \rho)$ and the pair (φ, ξ) agree. In particular, the adjoint gamma value of φ should be related to the formal degree of $\pi_x(\lambda, \rho)$, as proposed in [17]. As reformulated in [14], this expected property is the following.

Let μ_G be the absolute value of the Euler-Poincaré measure on $G(k)$ [14, 7.1]. We write $\mathrm{Deg}(\pi) = \deg_{\mu_G}(\pi)$ for the formal degree with respect to μ_G of a discrete series representation π. Then the pair (φ, ξ) attached to $\pi_x(\lambda, \rho)$ should satisfy

$$\mathrm{Deg}(\pi_x(\lambda, \rho)) \overset{?}{=} \frac{\dim \xi}{|A_\varphi|} \cdot \frac{\omega(\varphi_0)}{\gamma(\varphi_0)} \cdot \frac{\gamma(\varphi)}{\omega(\varphi)}, \tag{36}$$

where $\gamma(\varphi) = \gamma(\varphi, \hat{\mathfrak{g}}, 0)$ is the gamma value at $s = 0$ of the adjoint representation

$$\mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\varphi} {}^L G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\hat{\mathfrak{g}}),$$

$\omega(\varphi)$ is the adjoint root number and $\varphi_0 : \mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is the principal parameter (which corresponds to the Steinberg representation of $G(k)$). The right-hand side of (36) is of the form $\Gamma_\varphi(q)$ where $\Gamma_\varphi(x) \in \mathbb{Q}(x)$ is a rational function [14, Prop. 4.1].

We now analyze the conditions that equality in (36) would impose on $\gamma(\varphi)/\omega(\varphi)$. These will lead to strong conditions on φ itself.

Let $y \in \mathcal{A}(S, k)$ be a point such that $G(k)_{y,0}$ is an Iwahori subgroup of $G(k)$ contained in $G(k)_{x,0}$. Then the compact open subgroup $J_y = G(k)_{y,0+}$ contains J_x. From [14, (55)], the volume of J_y with respect to the measure μ_G can be expressed as

$$\mathrm{vol}(J_y) = q^{-(\dim G + \ell_\vartheta)/2} \cdot \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}.$$

where we recall that ℓ_ϑ is the rank of G over K. It follows that

$$\mathrm{vol}(J_x) = \mathrm{vol}(J_y) \cdot [J_y : J_x]^{-1} = q^{-(\dim G + \dim \mathsf{G}_x)/2} \cdot \frac{\gamma(\varphi_0)}{\omega(\varphi_0)}.$$

From Prop. 2.4, the formal degree of $\pi_x(\lambda, \rho)$ with respect to μ_G is (actually) given by

$$\mathrm{Deg}(\pi_x(\lambda, \rho)) = \frac{\dim \chi_{\lambda,\rho}}{|A_{x,\lambda}|} \cdot q^{(\dim G + \dim \mathsf{G}_x)/2} \cdot \frac{\omega(\varphi_0)}{\gamma(\varphi_0)}. \tag{37}$$

Equality of the right sides of (37) and (36) forces the L-function of φ to be trivial, so that φ is totally ramified (that is, $\hat{\mathfrak{g}}^{\varphi(\mathcal{I})} = 0$, where $\mathcal{I} \subset \mathcal{W}_k$ is the inertia subgroup) and then φ is trivial on $\mathrm{SL}_2(\mathbb{C})$ and we have [14, 3.2]

$$\frac{\gamma(\varphi)}{\omega(\varphi)} = q^{\alpha(\varphi)/2},$$

where $\alpha(\varphi)$ is the Artin conductor of the representation $\mathrm{Ad}\circ\varphi$ on $\hat{\mathfrak{g}}$. Therefore equality in (36) forces $\alpha(\varphi) = \dim G + \dim \mathsf{G}_x$. But, again since φ is totally ramified, we have $\alpha(\varphi) = \dim \hat{\mathfrak{g}} + b(\varphi) = \dim G + b(\varphi)$, where $b(\varphi)$ is the Swan conductor of $\mathrm{Ad}\circ\varphi$. Thus, we find that equality in (36) forces

$$b(\varphi) = \dim \mathsf{G}_x \tag{38}$$

and

$$\frac{\dim \xi}{|A_\varphi|} = \frac{\dim \chi_{\lambda,\rho}}{|A_{x,\lambda}|}. \tag{39}$$

Conversely, it is clear that (38) and (39) imply that equality holds in (36).

If p does not divide $|W|$, we can sharpen the predicted equality (38) and show that it is an extremal condition on φ, as follows. Let $\varphi(\mathcal{I}) = D_0 \geq D_1 \geq D_2 \geq \cdots$ be the lower ramification filtration of the inertial image of φ. Since $p \nmid |W|$, it follows from [4] that D_1 is contained in a maximal torus of $\hat{G}$, which we may assume is $\hat{T}$, and that D_0 is contained in the normalizer ${}^LN$ of $\hat{T}$ in ${}^LG$. We therefore have $D_0 = \langle \hat{n}_0 \rangle \ltimes D_1$, where $\hat{n}_0 \in {}^LN$ is an element whose order, say m_0, is prime to p. Let σ_0 be the projection of $\hat{n}_0$ to $W\vartheta$. Since $\hat{\mathfrak{g}}^{D_0} = 0$, it follows that $\hat{\mathfrak{g}}^{D_1} = \hat{\mathfrak{t}}$ and that $\hat{\mathfrak{t}}^{\hat{\sigma}_0} = 0$ [14, Prop. 5.4]. Then from the definition of the Swan conductor we have

$$b(\varphi) \geq \frac{\dim(\hat{\mathfrak{g}}/\hat{\mathfrak{t}})}{m_0} = \frac{|R|}{m_0},$$

with equality if and only if $D_2 = 1$.

On the other hand, via the Kac classification (see section 3.3), the automorphism $\mathrm{Ad}(\hat{n}_0) \in \mathrm{Aut}(\hat{\mathfrak{g}})$ corresponds to a point $y \in \mathcal{A}(T, K)$ such that $(\hat{G}^{\mathrm{Ad}(\hat{n}_0)})^\circ$ and G_y are reductive groups (over $\mathbb{C}$ and $\mathfrak{f}$, respectively) with the same root data. Since $\hat{\mathfrak{t}}^{\sigma_0} = 0$, it follows that

$$\dim \mathsf{G}_y = \dim \hat{\mathfrak{g}}^{\mathrm{Ad}(\hat{n}_0)} \geq |R/\sigma_0| \geq \frac{|R|}{m_0},$$

with equality if and only if $\langle \sigma_0 \rangle$ acts freely on R.

If $D_2 = 1$ and σ_0 is a $\mathbb{Z}$-regular automorphism of R, the lower bounds in these inequalities will be attained and then (38) will be satisfied exactly when $y \in W_{\mathsf{aff}}(R, \vartheta) \cdot x$. This last condition means that $m_0 = m$ is the order of x and the tame inertial generator $\hat{n}_0$ of the discrete parameter φ induces a $\mathbb{Z}$-regular elliptic automorphism σ_0 of R which is W-conjugate to the automorphism σ corresponding to x as in section 5.

7.2 A template for epipelagic parameters

Recall that e is the ramification degree of the minimal tame extension splitting G. Let m, f be positive integers such that e divides m, and m divides $q^f - 1$.

