Math 845
Notes on Lie Groups

Mark Reeder

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1 Quaternions and the three-dimensional sphere

1.1 Hamilton’s quaternions

The quaternion algebra \( \mathbb{H} \) is a four dimensional real vector space with basis \( 1, i, j, k \):
\[
\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k
\]
and multiplication rules
\[
ij = k, \quad jk = i, \quad ki = j, \quad i^2 = j^2 = k^2 = -1,
\]
extended to \( \mathbb{H} \) via the associative and distributive laws. The subalgebra \( \mathbb{R} = \mathbb{R}1 \) is the center of \( \mathbb{H} \), and every quaternion \( q \in \mathbb{H} \) may be expressed as
\[
q = t + xi + yj + zk
\]
for unique \( t, x, y, z \in \mathbb{R} \).

The conjugate of \( q = t + xi + yj + zk \) is the quaternion
\[
\bar{q} = t - xi - yj - zk.
\]
Thus, \( \mathbb{R} = \{ q \in \mathbb{H} : \bar{q} = q \} \). One checks that
\[
\bar{pq} = \bar{q}\bar{p},
\]
for all \( p, q \in \mathbb{H} \). The norm of \( q \) is
\[
N(q) = q\bar{q} \in \mathbb{R}
\]
One checks that \( N(q) = t^2 + x^2 + y^2 + z^2 \), for \( q = t + xi + yj + zk \). Hence \( N(q) \geq 0 \), with equality only for \( q = 0 \). One also checks that
\[
N(pq) = N(p)N(q).
\]
It follows that if \( q \neq 0 \) then \( N(q)^{-1} \cdot \bar{q} \) is a multiplicative inverse of \( q \) in \( \mathbb{H} \). Hence \( \mathbb{H} \) is a division algebra, that its set of nonzero elements
\[
\mathbb{H}^\times = \mathbb{H} - \{0\}
\]
is a group under quaternion multiplication, and that the norm $N$ is a homomorphism

$$N : \mathbb{H}^\times \rightarrow \mathbb{R}_{>0}^\times$$

from $\mathbb{H}^\times$ to the group $\mathbb{R}_{>0}^\times$ of positive real numbers under multiplication, whose kernel

$$\ker N = \{ q \in \mathbb{H}^\times : q\bar{q} = 1 \} = \{ t + xi + yj + zk \in \mathbb{H} : t^2 + x^2 + y^2 + z^2 = 1 \}$$

may be identified with the three-dimensional sphere $S^3 \subset \mathbb{R}^4$.

1.2 The Lie group $S^3$

From now on we write

$$S^3 = \{ q \in \mathbb{H}^\times : q\bar{q} = 1 \}.$$

Thus $S^3$ is a group under quaternion multiplication, fitting into the exact sequence

$$1 \rightarrow S^3 \rightarrow \mathbb{H}^\times \xrightarrow{N} \mathbb{R}_{>0}^\times \rightarrow 1.$$

The group $S^3$ contains the quaternion group

$$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$$

of order eight as a subgroup, so $S^3$ is nonabelian, and in fact the center of $S^3$ has just two elements:

$$Z(S^3) = \{ \pm 1 \},$$

since this is already the full center of $Q_8$. The aim for the rest of this section is to find the noncentral conjugacy classes in $S^3$.

The subgroup

$$T = \{ t + xi : t^2 + x^2 = 1 \} = \{ e^{i\theta} : \theta \in \mathbb{R} \}$$

is an abelian subgroup of $S^3$, isomorphic to $S^1$, the circle group. One checks that

$$T = C_{S^3}(i)$$

is the centralizer of $i$ in $S^3$. Let $N(T)$ be the normalizer of $T$ in $S^3$.

Lemma 1.1 We have $N(T) = T \cup Tj$. Thus $N(T)$ consists of two circles, which are cosets of $T$.

