The goal of these notes is to introduce the Killing form and explain some of its remarkable properties.

We begin with a brief review of the “world of ad.” Let $G$ be a Lie group with finite center; set $\mathfrak{g} = \mathfrak{t}_e G$. For $g \in G$ fixed, conjugation induces a map $\psi_g(h) = ghg^{-1}$ from $G$ to $G$. The differential of $\psi_g$ at $h$ induces a linear map $d\psi_g : T_h G \to T_{\psi_g(h)} G$; in particular, taking $h = e$ this becomes an endomorphism $d\psi_g|_{T_e G} : T_e G \to T_e G$. Call this endomorphism $\text{Ad}_g$; it respects the Lie algebra structure of $\mathfrak{g}$, and in particular we have $\text{Ad}_g[x, y] = [\text{Ad}_g x, \text{Ad}_g y]$ for all $g \in G$ and $x, y \in \mathfrak{g}$. The chain rule implies $\text{Ad}_g \text{Ad}_g' = \text{Ad}_{gg'}$, so we may define a group homomorphism $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ by $g \mapsto \text{Ad}_g$. This map is a closed morphism of $G$ onto the connected component of the identity in $\text{GL}(\mathfrak{g})$, with kernel contained in the center of $G$. The derivative of $\text{Ad}$, in turn, defines a map $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \simeq \text{End}(\mathfrak{g})$. The fiber of $\text{ad}$ over a fixed element $x \in \mathfrak{g}$ yields a map $\text{ad}_x : \mathfrak{g} \to \mathfrak{g}$. A straightforward but messy calculation shows that $\text{ad}_x(y) = [x, y]$; in particular, $\text{ad}_x(y)$ is bilinear in $x$ and $y$.

The Killing form is an $\mathbb{R}$-valued bilinear form on $\mathfrak{g} \times \mathfrak{g}$ defined by $B(x, y) = \text{tr}(\text{ad}_x \text{ad}_y)$. This object enjoys the following remarkable properties:

**Theorem 1.** The Killing form is symmetric, satisfies the invariance identity $B([x, y], z) = B(x, [y, z])$ for any $x, y, z \in \mathfrak{g}$, and is $\text{Ad}_g$-invariant, that is $B(\text{Ad}_g x, \text{Ad}_g y)$ for any $g \in G$ and $x, y \in \mathfrak{g}$.

**Theorem 2.** The Killing form is negative-definite if and only if $G$ is compact.

**Proof of Theorem 1.** Write $X = \text{ad}_x, Y = \text{ad}_y, Z = \text{ad}_z$. Symmetry follows from $\text{Tr}(XY) = \text{Tr}(YX)$. For invariance, we compute

$$B([x, y], z) = \text{Tr}((XY - YX)Z) = \text{Tr}(XYZ - YXZ) = \text{Tr}(XYZ - XZY) = \text{Tr}(X(YZ - ZY)) = B(x, [y, z]),$$

where in passing to the third line we have used the invariance of trace under cyclic permutations of order. For $\text{Ad}_g$-invariance, we first note the following lemma:

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1In particular, $\text{Ad}_g^{-1} = \text{Ad}_{g^{-1}}$ so $\text{Ad}_g$ is invertible
Lemma. We have $\text{Ad}_g \text{ad}_x \text{Ad}_g^{-1} = \text{ad}_{\text{Ad}_g x}$ for all $x \in \mathfrak{g}$ and $g \in G$.

Proof of lemma. Fix $y \in \mathfrak{g}$. We use the bracket to compute

$$
\text{Ad}_g \text{ad}_x \text{Ad}_g^{-1} y = \text{Ad}_g [x, \text{Ad}_g^{-1} y] = [\text{Ad}_g x, \text{Ad}_g \text{Ad}_g^{-1} y] = [\text{Ad}_g x, y] = \text{ad}_{\text{Ad}_g x} y,
$$

where in passing to the second line we have used the compatibility of $\text{Ad}_g$ with the bracket. Since $y$ is arbitrary, this proves the lemma. $\square$

Granted this lemma, we prove $\text{Ad}_g$-invariance by computing, for any $g \in G$,

$$
B(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) = \text{tr}(\text{Ad}_g \text{ad}_x \text{ad}_y \text{Ad}_g^{-1}) = \text{tr}(\text{Ad}_g \text{ad}_x \text{Ad}_g^{-1} \text{Ad}_g \text{ad}_y \text{Ad}_g^{-1}) = \text{tr}(\text{ad}_{\text{Ad}_g x} \text{ad}_{\text{Ad}_g y}) = B(\text{Ad}_g x, \text{Ad}_g y),
$$

where in passing to the second line we have used the conjugation invariance of trace. This proves Theorem 1. $\square$