We make some slight changes in notation. Let K and K_m now denote the unramified extensions of k of degrees f and and mf, respectively, and let $\mathfrak{F}, \mathfrak{F}_m$ be the residue fields of K, K_m. Let M/K be the abelian extension with norm group $N(M/K) = K^{\times m}$. We choose a root $\pi = \sqrt[m]{\varpi}$, where ϖ is a prime element in A_k. Then $M = K_m(\pi)$, and M/k is a tamely ramified Galois extension of degree $[M : k] = fm^2$. The Galois group $\Gamma = \mathrm{Gal}(M/k)$ has the presentation

$$\Gamma = \langle s, t : \ s^m = t^{mf} = 1, \ tst^{-1} = s^q \rangle, \tag{40}$$

where

$$s(\pi) = \zeta\pi, \qquad t(\pi) = \pi,$$

and s (resp. t) are the identity (resp. Frobenius) automorphisms on K_m. We identify $\mathfrak{F}_m$ with the residue field of M and regard $\mathfrak{F}_m^\times$ as a subgroup of $M^\times$. Let $U_M^n = 1 + \pi^n A_M$ be the higher unit groups of M. The choice of π gives an isomorphism (denoted by the same letter)

$$U_M^1/U_M^2 \xrightarrow{\pi} \mathfrak{F}_m^+, \qquad 1 + \pi a \mapsto a \mod \pi A_M. \tag{41}$$

With the natural action of Γ on U_M^1/U_M^2 and letting Γ act on $\mathfrak{F}_m$ via $s \cdot a = \zeta a$ and $t \cdot a = a^q$ for $a \in \mathfrak{F}_m$, the map π becomes an isomorphism of $\mathbb{F}_p[\Gamma]$-modules.

Let L/M be the abelian extension with norm group

$$N(L/M) = \langle \pi \rangle \times \mathfrak{F}_m^\times \times U_M^2,$$

so that $\mathrm{Gal}(L/M) \simeq U_M^1/U_M^2$. Then L/k is Galois, since $N(L/M)$ is preserved by Γ. And since $\mathrm{Gal}(M/K_m) = \langle s \rangle$ has trivial invariants in U_M^1/U_M^2, it follows [14, Lemma 6.1] that there is a splitting

$$\mathrm{Gal}(L/k) \simeq (U_M^1/U_M^2) \rtimes \Gamma \simeq \mathfrak{F}_m^+ \rtimes \Gamma,$$

where the first isomorphism is canonical and the second depends on the choice of π, as in (41).

Suppose $J \subset \hat{G}$ is a finite elementary abelian p-subgroup of $\hat{G}$, whose normalizer ${}^L N(J)$ in ${}^L G$ contains two elements $\hat{n}_s \in \hat{G}\vartheta$ of order m and $\hat{n}_t \in \hat{G}\phi$ satisfying $\hat{n}_t^{mf} = 1$, with the relation $\hat{n}_t \hat{n}_s \hat{n}_t^{-1} = \hat{n}_s^q$.

Then we have a homomorphism

$$\eta : \Gamma \to {}^L N(J), \quad s \mapsto \hat{n}_s, \quad t \mapsto \hat{n}_t, \tag{42}$$

and any Γ-equivariant homomorphism $\psi : \mathfrak{F}_m^+ \to J$ extends to a homomorphism

$$\psi \rtimes \eta : \mathrm{Gal}(L/k) \simeq \mathfrak{F}_m^+ \rtimes \Gamma \longrightarrow {}^L N(J), \quad (a, \gamma) \mapsto \psi(a) \cdot \eta(\gamma), \tag{43}$$

for $a \in \mathfrak{F}_m$ and $\gamma \in \Gamma$. Replacing J by $\operatorname{im} \psi$ if necessary, we may assume ψ is surjective. Then the Galois group $D = \operatorname{im}(\psi \rtimes \eta)$ has lower ramification filtration

$$D \ = \langle J, \hat{n}_s, \hat{n}_t \rangle \qquad D_0 = \langle J, \hat{n}_s \rangle \qquad D_1 = J, \qquad D_2 = 1,$$

and the Swan conductor of the adjoint representation $\mathrm{Gal}(L/k) \xrightarrow{\psi \rtimes \eta} {}^L N(J) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\hat{\mathfrak{g}})$ is given by

$$b(\psi \rtimes \eta) = \frac{\dim(\hat{\mathfrak{g}}/\hat{\mathfrak{g}}^J)}{m},$$

where $\hat{\mathfrak{g}}^J$ is the fixed-point subalgebra of J in $\hat{\mathfrak{g}}$. To have $\hat{\mathfrak{g}}^{D_0} = 0$ (total ramification), $\hat{\mathfrak{g}}^J$ must be the Lie algebra of a torus (possibly trivial) in $\hat{G}$.

If p is not a torsion prime for $\hat{G}$ then J is contained in a maximal torus of $\hat{G}$, by [36, 2.28], so

$$b(\psi \rtimes \eta) = \frac{\dim(\hat{\mathfrak{g}}/\hat{\mathfrak{t}})}{m} = \frac{|R|}{m},$$

in this case.

7.3 Parameters arising from Cartan subspaces

Let $x \in \mathcal{A}(S, k)$ be a rational point of order m such that $\check{\mathsf{V}}_x = \check{\mathsf{V}}_{x,1/m}$ contains stable functionals for the action of G_x. In this section we will attach a discrete Langlands parameter to each $\mathsf{G}_x(\mathfrak{f})$-orbit of stable functionals in $\check{\mathsf{V}}_x(\mathfrak{f})$, assuming that the characteristic p of the residue field $\mathfrak{f}$ is not a torsion prime for $\hat{G}$, and that $p \nmid 2m$.

Let $\mathfrak{c} \subset \check{\mathsf{V}}_x$ be a Cartan subspace for which $F(\mathfrak{c}) = \mathfrak{c}$. Then $\mathfrak{c}$ is defined over $\mathfrak{f}$ and we have $\mathfrak{c}(\mathfrak{f}) = \mathfrak{c}^F$. The point x and choice of root of unity $\zeta \in \mathfrak{F}^\times$ determine an automorphism θ (see (23)) of the Lie algebra $\mathfrak{L} = \mathrm{Lie}(\mathsf{G})$ and from Thm. 4.1, we have an isomorphism

$$f_1 : \check{\mathsf{V}}_x \xrightarrow{\sim} \mathfrak{L}(\theta, \zeta)$$

sending $\mathfrak{c}$ to the eigenspace $\mathfrak{s}(\theta, \zeta)$, where $\mathfrak{s}$ is the unique Cartan subalgebra of $\mathfrak{L}$ centralizing $f_1(\mathfrak{c})$. The G-invariant pairing on $\mathfrak{L}$ restricts to a pairing

$$\mathfrak{s}(\theta, \zeta^{-1}) \otimes \mathfrak{c} \simeq \mathfrak{s}(\theta, \zeta^{-1}) \otimes \mathfrak{s}(\theta, \zeta) \longrightarrow \mathfrak{F}.$$

We identify the Cartan subalgebra $\mathfrak{t}$ of section 6.1 with $\mathfrak{F} \otimes \check{X}$; its dual space is $\hat{\mathfrak{t}} := \mathfrak{F} \otimes X$. Recall that the map (35) determines a W-congugacy class of pairs $(\sigma, \tau) \in M(\vartheta, \phi)$, arising from an element $g \in \mathsf{G}$ such that $\mathrm{Ad}(g)\mathfrak{t} = \mathfrak{s}$, and $\mathrm{Ad}(g)$ interwines σ and F_τ on $\mathfrak{t}$ with θ and F on $\mathfrak{s}$. In particular, $\mathrm{Ad}(g)$ gives an isomorphism $\mathfrak{t}(\sigma, \zeta^{-1}) \to \mathfrak{s}(\theta, \zeta^{-1})$. By duality, we get an isomorphism

$$\mathfrak{c} \xrightarrow{\sim} \hat{\mathfrak{t}}(\sigma, \zeta), \tag{44}$$

which intertwines F on $\mathfrak{c}$ and F_τ on $\hat{\mathfrak{t}}(\sigma, \zeta)$. Thus, we get an isomorphism on rational points

$$\mathfrak{c}(\mathfrak{f}) = \mathfrak{c}^F \xrightarrow{\sim} \hat{\mathfrak{t}}(\sigma, \zeta)^{F_\tau}. \tag{45}$$

Recall that σ is a $\mathbb{Z}$-regular elliptic element in $\mathrm{Aut}(R, \Delta)$. Since $\hat{G}$ is adjoint, all lifts of σ in ${}^LN$ are $\hat{T}$-conjugate and have order m [33, Prop. 8]. We fix one such lift $\hat{n}_\sigma \in {}^LN$. For any lift $\hat{n}'_\tau \in {}^LN$ of τ, we will have $\hat{n}'_\tau \cdot \hat{n}_\sigma \cdot (\hat{n}'_\tau)^{-1} \cdot y = \hat{n}_\sigma^q$, for some $y \in \hat{T}$. By the ellipticity of σ, there is $z \in \hat{T}$ such that $z^\sigma \cdot z^{-1} = y^\tau$. We choose the new lift $\hat{n}_\tau := \hat{n}'_\tau \cdot z$, so that $\hat{n}_\tau \cdot \hat{n}_\sigma \cdot \hat{n}_\tau^{-1} = \hat{n}_\sigma^q$.