Proof: The elements of order four in $T$ are just $\pm i$. Hence if $q \in N(T)$ we have either $qiq^{-1} = i$ or $qiq^{-1} = -i$. The former means that $q \in T$. Assume that $qiq^{-1} = -i$. We note that $jiq^{-1} = -i$ as well, so $qj^{-1} \in C_{S^3}i = T$, which means $q \in Tj$. $lacksquare$

We note that $Q_8 < N(T)$ and that

$$jsj^{-1} = \bar{s} = s^{-1} \quad \text{for all} \quad s \in T.$$
Also \( T_j = \{ t_j + xk : t^2 + x^2 = 1 \} \) lies on the **equatorial two sphere**

\[
C_0 := \{ xi + yj + zk : x^2 + y^2 + z^2 = 1 \} \subset S^3.
\]

The meaning of the subscript “0” is as follows.

As we have defined the norm of a quaternion \( q \) to be \( N(q) = q\overline{q} \), so we define the **trace** of \( q \) to be

\[
\tau(q) = \frac{1}{2}(q + \overline{q}).
\]

Note that \( \tau : \mathbb{H} \to \mathbb{R} \) because \( \tau(q) = \tau(q) \).

In fact we have

\[
\tau(t + xi + yj + zk) = t.
\]

**Lemma 1.2** For \( q \in S^3 \) and all \( p \in \mathbb{H} \) we have \( \tau(qp^{-1}) = \tau(p) \).

**Proof:** Since \( q \in S^3 \) we have \( q^{-1} = \overline{q} \). We compute

\[
\tau(qp^{-1}) = \tau(q\overline{p}) = \frac{1}{2}(q\overline{p} + \overline{q}\overline{p}) = \frac{1}{2}(q\overline{p} + \overline{q}\overline{p}) = \frac{1}{2}(q\overline{p} + q\overline{p}) = q\tau(p)\overline{q}.
\]

Since \( \tau(p) \in \mathbb{R} \) it commutes with \( q \), so we have

\[
\tau(qp^{-1}) = \tau(p)q\overline{q} = \tau(p),
\]

again because \( q \in S^3 \).

By Lemma 1.2, the restriction of \( \tau \) to \( S^3 \) is a function

\[
\tau : S^3 \to [-1, 1]
\]

whose level sets

\[
C_t = \{ p \in S^3 : \tau(p) = t \}
\]

are preserved under conjugation by \( S^3 \). For \( t = 0 \), the level set is the equatorial two-sphere \( C_0 \) mentioned above. We have

\[
C_0 = S^3 \cap \mathbb{H}_0,
\]

where

\[
\mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \{ p \in \mathbb{H} : \tau(p) = 0 \}.
\]

more generally, for fixed \( t \in [-1, 1] \), the level set

\[
C_t = t + \{ xi + yj + zk : x^2 + y^2 + z^2 = 1 - t^2 \}
\]

is a translate of the sphere of radius \( \sqrt{1 - t^2} \) in \( \mathbb{H}_0 \). Here we are invoking the inner (dot) product on \( \mathbb{H}_0 \) for which \( \{ i, j, k \} \) is an orthonormal basis. We may think of \( C_t \) as a sphere of constant latitude in \( S^3 \). \(^1\)

Thus, \( S^3 \) is the disjoint union of its latitude spheres:

\[
S^3 = \bigsqcup_{t \in [-1, 1]} C_t.
\]  

\(^1\)Of course \( C_1 = \{1\} \) and \( C_{-1} = \{-1\} \) are spheres of zero radius.
Proposition 1.3 For each \( t \in [-1, 1] \), the latitude sphere \( C_t \) is a single conjugacy class in \( S^3 \). Hence (1) is the partition of \( S^3 \) into conjugacy classes.