Proof of Theorem 2. Suppose $B(x, x)$ is negative definite; write $A = \text{ad}_x$. Then by the $\text{Ad}_g$-invariance of $B$, the closed map $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ has image contained in the orthogonal group $O(\mathfrak{g}, -B)$, which is compact. This map has fibers contained in a fixed finite subgroup $N \triangleleft Z(G)$, with $\text{Ad}(G) \simeq G/N$ tautologically. Now, given an $N$-invariant open covering $U_i$ of $G$, push this forward to an open covering $\tilde{U}_i$ of $\text{Ad}(G)$; by the compactness of $\text{Ad}(G)$, this admits a finite subcover $\tilde{U}_i^0$. Pulling this back to $G$, the preimages $\text{Ad}^{-1}(\tilde{U}_i^0)$ are open, and the finite set $n \cdot \text{Ad}^{-1}(\tilde{U}_i^0)$, $n \in N$ is a finite subcover of $U_i$. Done. (Alternate proof: Translating $-B(x, x)$ around $G$ defines a $G$-bi-invariant Riemannian metric whose scalar curvature is bounded below by a nonnegative quantity; hence $G$ is compact by the Bonnet-Myers theorem.)

Now, suppose $G$ is compact. We may equip $\text{GL}(\mathfrak{g})$ with an $\text{Ad}$-invariant positive definite inner product. Indeed, given any basis for $\mathfrak{g}$, define $\langle , \rangle$ to be the inner product on $\mathfrak{g}$ for which this basis is orthonormal. Let $\mu(g)$ be a right Haar measure on $G$. Then

$$
\beta(v, w) := \int_G \langle \text{Ad}_g v, \text{Ad}_g w \rangle d\mu(g)
$$

is positive-definite and $\text{Ad}$-invariant by construction. Thus $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ has its image contained in $O(\mathfrak{g}, \beta)$, so its derivative $\text{ad}$ has image contained in $o(\mathfrak{g}, \beta)$; that is to say $\beta(Av, w + \beta(v, Av) = 0$ for any $v, w \in \mathfrak{g}$. Extend $\beta$ to a Hermitian form on $\mathfrak{g} \otimes \mathbb{C}$; by the spectral theorem, $A$ is unitarily equivalent to a diagonal matrix. Let $v$ be an eigenvector of $A$, with eigenvalue $\lambda$; then we compute

$$
\lambda \beta(v, v) = \beta(Av, v) = -\beta(v, Av) = -\overline{\lambda} \beta(v, v),
$$
so $\lambda$ is purely imaginary, and $B(x,x) = \text{tr}(A^2) = \sum \lambda^2 \leq 0$. □

It seems rather remarkable that the Killing form can detect a global phenomenon such as compactness, defined as it is using information given at a single point.

The Killing form also plays an important role in the classification of semisimple Lie groups. More precisely, let $G$ be a non-compact semisimple Lie group. The Killing form $B(x,y)$ is thus indefinite. The following result is essentially due to Cartan.

**Theorem 3.** There exists an element $\theta \in \text{Aut}(\mathfrak{g})$, unique up to inner automorphisms, such that $B_\theta(x,y) := B(x,\theta y)$ is negative-definite and is the Killing form of a compact Lie group $G_c$ whose Lie algebra $\mathfrak{g}_c$ matches $\mathfrak{g}$ after complexification, that is $\mathfrak{g} \otimes \mathbb{C} \simeq \mathfrak{g}_c \otimes \mathbb{C}$.

The element $\theta$ is the Cartan involution. More precisely, let $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C}$ be the complexified Lie algebra of $G$. Complex conjugation induces an involution $\iota: \mathfrak{g}_C \to \mathfrak{g}_C$ whose fixed points are precisely $\mathfrak{g}$. Twisting this involution by $\theta$ (extended to $\mathfrak{g}_C$ by linearity) yields $\mathfrak{g}_c$ as the fixed point set of $\iota \circ \theta$. For example, if $G = \text{SL}_n(\mathbb{R})$, then $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ and the Killing form is $B(x,y) = 4\text{tr}(xy)$ (here we are thinking of $x$ and $y$ as traceless real matrices). The Cartan involution is $\theta x = -x^t$; one may easily check that $B(x,\theta x) = -4\text{tr}(xx^t)$ is negative definite. The fixed point set of $\iota \circ \theta$ on $\mathfrak{g}_C = \mathfrak{sl}_n(\mathbb{C})$ is $\mathfrak{g}_c = \{ x \in \mathfrak{sl}_n(\mathbb{C}) | x + x^t = 0 \} = \mathfrak{su}_n$, and $G_c = \text{SU}_n$. 