Let f be the order of τ in W. Then $\sigma = \tau^f \sigma \tau^{-f} = \sigma^{q^f}$, so $m \mid q^f - 1$ and $\hat{\tau}^f \in \hat{T}^\sigma$. The latter is a finite abelian group of exponent dividing m [32, 2.1], so $\hat{\tau}^{fm} = 1$. As in section 7.2 we have a homomorphism $\eta : \Gamma \longrightarrow {}^LN$, sending $s \mapsto \hat{n}_\sigma$ and $t \mapsto \hat{n}_\tau$. This gives an action of Γ on $\hat{T}$, whereby s acts via σ and t acts via τ.

Let $\hat{T}[p] := \{t \in \hat{T} : \ t^p = 1\}$ be the p-torsion subgroup of $\hat{T}$. Since $X = \mathrm{Hom}(\mathbb{C}^\times, \hat{T})$, we have an isomorphism

$$\mathbb{F}_p \otimes X \xrightarrow{\sim} \hat{T}[p], \qquad a \otimes \lambda \mapsto \lambda(z^a), \tag{46}$$

where $z = \exp(2\pi\sqrt{-1}/p)$.

Let $h(x) \in \mathbb{F}_p[x]$ be the minimal polynomial of ζ over $\mathbb{F}_p$. The group J of section 7.2 will be the subgroup

$$\hat{T}[p, \zeta] := \{y \in \hat{T}[p] : \ h(\sigma)y = 1\},$$

which we will henceforth identify with the kernel of $h(\sigma)$ in $\mathbb{F}_p \otimes X$, via (46). This subspace of $\hat{T}[p]$ is preserved by σ and τ. Thus, $\hat{T}[p, \zeta]$ becomes an $\mathbb{F}_p[\Gamma]$ module, on which t^f acts trivially.

Lemma 7.1. *Let* $\mathrm{tr} : \mathfrak{F}_m \to \mathbb{F}_p$ *be the trace map. There is a canonical isomorphism*

$$\hat{\mathfrak{t}}(\sigma, \zeta)^{F_\tau} \simeq \mathrm{Hom}_\Gamma\left(\mathfrak{F}_m, \hat{T}[p, \zeta]\right), \qquad \lambda \mapsto \psi'_\lambda,$$

such that $\langle \psi'_\lambda(b), \check{\omega} \rangle = \mathrm{tr}\,(b\langle \lambda, \check{\omega} \rangle)$, *for all* $b \in \mathfrak{F}_m$ *and* $\check{\omega} \in \check{X}$.

Proof. Non-degeneracy of the form $(a, b) = \mathrm{tr}(ab)$ implies that the map $\lambda \mapsto \psi'_\lambda$ gives an isomorphism

$$\mathfrak{F}_m \otimes X \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p}(\mathfrak{F}_m, \hat{T}[p]).$$

Any F_τ-fixed element of $\hat{\mathfrak{t}}$ lies in $\mathfrak{F}_m \otimes X$. One checks that for all $b \in \mathfrak{F}_m$ we have $F_\tau(\lambda) = \lambda$ if and only if $\psi'_\lambda(b^q) = \tau(\psi'_\lambda(b))$, and that $\sigma(\lambda) = \zeta\lambda$ if and only if $\psi'_\lambda(\zeta b) = \sigma(\psi'_\lambda(b))$. The lemma follows. □

Combining Lemma 7.1 with (45), we obtain an isomorphism

$$\mathfrak{c}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{Hom}_\Gamma\left(\mathfrak{F}_m, \hat{T}[p,\zeta]\right), \tag{47}$$

which we denote by $\lambda \mapsto \psi_\lambda$, for $\lambda \in \mathfrak{c}(\mathfrak{f})$.

Lemma 7.2. *If $\lambda \in \mathfrak{c}(\mathfrak{f})$ is stable then $\hat{T}$ is the full centralizer in $\hat{G}$ of the image $\psi_\lambda(\mathfrak{F}_m)$.*

Proof. A stable vector $\lambda \in \mathfrak{c}$ is a regular semisimple element in $\mathfrak{L} = \mathrm{Lie}(\mathsf{G})$ [33, Lemma 13]. As above, let $\mathfrak{s}$ be the Cartan subalgebra centralizing λ and let $g \in \mathsf{G}$ be such that $\mathrm{Ad}(g)\mathfrak{t} = \mathfrak{s}$. Then $\mathrm{Ad}(g^{-1})\lambda \in \mathfrak{t}$ has trivial stabilizer in $W(\mathfrak{t})$, the Weyl group of $\mathfrak{t}$ in G. The isomorphism $\mathfrak{t} \simeq \hat{\mathfrak{t}}$ induced by the G-invariant pairing on $\mathfrak{L}$ is W-equivariant, so the isomorphism (44) sends λ to an element $\lambda' \in \hat{\mathfrak{t}}(\sigma,\zeta)$ with trivial stabilizer in $W(\mathfrak{t})$. This means that $\langle\lambda', \check{\alpha}\rangle \neq 0$ for all co-roots $\check{\alpha} \in \check{R}$. Hence $\langle\psi_\lambda(b), \check{\alpha}\rangle = \mathrm{tr}(b\langle\lambda', \check{\alpha}\rangle) \neq 0$ for some $b \in \mathfrak{F}_m$. Since $\check{R}$ is the set of roots of $\hat{T}$ in $\hat{G}$ and we have assumed p is not a torsion prime for $\hat{G}$, it follows from [36, 2.25] that $\hat{T}$ is the full centralizer of $\psi_\lambda(\mathfrak{F})$, as claimed. □

Now take $\lambda \in \mathfrak{c}(\mathfrak{f})$ to be a stable vector, with corresponding map $\psi_\lambda \in \mathrm{Hom}_\Gamma\left(\mathfrak{F}_m, \hat{T}[p,\zeta]\right)$, as in (47). As in (43) we define

$$\varphi_\lambda := \psi_\lambda \rtimes \eta : \mathrm{Gal}(L/k) \longrightarrow {}^LN \subset {}^LG,$$

and we set

$$A_{\varphi_\lambda} = C_{\hat{G}}(\varphi_\lambda).$$

From Lemma 7.2, we have $C_{\hat{G}}(\mathrm{im}\,\psi_\lambda) = \hat{T}$, so

$$A_{\varphi_\lambda} = \hat{T}^{\sigma,\tau} = \hat{T}^\Gamma.$$

Since σ is elliptic, the group $\hat{T}^\sigma$ is finite, so φ_λ is a discrete Langlands parameter whose image $D = \mathrm{im}\,\varphi_\lambda \subset {}^LN$ has lower filtration groups

$$D_0 = \langle\hat{n}_\sigma\rangle \ltimes \mathrm{im}\,\psi_\lambda, \quad D_1 = \mathrm{im}\,\psi_\lambda, \quad D_2 = 1.$$

The Swan conductor of the adjoint representation

$$\mathrm{Ad}\circ\varphi_\lambda : \mathrm{Gal}(L/k) \longrightarrow {}^LG = \mathrm{Aut}(\hat{\mathfrak{g}})$$

is therefore given by

$$b(\varphi_\lambda) = \frac{\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{t}}}{[D_0 : D_1]} = \frac{|R|}{m} = \dim \hat{\mathfrak{g}}^{\hat{n}_\sigma} = \dim \mathsf{G}_0.$$

Since G_0 is isomorphic to the reductive quotient G_x of the parahoric subgroup $G(K)_{x,0}$, the expected equality (38) holds for φ_λ.