Proof: It must be shown that \( S^3 \) acts transitively on each latitude sphere \( C_t \). We first prove this for \( C_0 \). Following Euler, we write
\[
e^{i\theta} = \cos \theta + i \sin \theta, \quad e^{j\theta} = \cos \theta + j \sin \theta, \quad e^{k\theta} = \cos \theta + k \sin \theta.
\]
Thus we have three subgroups \( T_i, T_j, T_k < S^3 \), all isomorphic to \( S^1 \), given by
\[
T_i = \{ e^{i\theta} : \theta \in \mathbb{R} \}, \quad T_j = \{ e^{j\theta} : \theta \in \mathbb{R} \}, \quad T_k = \{ e^{k\theta} : \theta \in \mathbb{R} \}.
\]
Everything we have said about \( T_i \) (previously called \( T \)) holds for the other subgroups. Their normalizers are
\[
N(T_i) = T_i \cup T_{ij}, \quad N(T_j) = T_j \cup T_{jk}, \quad N(T_k) = T_{ki}.
\]
The nontrivial cosets \( T_i j, T_j k, T_k i \) are three orthogonal great circles on the two-sphere \( C_0 \). Conjugation by \( j, k, i \) on \( T_i T_j \), \( T_j k \), \( T_k i \) is inversion, meaning that
\[
j e^{i\theta} j^{-1} = j e^{-i\theta}, \quad k e^{j\theta} k^{-1} = e^{-j\theta}, \quad i e^{k\theta} i^{-1} j = e^{-k\theta}.
\]
It follows that \( T_i \) conjugates the coset \( T_i j \) to itself, and likewise for \( T_j \) with \( T_j k \), and \( T_k \) with \( T_k i \). Explicitly, we have
\[
e^{i\theta} \cdot e^{i\alpha} j \cdot e^{-i\theta} = e^{i(\alpha + 2\theta)} j, \quad e^{j\theta} \cdot e^{j\alpha} k \cdot e^{-j\theta} = e^{j(\alpha + 2\theta)} k, \quad e^{k\theta} \cdot e^{k\alpha} j \cdot e^{-k\theta} = e^{k(\alpha + 2\theta)} i.
\]
Now take a point \( p \in C_0 \) and write it in spherical coordinates:
\[
p = \sin \phi \cos \theta i + \sin \phi \sin \theta j + \cos \phi k.
\]
If we view \( k \) as the north pole of \( C_0 \) then conjugation by \( e^{j\phi/2} \) sends \( k \) down to a point \( p' \) on the same latitude as \( p \), and then conjugation by \( e^{k\theta/2} \) sends \( p' \) over to \( p \). In other words, we have
\[
e^{k\theta/2} e^{j\phi/2} \cdot k \cdot e^{-j\phi/2} e^{-k\theta/2} = p.
\]
This proves that \( S^3 \) acts transitively on \( C_0 \) by conjugation.

Now for any \( t \in (-1, 1) \), define \( f_t : C_0 \to C_t \) by
\[
f_t(p) = t + \sqrt{1 - t^2} p.
\]
Then \( f_t \) is bijective, with inverse
\[
f_t^{-1}(q) = \frac{q - t}{\sqrt{1 - t^2}},
\]
and for all \( q \in S^3 \) we have \( f_t(qp q^{-1}) = f_t(p) \). Now the transitivity on \( C_t \) follows from the transitivity on \( C_0 \), completing the proof. \( \blacksquare \)

We can write each \( t \in [-1, 1] \) as \( t = \cos \theta \) for \( \theta \in [0, \pi] \), and we have the
**Corollary 1.4** For $0 < \theta < \pi$, the conjugacy class $C_{\cos \theta}$ meets each of $T_i, T_j, T_k$ in two mutually inverse points. Namely,

\[ C_{\cos \theta} \cap T_i = \{ e^{i\theta}, e^{-i\theta} \}, \quad C_{\cos \theta} \cap T_j = \{ e^{j\theta}, e^{-j\theta} \}, \quad C_{\cos \theta} \cap T_k = \{ e^{k\theta}, e^{-k\theta} \}. \]

**Proof:** The sets on the right hand side of each asserted equality consist of the points in $T_i, T_j, T_k$ whose trace is $\cos \theta$. ■

We now understand conjugacy classes of points in $S^3$. The next step is conjugacy of circles. More precisely, by “circle” we mean a subgroup $S < S^3$ such that $S \simeq S^1$ via a continuous group isomorphism.

**Lemma 1.5** For $\theta \in \mathbb{R}$, the subgroup $\langle e^{i\theta} \rangle$ of $S^1$ generated by $e^{i\theta}$ is finite if $\theta \in 2\pi \mathbb{Q}$ and is dense in $S^1$ if $\theta \notin 2\pi \mathbb{Q}$.

