Proposition I.21

If on one of the sides of a triangle, from its extremities, there be constructd two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

Claim 1. \( AB + AC > CD + BD \).

Proof

We can extend line BD to point E on AC. [Post. 1.2]

The triangle inequality on triangle ABE implies

\[ AB + AE > BE. \]  \[\text{Prop. 1.20}\]

Adding both CE to both sides, we have

\[ AB + AC = AB + AE + CE > BE + CE. \]  \[\text{C. N. 2}\]

On the other hand, the triangle inequality on triangle CED further implies that

\[ EC + ED > CD \]  \[\text{Prop. 1.20}\]

Adding BD to both sides, we have

\[ EC + BE = EC + ED + BD > CD + BD. \]  \[\text{C. N. 2}\]

Thus, since \( AB + AC > BE + EC \) and \( EC + BE > CD + BD \), this implies

\[ AB + AC > CD + BD. \]  \[\text{Q.E.D.}\]

Claim 2. \( \angle BDC > \angle BAC \).

Proof

Since \( \angle BDC \) is extior to triangle CED, this implies

\[ \angle BDC > \angle CED. \]  \[\text{Prop. 1.16}\]

Since \( \angle CEB \) is extior to triangle BAC, this implies

\[ \angle CEB > \angle BAC. \]  \[\text{Prop. 1.16}\]

Thus, by transitivity of inequalities, this implies

\[ \angle BDC > \angle BAC. \]  \[\text{Q.E.D.}\]