Finally, since G is simply-connected, so is the group G. It follows that $\mathsf{G}_0 = \mathsf{G}^\theta$, so the stabilizer $\{g \in \mathsf{G}_0 : \ g\cdot\lambda = \lambda\} = \mathsf{S}^\theta$, where $\mathsf{S} = C_{\mathsf{G}}(\mathfrak{c})$ is the maximal torus of G with Lie algebra $\mathfrak{s}$. The map $\mathrm{Ad}(g) : \mathfrak{t} \to \mathfrak{s}$ (see (6.1)) gives an isomorphism $\mathsf{T}^\sigma \to \mathsf{S}^\theta$ intertwining F on S^θ with F_τ on T^σ.

It follows that the group $A_{x,\lambda}$ (see (6)) is given by

$$A_{x,\lambda} = \left(\mathsf{S}^\theta\right)^F \simeq \left(\mathsf{T}^\sigma\right)^{F_\tau}.$$

This group has the same cardinality as the invariants of F_τ^* in the coinvariants X_σ, where $X = X^*(\mathsf{T})$, and F_τ^*, the adjoint of F_τ, acts on X via $q\tau^{-1}$. It follows that

$$|A_{x,\lambda}| = |X_\sigma^{\tau=q}|. \tag{48}$$

On the other hand, we have isomorphisms

$$\hat{T}^\sigma \overset{\exp}{\longleftarrow} (1-\sigma)^{-1}X/X \overset{1-\sigma}{\longrightarrow} X_\sigma.$$

The relation $\tau\sigma\tau^{-1} = \sigma^q$ implies

$$\tau^{-1}\sigma_q \circ (1-\sigma) = (1-\sigma)\circ\tau^{-1},$$

where $\sigma_q := 1 + \sigma + \cdots + \sigma^{q-1}$ acts via q on X_σ. Thus, $1-\sigma$ intertwines $q\tau^{-1}$ on X_σ with τ^{-1} on $\hat{T}^\sigma$, and we have

$$|X_\sigma^{\tau=q}| = |\hat{T}^{\sigma,\tau}| = |A_{\varphi_\lambda}|. \tag{49}$$

Combining (48) and (49), we obtain the equality

$$|A_{x,\lambda}| = |A_{\varphi_\lambda}|$$

between the cardinalities of the isotropy group in G_0 of the stable functional $\lambda \in \mathfrak{c}(\mathfrak{f})$ and the centralizer in $\hat{G}$ of the corresponding discrete parameter φ_λ. As both $A_{x,\lambda}$ and A_{φ_λ} are abelian, we have shown that the expected property (39) holds for the parameter φ_λ.

7.4 Simple supercuspidal parameters for $\mathrm{SU}_p(\mathbb{Q}_p)$

Let $p = 2n+1$ be an odd prime. In this section we construct parameters for the simple supercuspidal representations (section 2.6) of the special unitary group $G = \mathrm{SU}_p(E/\mathbb{Q}_p)$ splitting over a ramified quadratic extension $E/\mathbb{Q}_p$.

The point $x \in \mathcal{A}(S,k)$ is the barycenter of an alcove, with all Kac coordinates $s_i = 1$, and has order $h_\vartheta = 2p$. The group G_x is a maximal torus in the orthogonal group $\mathrm{SO}_p(\overline{\mathbb{F}}_p)$ and $\check{\mathsf{V}}_{x,1/2p}$ is a direct sum of lines whose characters are the gradients of the simple affine roots $-2x_1, x_1-x_2, \ldots, x_{n-1}-x_n, x_n$. The stable points are those whose component in each line is nonzero and they have trivial isotropy groups in G_x. It follows that there are $p-1$ simple supercuspidal representations of $G(k)$.

Let $M/\mathbb{Q}_p$ be the tame extension constructed in section 7.2, for $m = 2(p+1)$ and $f = 2$. Here m is no longer the order of the point x. The quadratic character $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \{\pm 1\}$ corresponding to E factors through the Galois group $\Gamma = \mathrm{Gal}(M/\mathbb{Q}_p) = \langle s, t\rangle$. Explicitly, we have $\chi(s) = -1$ (since $E/\mathbb{Q}_p$ is ramified) and

$$\chi(t) = \begin{cases} +1 & \text{if} \quad E = \mathbb{Q}_p(\sqrt{p}) \\ -1 & \text{if} \quad E = \mathbb{Q}_p(\sqrt{up}), \end{cases}$$

where $u \in \mathbb{Z}_p^\times$ is a non-square modulo p.

In the dual group $\hat{G} = \mathrm{PGL}_p(\mathbb{C})$ there is a Jordan subgroup $J \subset {}^L G$ of type (p,p), generated by (the images of) matrices A, B given in terms of the standard basis $\{e_j\}$ by

$$Ae_j = z^j e_j, \qquad Be_j = e_{j+1},$$

where $z = \exp(2\pi\sqrt{-1}/p)$ and subscripts are read modulo p. The finite group J is its own centralizer in $\hat{G}$, and in particular is not contained in a maximal torus of $\hat{G}$ (see [3, 6.4] or [23, chap. 3]).

The L-group of G is

$${}^L G = \mathrm{PGL}_p(\mathbb{C}) \rtimes \langle \vartheta \rangle = \mathrm{Aut}(\mathfrak{sl}_p(\mathbb{C})),$$

where ϑ is an outer pinned involution. The action on J by its normalizer ${}^L N(J)$ in ${}^L G$ induces an isomorphism

$${}^L N(J)/J \simeq \mathrm{SL}^{\pm}(J) = \{g \in \mathrm{GL}(J) : \det(g) = \pm 1\},$$

sending the normalizer $N_{\hat{G}}(J)$ onto $\mathrm{SL}(J) \simeq \mathrm{SL}_2(\mathbb{F}_p)$.

Fix an element $\zeta \in \mathbb{F}_{p^2}^\times$ of order $m = 2(p+1)$. We regard $\mathbb{F}_{p^2}$ as a vector space over $\mathbb{F}_p$ and ζ as an element of $\mathrm{GL}(\mathbb{F}_{p^2})$ via multiplication. Let $j : \mathbb{F}_{p^2} \to J$ be an isomorphism of $\mathbb{F}_p$-vector spaces. Up to conjugacy by $\mathrm{SL}_2(p)$ there are $p-1$ choices for j, corresponding to the value of the determinant of the change of basis matrix. Each choice of j induces an isomorphism

$$j_* : \mathrm{GL}(\mathbb{F}_{p^2}) \longrightarrow \mathrm{GL}(J)$$

giving rise to two elements of $\mathrm{GL}(J)$:

$$\sigma_j = j_*(\zeta), \qquad \phi_j = j_*(\mathrm{Frob}_p)$$

where Frob_p is the p^{th}-power map on $\mathbb{F}_{p^2}$. One checks that $\det(\sigma_j) = \det(\phi_j) = -1$. [3]

We next define

$$\tau_j = \begin{cases} \sigma_j\phi_j & \text{if} \quad E = \mathbb{Q}_p(\sqrt{p}) \\ \phi_j & \text{if} \quad E = \mathbb{Q}_p(\sqrt{up}). \end{cases}$$

Then $\tau_j\sigma_j\tau_j^{-1} = \sigma_j^p$, and $\tau_j^4 = 1$, so we have a homomorphism

$$\eta_j : \Gamma \longrightarrow {}^L N(J) \qquad s \mapsto \sigma_j, \quad t \mapsto \tau_j,$$

such that $\chi(t) = \det(\tau_j)$, as required to get a parameter for $\mathrm{SU}_p(E/\mathbb{Q}_p)$. In this way, J becomes a Γ-module.