**Proof:** The group $A = \langle e^{i\theta} \rangle$ is finite if and only if $e^{in\theta} = 1$ for some $n \in \mathbb{Z}$, which is equivalent to having $\theta \in 2\pi \mathbb{Q}$. So if $\theta \notin 2\pi \mathbb{Q}$, the subgroup $A$ is infinite. We prove that $A$ is in fact dense in $S^1$, as follows.

Let $\epsilon > 0$ and subdivide $S^1$ into equal arcs, starting at 1, of length at most $\epsilon$. In the infinite set $A$ there exist distinct points $e^{in\theta}$ and $e^{im\theta}$, with $m \neq n$, lying the same arc. Since $A = e^{-im\theta} A$, it contains the point $e^{i(n-m)\theta}$ lying in an arc having 1 as an endpoint. The subgroup generated by $e^{i(n-m)\theta}$ is contained in $A$ and meets every arc. Hence $A$ is dense in $S^1$. ■

We revert to the notation $T = T_i = \{ e^{i\theta} : \theta \in \mathbb{R} \}$.

**Proposition 1.6** Every circle in $S^3$ is conjugate to $T$.

**Proof:** Let $S$ be a circle in $S^3$. This means $S$ is a subgroup of $S^3$ and we have a continuous group isomorphism $f : S^1 \to S$. Let $s = f(e^{i\theta})$, where $\theta \in \mathbb{R} - 2\pi \mathbb{Q}$. Then $\langle s \rangle$ is dense in $S$, by Lemma 1.5 and the continuity of $f$. By Cor. 1.4, there exists $q \in S^3$ such that $qsq^{-1} \in T$. The conjugate element $qsq^{-1}$ also has infinite order, hence the subgroup $\langle qsq^{-1} \rangle$ is dense in $T$. Letting $X$ denote the closure of a subset $X \subset S^3$, we have

\[ qsq^{-1} = q \langle s \rangle q^{-1} = \langle qsq^{-1} \rangle = T. \]

**1.2.1 Binary Tetrahedral and Octahedral groups**

The relations in the quaternion group show that $Q_8$ has an automorphism of order three sending $i \mapsto j \mapsto k \mapsto i$. There are also automorphisms such as $i \mapsto -i$, $j \leftrightarrow k$. Can these automorphisms be realized by conjugation in $S^3$?

---

Footnote: I thank Matt Sarmiento for pointing out a mistake in an earlier version of this section.
Proposition 1.7 There are exactly two elements \( q \in S^3 \) which satisfy
\[
qiq^{-1} = j, \quad qjq^{-1} = k, \quad qkq^{-1} = i,
\]
namely \( \pm \frac{1}{2} (1 + i + j + k) \), which have orders six (+) and three (−).

Proof: Letting \( q = t + xi + yj + zk \) and rewriting the equations as
\[
qi = jq, \quad qj = kq, \quad qk = iq,
\]
and equating coefficients, we find that \( t = x = y = z \), so it suffices to determine \( t = \tau(q) \). Since we must have \( t^2 + x^2 + y^2 + z^2 = 1 \), it follows that \( t = \pm \frac{1}{2} \). Since all elements of \( S^3 \) with a given \( t \) are conjugate and \( \cos(\pi/3) = 1/2 \) while \( \cos(2\pi/3) = -1/2 \), we see that \( \frac{1}{2} (1 + i + j + k) \) has order six while \( -\frac{1}{2} (1 + i + j + k) \) has order three. \( \blacksquare \)

The 16 quaternions \( q = \frac{1}{2} (\pm 1 \pm i \pm j \pm k) \in S^3 \), with all possible combinations of signs, along with \( Q_8 \) itself, comprise a 24-element subgroup
\[
G_{24} = \{ \pm 1, \pm i, \pm j, \pm k \} \cup \left\{ \frac{\pm 1 \pm i \pm j \pm k}{2} \right\},
\]
and \( Q_8 \) is the unique (hence normal) Sylow 2-subgroup of \( G_{24} \). The group \( G_{24} \) is usually called the binary tetrahedral group for reasons that will become clear in the next chapter. One can show that
\[
G_{24} \cong SL_2(\mathbb{Z}/3\mathbb{Z}),
\]
the group of \( 2 \times 2 \) matrices over \( \mathbb{Z}/3\mathbb{Z} \) with determinant = 1.