As $\zeta^{p+1} = -1$, it follows from Hilbert's Theorem 90 that $\zeta = v^{1-p}$ for some $v \in \mathbb{F}_{p^4}^\times$. The residue field $\mathfrak{F}_m$ of M has cardinality $p^{2m} = p^{4(p+1)}$ and therefore contains v. Let $\mathrm{tr} : \mathfrak{F}_m \to \mathbb{F}_{p^2}$ be the trace map and define

$$\psi : U_M^1/U_M^2 \longrightarrow \mathbb{F}_p^2 \quad \text{by} \quad \psi(1+a\pi) = \begin{cases} \mathrm{tr}(v\bar{a}) & \text{if} \quad E = \mathbb{Q}_p(\sqrt{p}) \\ \mathrm{tr}(\bar{a}) & \text{if} \quad E = \mathbb{Q}_p(\sqrt{up}). \end{cases}$$

One checks that the map $\psi_j := j \circ \psi$ is a Γ-equivariant surjection

$$\psi_j : U_M^1/U_M^2 \longrightarrow J,$$

so as in section 7.2 we have a parameter

$$\varphi_j = \psi_j \rtimes \eta_j : (U_M^1/U_M^2) \rtimes \Gamma = \mathrm{Gal}(M/\mathbb{Q}_p) \longrightarrow {}^L G.$$

Since $J = C_{\hat{G}}(J)$ it follows that φ_j has trivial centralizer in $\hat{G}$ and adjoint Swan conductor

$$b(\varphi_j, \hat{\mathfrak{g}}) = \frac{\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{g}}^J}{2(p+1)} = \frac{p-1}{2} = \ell_\vartheta,$$

in accordance with (38) and (39).

[3] The element σ_j belongs to the class in ${}^L G$ of "Macdonald-Kac" elements, with Kac coordinates $2 \Rrightarrow 1\ 1\ \cdots\ 1\ 1\ \Rrightarrow 1$.

7.5 Examples in G_2

Let G be a group over k of type G_2, with simple roots α_1, α_2, where α_2 is short. Let $x = \frac{1}{3}\check{\omega}_1$, where $\check{\omega}_i$ are the fundamental co-weights dual to α_i. The Kac coordinates of x are given in the diagram

$$1\,1 \Rrightarrow 0,$$

and $\mathsf{G}_x = \mathrm{GL}_2(U)$, where U is a two-dimensional vector space over $\mathfrak{F}$.

We will assume that $p \neq 3$. (The case $p = 3$ is more complicated, but can be handled by a modification of our present methods.) Since $p \neq 3$, the G_x-representation $\check{\mathsf{V}}_x = \check{\mathsf{V}}_{x,1/3}$ is given by:

$$\check{\mathsf{V}}_x \simeq \left[\det{}^2 \otimes P_3(U)\right] \oplus \left[\det{}^{-1} \otimes P_0(U)\right], \tag{50}$$

where $P_d(U)$ is the space of homogeneous polynomials of degree d on U, with the natural action of $\mathsf{G}_x = \mathrm{GL}(U)$. Choosing coordinates, we may regard a vector in $\check{\mathsf{V}}_x$ as a pair (f, z), where $f = f(x, y)$ is a binary cubic polynomial over $\mathfrak{F}$ and $z \in \mathfrak{F}$. Using the Hilbert-Mumford criterion [29, p.41], one checks that (f, z) is stable if and only if $z \neq 0$ and f has three distinct roots in the projective line.

If $p > 3$ then x arises from the square of a Coxeter element $c \in W(G_2)$ (see section 5.1), and a Cartan subspace $\mathfrak{c}$ is a line in $\check{\mathsf{V}}_x$, on which the little Weyl group $W(\mathfrak{c}, \theta) \simeq \langle c \rangle \simeq \boldsymbol{\mu}_6$ acts as multiplication by sixth roots of unity (here θ is a lift of c^2 in $\mathsf{G} = G_2(\mathfrak{F})$). The results of Vinberg-Levy theory (recalled in section 5) and Thm. 4.1 show that each nonzero closed G_x-orbit in $\check{\mathsf{V}}_x$ meets $\mathfrak{c}$ in exactly six points which are permuted freely and transitively by $W(\mathfrak{c}, \theta)$, and that the algebra of G_x-invariant polynomials on $\check{\mathsf{V}}_x$ is generated by a degree-six polynomial on $\check{\mathsf{V}}_x$.

Indeed, the polynomial given by

$$\Delta(f, z) = \mathrm{disc}(f) \cdot z^2,$$

where $\mathrm{disc}(f)$is the discriminant of f, is invariant under G_x and has degree six; for if we write $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ then $\mathrm{disc}(f)$ becomes the quartic polynomial

$$\mathrm{disc}(f) = b^2c^2 + 18abcd - 4ac^3 - 4db^3 - 27a^2d^2.$$

Since $\mathrm{disc}(f) \neq 0$ precisely when f has distinct zeros in the projective line, our discussion above shows that the stable points in $\check{\mathsf{V}}_x$ are precisely those where $\Delta \neq 0$. Moreover, the level set $\Delta^{-1}(h)$ is a single G_x-orbit, for each nonzero value $h \in \mathfrak{F}$. Indeed, if $\Delta(v) = h = \Delta(v')$, then v, v' are stable, hence are G_x-conjugate to some points $v_0, v_0' \in \mathfrak{c}$. Since $\dim \mathfrak{c} = 1$ we have $v_0' = tv_0$ for some $t \in \mathfrak{F}^\times$, so that $0 \neq \Delta(v_0) = \Delta(v_0') = t^6 \cdot \Delta(v_0)$ and we have $t \in \boldsymbol{\mu}_6$. Thus there is $w \in W(\mathfrak{c}, \theta)$ such that $w \cdot v_0' = v_0$, so v, v' are G_x-conjugate, as claimed. B. Gross has shown us how this classification of stable orbits can also be proved using the classical theory of binary cubic forms.

The isotropy group in G_x of a stable vector (f, z) is $\mathrm{Stab}_{\mathsf{G}_x}(f) \cap \mathrm{SL}(U)$. And $\mathrm{Stab}_{\mathsf{G}_x}(f)$ is the symmetric group S_3 (permuting the three roots of f), on which the determinant equals the sign character of S_3. It follows that the isotropy groups in G_x of stable points are isomorphic $\boldsymbol{\mu}_3$ over $\mathfrak{F}$. Hence the isotropy group $A_{x,\lambda}$ in $\mathsf{G}_x(\mathfrak{f})$ of a stable rational functional $\lambda = (f, z) \in \check{\mathsf{V}}(\mathfrak{f})$ is a subgroup of $\boldsymbol{\mu}_3$.

The parameter φ_λ, from the construction of section 7.3, has image in ${}^LG = \hat{G} = G_2(\mathbb{C})$ of the form

$$D_1 \rtimes \langle \hat{\sigma}, \hat{\tau} \rangle \subset N_{\hat{G}}(\hat{T}),$$

with wild inertia group $D_1 \subset \hat{T}[p]$ of order p^r (where $r \in \{1, 2\}$ is the order of p in $\mathbb{Z}/3^\times$), inertia group $D_0 = D_1 \rtimes \langle \hat{\sigma} \rangle$, and Frobenius image $\hat{\tau}$.

Now σ is a Coxeter element in the Weyl group of the subgroup $\hat{H} \simeq \mathrm{SL}_3(\mathbb{C})$ containing $\hat{T}$ and we have

$$\mathrm{im}\, \varphi_\lambda \subset N_{\hat{G}}(\hat{T}) \subset N_{\hat{G}}(\hat{H}),$$

where $N_{\hat{G}}(\hat{H}) = \hat{H} \cdot 2$ is the normalizer of $\hat{H}$, and

$$A_{\varphi_\lambda} = \hat{Z}^\tau,$$

the fixed points of τ acting on the center $\hat{Z}$ of $\hat{H}$ by conjugation. One checks that

$$\Delta(\lambda) \in \mathfrak{f}^{\times 2} \quad \Leftrightarrow \quad A_{x,\lambda} = \boldsymbol{\mu}_3 \quad \Leftrightarrow \quad \hat{Z}^\tau = \hat{Z} \quad \Leftrightarrow \quad \mathrm{im}\, \varphi_\lambda \subset \hat{H}. \tag{51}$$

It follows that for any character $\rho : A_{\varphi_\lambda} \to \mathbb{C}^\times$ the pair (φ_λ, ρ) may be regarded as a simple wild parameter [14, section 6] for the group $H(k)$, where

$$H = \begin{cases} \mathrm{PGL}_3 & \text{if } \hat{Z}^\tau = \hat{Z} \text{ and } \rho = \mathrm{triv} \\ \mathrm{PD}_3 & \text{if } \hat{Z}^\tau = \hat{Z} \text{ and } \rho \neq \mathrm{triv} \\ \mathrm{PU}_3 & \text{if } \hat{Z}^\tau = 1 \text{ and } \rho = \mathrm{triv} \text{ (necessarily).} \end{cases}$$

Thus, (φ_λ, ρ) also corresponds to a simple supercuspidal representation $\pi_H(\varphi_\lambda, \rho)$ of $H(k)$.

On $G_2(k)$, the induced representation

$$\pi_x(\lambda) = \bigoplus_{\rho \in \mathrm{Irr}(\hat{Z}^\tau)} \pi_x(\lambda, \rho)$$

is a sum of three or one irreducible epipelagic supercuspidal representations, according to whether the equivalent conditions (51) hold or not. The conjectures of [15] predict that $\pi_x(\lambda, \rho)$ is a lift of $\pi_H(\varphi_\lambda, \rho)$ under the exceptional theta correspondence for $G_2 \times H$ arising from the minimal representation of an unramified group of type E_6. [4]

Now assume $p = 2$, a torsion prime for G_2. The description (50) of $\check{\mathsf{V}}_x$ in terms of binary cubics is still valid, but now the discriminant becomes a square:

$$\mathrm{disc}(ax^3 + bx^2y + cxy^2 + dy^3) = (ad + bc)^2,$$

so we now have a cubic polynomial invariant

$$\delta(f, z) = (ad + bc)z,$$

with $\delta^2 = \Delta$. The stable points in $\check{\mathsf{V}}_x$ are those with nonzero value of δ. But the sign character is trivial when $p = 2$, so the isotropy groups of stable points are now isomorphic to S_3 over $\mathfrak{F}$. If $\lambda \in \check{\mathsf{V}}_x(\mathfrak{f})$ is a stable rational point then the rational stabilizer $A_{x,\lambda}$ is a subgroup of S_3.

Consider the case $k = \mathbb{Q}_2$, where $\mathsf{G}_x(\mathfrak{f}) = \mathrm{GL}_2(\mathbb{F}_2) = S_3$. There are six stable vectors in $\check{\mathsf{V}}_x(\mathbb{F}_2)$; they correspond to elements of $\mathrm{GL}_2(\mathbb{F}_2)$ via $ax^3 + bx^2y + cxy^2 + dy^3 \leftrightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$, under which the action of $\mathrm{GL}_2(\mathbb{F}_2)$ on the set of stable vectors corresponds to its action on itself by conjugation. Thus the $\mathrm{GL}_2(\mathbb{F}_2)$-orbits of stable vectors in $\check{\mathsf{V}}_x(\mathbb{F}_2)$ are in canonical bijection with the conjugacy classes in S_3.

Up to conjugacy, $G_2(\mathbb{C})$ has a unique closed subgroup isomorphic to $\mathrm{SO}_3(\mathbb{C}) \times S_3$; the first factor is the image of the subregular homomorphism $\varphi' : \mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$, and the second factor S_3 is the full centralizer of the first factor. Let $\varphi : \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2) \to \mathrm{SO}_3(\mathbb{C})$ be the unique simple wild parameter [14, 6.3]. Then φ has image S_4 and inertial image A_4. One can check [5] that the centralizer of this S_4 in $G_2(\mathbb{C})$ is just the S_3 centralizing the $\mathrm{SO}_3(\mathbb{C})$. For each $s \in S_3$ we have an unramified twist

$$\varphi_s = \varphi \times \chi_s : \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2) \to \mathrm{SO}_3(\mathbb{C}) \times S_3,$$

where χ_s is the unramified character such that $\chi_s(\mathrm{Frob}) = s$. The centralizer $A_{\varphi_s} = C_{\hat{G}}(\mathrm{im}\,\varphi_s) = C_{S_3}(s) \simeq A_{x,\lambda}$, where $\lambda \in \check{\mathsf{V}}_x(\mathbb{F}_2)$ is a stable vector whose orbit corresponds to the class of s, as above.

[4] See also [12], which verifies the conjectures in [15] for non-supercuspidal representations of H.

[5] Consider the restriction of the irreducible 7-dimensional representation of $G_2(\mathbb{C})$ to $\mathrm{SO}_3(\mathbb{C})$.

The image of wild inertia under φ_s is the Klein subgroup $K_4 < A_4$, whose centralizer in $\hat{G}$ is $\hat{T}$ extended by inversion. The Swan conductor does not see this extension: we still have

$$b(\varphi_s) = \frac{\dim \hat{\mathfrak{g}} - \dim \hat{\mathfrak{t}}}{[A_4 : K_4]} = 4 = \dim \mathsf{G}_x.$$

Thus the expected equalities (38) and (39) continue to hold when $p = 2$.

Denoting stable orbits in $\check{\mathsf{V}}_x(\mathbb{F}_2)$ by partitions of 3, we have three induced representations decomposing into irreducible epipelagic supercuspidal representations of $G_2(\mathbb{Q}_2)$ as follows:

$$\begin{aligned} \pi_x(111) &= \pi_x(111, \mathrm{triv}) \oplus 2 \cdot \pi_x(111, \mathrm{refl}) \oplus \pi_x(111, \mathrm{sgn}) \\ \pi_x(21) &= \pi_x(21, \mathrm{triv}) \quad \oplus \pi_x(21, \mathrm{sgn}) \\ \pi_x(3) &= \pi_x(3, \mathrm{triv}) \quad \oplus \pi_x(3, \omega) \qquad \oplus \pi_x(3, \omega^{-1}), \end{aligned}$$

where $\omega : \mu_3 \to \mathbb{C}^\times$ is a character of order three.

We note this is parallel to the three *unipotent* L-packets attached to the *unramified* discrete parameters

$$\varphi'_s = \varphi' \times \chi_s : \mathrm{SL}_2(\mathbb{C}) \times \mathrm{Gal}(\bar{\mathbb{Q}}_2/\mathbb{Q}_2) \longrightarrow \mathrm{SO}_3(\mathbb{C}) \times S_3 \subset G_2(\mathbb{C}),$$

as in [26] and [30].

8 Semistable vectors

An irreducible admissible representation of $G(k)$ has **positive depth** if it contains no nonzero vectors invariant under $G(k)_{x,0+}$, for any $x \in \mathcal{B}(G,k)$. In [27], Moy and Prasad proved the fundamental result that for every irreducible admissible positive-depth representation π of $G(k)$ there is a pair $(x,r) \in \mathcal{A}(S,k) \times \mathbb{R}_{>0}$ such that π contains a character χ_λ of $G(k)_{x,r}/G(k)_{x,r+}$ such that $\lambda \in \check{\mathsf{V}}_{x,r}(\mathfrak{f})$ is semistable. (This is a reformulation of the original statement in [27], see [11, Lemma 4.1.2].) It is known that if $\check{\mathsf{V}}_{x,r}$ has semistable vectors for the action of G_x then r is a rational number [2, 3.2].

The aim of this section is to classify the pairs (x,r) for which $\check{\mathsf{V}}_{x,r}(\mathfrak{f})$ contains semistable vectors under G_x. We will use Thm. 4.1 and [33, Thm. 29]. The latter result is only valid for inner automorphisms of the Lie algebra $\mathfrak{L}$. This means in our present application of it that we must assume G is split over K. As we have seen, there are also restrictions on the residual characteristic p which depend on the denominator m of r. But m is not known *a priori*, so we must first find a universal bound on m which is independent of p. For this issue there is no distinction between $\check{\mathsf{V}}_{x,r}$ and $\mathsf{V}_{x,r}$, so we shall consider the latter.

8.1 A necessary condition for semistability

In this section we show, without any conditions on p, that if $\check{\mathsf{V}}_{x,r}$ contains semistable vectors then the denominator of r is bounded by the Coxeter number h of G.

We will assume G is split over K (since that will eventually be required anyway), and that G is absolutely simple. In particular T is now a maximal torus in G and the root system R is irreducible.

We have $\vartheta = 1$ and we now write $\mathcal{A} := \mathcal{A}(T, K)$. Using the basepoint $x_0 \in \mathcal{A}$, we identify $\mathcal{A} = \mathbb{R} \otimes \check{X}$. Thus we identify each root $\alpha \in R$ with the affine function $\tilde{\alpha}$, and the system of affine K-roots is now written as

$$\Psi = R + \mathbb{Z} = \{\alpha + n : \ \alpha \in R, \ n \in \mathbb{Z}\}.$$

We can also scale points in $\mathcal{A}$ with respect to x_0. That is, for $t \in \mathbb{R}$ and $x \in \mathcal{A}$, we define $tx = t(x - x_0) + x_0$.

For each $(x, r) \in \mathcal{A} \times \mathbb{R}$, the set

$$R_{x,r} = \{\alpha \in R : \ \alpha(x) \in r + \mathbb{Z}\}$$

is the set of nonzero weights of $\mathsf{V}_{x,r}$ under the torus $\mathsf{T} := T(K)_0/T(K)_{0+}$ (section 2.3).

We now fix $(x, r) \in \mathcal{A} \times \mathbb{Q}$, and write $r = k/m$ with $\gcd(k, m) = 1$. We will prove:

Proposition 8.1. *Assume that $\mathsf{V}_{x,r}$ has semistable vectors under G_x. Then $m \leq h$, the Coxeter number of G.*

Before beginning the proof we need a few more preparatory remarks.

First, suppose that $R_{x,r}$ is empty. In this case, since $\mathsf{V}_{x,r}$ is nonzero (by the definition of semistability) , we must have $\mathsf{V}_{x,r} = T(K)_r/T(K)_{r+} \neq 0$, which forces $r \in \mathbb{Z}$, the valuation group of K. So in this case we have $m = 1 \leq h$. Henceforth we assume $R_{x,r}$ is nonempty.

Next, in contrast to Prop. 3.2, the hypotheses of Prop. 8.1 impose no obvious relation between x and m. In fact, x need not even be rational with respect to Ψ. However, x *is* rational with respect to a certain affine root subsystem of Ψ, which we shall soon describe.

For any root subsystem $R' \subset R$, the set

$$\Psi' := R' + \mathbb{Z} = \{\alpha + n : \ \alpha \in R', \ n \in \mathbb{Z}\}$$

is an affine root subsystem of Ψ. The collection of hyperplanes $\{\psi^{-1}(0) : \ \psi \in \Psi'\}$ partitions $\mathcal{A}$ into Ψ'**-facets** [5, V.2]. Note that Ψ'-facets are unbounded if R' has smaller rank than R.

Given $y \in \mathcal{A}$, let $\mathcal{C}'$ be any Ψ'-alcove containing y in its closure, and let $\Pi' \subset \Psi'$ be the set of $\psi \in \Psi'$ whose vanishing hyperplanes $\psi^{-1}(0)$ are the walls of $\mathcal{C}'$. We say that y is a **Ψ'-barycenter** if all $\psi \in \Pi'$ not vanishing at y take the same value. From [5, p.80] it follows that this condition is independent of the choice of $\mathcal{C}'$ containing y in its closure. The Ψ'-barycenters contained in a given Ψ'-facet form an affine space of dimension equal to the co-rank of R' in R.

We say that a point $y \in \mathcal{A}$ is **Ψ'-rational** if $\psi(y) \in \mathbb{Q}$ for all $\psi \in \Psi'$. When this holds, we define the **Ψ'-order** of y to be the smallest positive integer n such that $\psi(y) \in \frac{1}{n}\mathbb{Z}$ for all $\psi \in \Psi'$.

Returning now to $\mathsf{V}_{x,r}$ with $r = k/m$ in lowest terms, let j be a multiplicative inverse of k modulo m, let y be the scaled point

$$y := jx$$

and consider $R_{y,1/m} = \{\alpha \in R : \ \alpha(y) \in \frac{1}{m} + \mathbb{Z}\}$. One checks that

$$R_{x,r} \subset R_{y,1/m}. \tag{52}$$

Since we have assumed $R_{x,r}$ is nonempty, it follows that $R_{y,1/m}$ is nonempty.

Now $R_{y,1/m}$ is contained in the root subsystem

$$R' := \{\alpha \in R : \ \alpha(y) \in \tfrac{1}{m}\mathbb{Z}\}.$$

Let $\Psi' = R' + \mathbb{Z}$ be the corresponding affine root subsystem, as above. By construction we have $\Psi'(y) \subset \frac{1}{m}\mathbb{Z}$, so y is Ψ'-rational of order at most m. And since $R_{y,1/m} \neq \varnothing$, it follows that there exists $\psi \in \Psi'$ such that $\psi(y) = 1/m$. Therefore, m is the Ψ'-order of y.

Decompose R' into its irreducible components R'_i:

$$R' = R'_1 \cup \cdots \cup R'_t.$$

This gives a corresponding decomposition

$$\Psi' = \Psi'_1 \cup \cdots \cup \Psi'_t,$$

where $\Psi'_i = R'_i + \mathbb{Z}$. Finally, we have a decomposition

$$\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1 \times \cdots \times \mathcal{A}_t,$$

such that if $\psi \in \Psi'_i$ for some $1 \leq i \leq t$, and $z = (z_0, z_1, \ldots, z_t) \in \mathcal{A}$, then the value $\psi(z)$ depends only on the component z_i; we denote this value by $\psi(z) = \psi(z_i)$. Thus, Ψ'_i is an affine root system on $\mathcal{A}_i$, whose gradient root system R'_i is irreducible.

Let $y = (y_0, y_1, \ldots, y_t)$, and let m_i be the Ψ'_i-order of y_i, for $1 \le i \le t$. Since, for each $\psi \in \Psi'_i$, we have $\psi(y_i) = \psi(y) \in \frac{1}{m}\mathbb{Z}$, it follows that

$$m_i \le m. \tag{53}$$

Let $I = \{i : 1 \le i \le t,\ R'_i \cap R_{y,1/m} \ne \varnothing\}$. Note that I itself is nonempty, since $R_{y,1/m}$ is nonempty. Let $i \in I$ and let $\alpha \in R'_i \cap R_{y,1/m}$. Then $\alpha(y) = 1/m - n$ for some $n \in \mathbb{Z}$, so the affine root $\psi = \alpha + n$ belongs to Ψ'_i and we have $\psi(y) = 1/m$. Combined with (53), it follows that

$$m_i = m, \qquad \text{for any} \quad i \in I. \tag{54}$$

We claim there exists $i \in I$ for which y_i is a Ψ'_i-barycenter. For if not, then by (54), we can apply Lemma 3.1 to y_i, Ψ'_i for every $i \in I$. Hence there exists $\check{\lambda}_i$ in the co-root lattice of R'_i such that $\langle \alpha, \check{\lambda}_i \rangle > 0$ for all $\alpha \in R'_i \cap R_{y,1/m}$. As $\langle \beta, \check{\lambda}_i \rangle = 0$ for all $\beta \in R'_j$ with $j \ne i$, it follows that the sum

$$\check{\lambda} := \sum_{i \in I} \check{\lambda}_i$$

satisfies $\langle \alpha, \check{\lambda} \rangle > 0$ for all $\alpha \in R_{y,1/m}$. By (52) this also holds for all $\alpha \in R_{x,r}$, contradicting the assumption that $\mathsf{V}_{x,r}$ has semistable vectors.

Choose $i \in I$ such that y_i is a Ψ'_i-barycenter. The Ψ'_i-order $m_i = m$ of y_i is at most the Coxeter number h_i of R'_i. From [5, V.6.2] or inspection of cases, we have $h_i \le h$. Thus, $m \le h$, and Prop. 8.1 is proved.

8.2 Existence of semistable vectors

In this section we determine the pairs (x, r) such that $\check{\mathsf{V}}_{x,r}$ contains semi-stable vectors, under the assumption that G is split over an unramified extension of k and that the residual characteristic p is greater than the Coxeter number h of G.

We need one more preliminary result.

Lemma 8.2. *Let θ be an inner automorphism of $\mathfrak{L}$, whose order m is prime to p, let ζ be a root of unity of order m, and let $\mathfrak{c} \subset \mathfrak{L}(\theta, \zeta)$ and $\mathfrak{c}' \subset \mathfrak{L}(\theta^j, \zeta)$ be Cartan subspaces, where $(j, m) = 1$. Then* $\dim \mathfrak{c} = \dim \mathfrak{c}'$.

Proof. There is a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{L}$ such that $\theta(\mathfrak{s}) = \mathfrak{s}$ and $\mathfrak{c} = \mathfrak{s} \cap \mathfrak{L}(\theta, \zeta)$ [33, 3.1]. The restriction $\sigma := \theta|_{\mathfrak{s}}$ is an element of the Weyl group $W(\mathfrak{s})$ of $\mathfrak{s}$, and $\sigma^j = \theta^j|_{\mathfrak{s}}$. The group algebra of $W(\mathfrak{s})$ splits over $\mathbb{Q}$, so σ and σ^j are conjugate in $W(\mathfrak{s})$. Hence the ζ-eigenspaces $\mathfrak{c} = \mathfrak{s}(\sigma, \zeta)$ and $\mathfrak{s}(\sigma^j, \zeta)$ have the same dimension. Since $\mathfrak{s}(\sigma^j, \zeta)$ is abelian and consists of semisimple elements in $\mathfrak{L}$, it is contained in a Cartan subspace of $\mathfrak{L}(\theta^j, \zeta)$, so we have $\dim \mathfrak{c} \leq \dim \mathfrak{c}'$.

Let k be a multiplicative inverse of j modulo m. Replacing the pair (θ, θ^j) by $(\theta^j, \theta^{jk}) = (\theta^j, \theta)$ in the above argument, we get $\dim \mathfrak{c}' \leq \dim \mathfrak{c}$ and the lemma is proved. □

Remark: The automorphisms θ and θ^j need not be conjugate in $\mathrm{Aut}(\mathfrak{L})$.

Let ℓ be the absolute rank of G. For any proper subset $J \subsetneq [0, \ell]$ let $\Delta_J = \{\alpha_j : j \in J\}$ and $\check{\Delta}_J = \{\check{\alpha}_j : j \in J\}$ be the corresponding sets of simple roots and coroots; these generate root subsystems $R_J \subset R$ and $\check{R}_J \subset \check{R}$, respectively. Let $\Delta_J^\perp \subset \mathbb{R} \otimes \check{X}$ be the annihilator of Δ_J in the real vector space $\mathbb{R} \otimes \check{X}$ and let $\check{\rho}_J \in \frac{1}{2}\check{X}$ be one half of the sum of the coroots $\check{\alpha} \in \check{R}_J$ which are positive with respect to $\check{\Delta}_J$.

Let W_J be the subgroup of W generated by the reflections from Δ_J. The ring of W_J-invariant polynomials on the reflection representation of W_J is a polynomial ring with homogeneous generators $f_1, \dots, f_{|J|}$ whose degrees are uniquely determined; these are the **degrees of** W_J.

Theorem 8.3. *Let $r = k/m \geq 0$ be a rational number (in lowest terms) and let $x \in \mathcal{A}$. Then $\check{\mathsf{V}}_{x,r}$ has semistable vectors under G_x if and only if there exists a subset $J \subset [1, \ell]$ such that the following two conditions hold:*

1. *m divides a degree of W_J;*
2. *x is $W_{\mathsf{aff}}(R)$-conjugate to an element of the affine subspace $x_0 + \frac{1}{m}\check{\rho}_J + \Delta_J^\perp \subset \mathcal{A}$.*

Proof. Suppose that $\check{\mathsf{V}}_{x,r}$ contains semistable vectors, with $r = k/m$ in lowest terms. As in the proof of Prop. 8.1, let $R' = \{\alpha \in R : \alpha(x) \in \frac{1}{m}\mathbb{Z}\}$ and let G' be the subgroup of G generated by T and the root groups U_α with $\alpha \in R'$. The representation pairs $(\mathsf{G}_x, \check{\mathsf{V}}_{x,r})$ and $(\mathsf{G}'_x, \check{\mathsf{V}}'_{x,r})$ are isomorphic, so that $\check{\mathsf{V}}'_{x,r}$ has semistable vectors for G'_x. Passing to a suitable irreducible component of R', we may assume R' is irreducible and that x has Ψ'-order m, where $\Psi' = R' + \mathbb{Z}$.

Since $p > h$, and h is at least the Coxeter number of R', Prop. 8.1 ensures that p is large enough so that Thm. 4.1 is valid for G'. Thus, x gives an automorphism θ of the Lie algebra $\mathfrak{L}' = \mathrm{Lie}(\mathsf{G}')$ as in (23), and we have an isomorphism of representation pairs

$$(\mathsf{G}'_x, \check{\mathsf{V}}'_{x,r}) \simeq \left(\mathsf{H}', \mathfrak{L}'(\theta, \zeta^k)\right),$$

where H' is the identity component of the group of fixed-points of θ in G', the analogue of G for G' (cf.(20)). Since $\mathfrak{L}'(\theta,\zeta^k)=\mathfrak{L}'(\theta^j,\zeta)$, Lemma 8.2 (applied to $\mathfrak{L}'$) implies that the Cartan subspaces of $\big(\mathsf{H}',\mathfrak{L}'(\theta,\zeta^k)\big)$ and $(\mathsf{H}',\mathfrak{L}'(\theta,\zeta))$ have the same dimension. Thus we may assume that $k=1$.

The roots in R' are precisely those of the centralizer in G of θ^m. It follows that replacing x by a conjugate under $W_{\mathsf{aff}}(R)$, we may assume that $R'=R_I$, for some proper subset $I\subsetneq[0,\ell]$. Applying Thm. 29 of [33] to R_I, there is a subset $J\subset I$ such that m divides a degree of W_J and x is $W_{\mathsf{aff}}(R_I)$-conjugate to $x_0+\frac{1}{m}\check{\rho}_J+v$, where $v\in\Delta_J^\perp$. Hence m divides a degree of W_{J_1}, where R_{J_1} is some irreducible component of R_J, and we may write $x=x_0+\frac{1}{m}\check{\rho}_{J_1}+v_1$, with $v_1\in\Delta_{J_1}^\perp$. Replacing R_J by R_{J_1}, we may assume that R_J is irreducible, and that $x=x_0+\frac{1}{m}\check{\rho}_J+v$, with $v\in\Delta_J^\perp$.

It remains only to show that further conjugation will put $J\subset[1,\ell]$. In other words, we must show that R_J, which is *a priori* a parabolic root subsystem of R_I, is in fact a parabolic root subsystem of R. This can be checked case-by-case, as follows. Suppose there exists (I,J) constructed as above such that R_J is not a parabolic subsystem of R. Then J must contain the indices of all special nodes of the affine Dynkin diagram for G. Since R_J is irreducible, R can only have one of the types B_n,G_2,F_4,E_7,E_8. If J has type A_k it cannot be contained in a subset of type A_{k+1}, lest it be parabolic. In the (few) remaining cases one finds a root $\alpha\in R-R_I$ for which $\alpha(x)\in\frac{1}{m}\mathbb{Z}$, contradicting the definition of R_I. Thus R_J is indeed parabolic in R and this completes the proof of the "only if" direction of Thm. 8.3.

For the other direction, suppose $x=x_0+\frac{1}{m}\check{\rho}_J+v$ where $J\subset[1,\ell]$, m divides a degree of W_J, and $v\in\Delta_J^\perp$. Now, m divides the degree of some component J_1 of J, and we can write $x=x_0+\frac{1}{m}\check{\rho}_{J_1}+v_1$, with $v_1\in\Delta_{J_1}^\perp$. Thus we reduce to the case that J is irreducible. Projecting x to $\mathcal{A}/\Delta_J^\perp$, we reduce to the case that $J=[1,\ell]$, and that $x=x_0+\frac{1}{m}\check{\rho}$, where m divides a degree of W. The classification of positive rank gradings [25], [33] shows that $\mathfrak{L}(\theta,\zeta)$ has nonzero Cartan subspaces, where $\theta=\mathrm{Ad}(\check{\rho}(\zeta))$. From Lemma 8.2 again, it follows that $\mathfrak{L}(\theta^j,\zeta)=\mathfrak{L}(\theta,\zeta^k)$ has nonzero Cartan subspaces, hence has nonzero closed orbits. It then follows from Thm. 4.1 that $\check{V}_{x,k/m}$ has semistable vectors, completing the proof of Thm. 8.3. □

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