Now \( G_{24} < N(Q_8) \), the normalizer of \( Q_8 \) in \( S^3 \). To determine the size of \( N(Q_8) \) we first note that conjugation gives a map
\[
N(Q_8) \to \text{Aut}(Q_8)
\]
whose kernel is \( \{ \pm 1 \} \). Next, \( |\text{Aut}(Q_8)| = 24 \) because \( \text{Aut}(Q_8) \) acts transitively on the set of order-four subgroups of \( Q_8 \) and one checks there are exactly eight automorphisms stabilizing \( \langle i \rangle \). (With a little more work one can show that \( \text{Aut}(Q_8) \cong S_4 \).) It follows that \( |N(Q_8)| \in \{ 24, 48 \} \). But one checks \( (1 + i)/\sqrt{2} \in N(Q_8) - G_{24} \), so \( |N(Q_8)| = 48 \) and we see that every automorphism of \( Q_8 \) arises via conjugation in \( N(Q_8) \).

One can also check that \( N(Q_8) = N(G_{24}) = N(N(Q_8)) \). Hence from now on we write
\[
G_{48} = N(Q_8).
\]

Explicitly, the elements of \( G_{48} \) outside \( G_{24} \) are given by
\[
G_{48} - G_{24} = \left\{ \frac{\pm 1 \pm i}{\sqrt{2}}, \frac{\pm 1 \pm j}{\sqrt{2}}, \frac{\pm 1 \pm k}{\sqrt{2}} \right\} \cup \left\{ \frac{\pm i \pm j}{\sqrt{2}}, \frac{\pm j \pm k}{\sqrt{2}}, \frac{\pm k \pm i}{\sqrt{2}} \right\}
\]
again with all possible choices of signs. For reasons that will become clear, the group \( G_{48} \) is usually called the binary octahedral group.

Note that \( G_{48} \) is not isomorphic to \( GL_2(\mathbb{Z}/3\mathbb{Z}) \): the latter has too many involutions to be a subgroup of \( S^3 \). However, both \( G_{48} \) and \( GL_2(3) \) are two-fold covers of \( S_4 \), the former via its action on \( Q_8 \) and the latter via its action on the projective line over \( \mathbb{Z}/3\mathbb{Z} \).
1.3 The exponential map for $S^3$

The exponential map gives a canonical parametrization of compact Lie groups. Thus, the circle group $S^1$ is parametrized by exponentiating the purely imaginary complex numbers $i\mathbb{R}$. Hence,

$$S^1 = \{ e^z : z \in i\mathbb{R} \}, \quad \text{where} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We can write each purely imaginary complex number $z$ uniquely as $z = \pm i\theta$, where $\theta = |z| \geq 0$, and we have Euler’s formula

$$e^z = e^{\pm i\theta} = \cos \theta \pm i \sin \theta,$$

as one computes by expanding the exponential series. This parameterization of $S^1$ sends the open segment $(-\pi, \pi)i = \{ \pm \theta i : 0 \leq \theta < \pi \} = [0, \pi) \cdot (S^1 \cap i\mathbb{R})$ bijectively onto $S^1 - \{-1\}$ and both values $\pm \pi i$ are sent to $-1 \in S^1$. Thus, the map $z \mapsto e^z$ glues the ends of the closed segment $[-\pi, \pi]i$ together, forming a circle $S^1$.

Likewise, we can parametrize the 3-sphere $S^3 \subset \mathbb{H}$, by exponentiating the pure quaternions: $\mathbb{H}_0 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Thus, we define

$$\exp : \mathbb{H}_0 \longrightarrow S^3 \quad \text{by} \quad \exp(v) = \sum_{n=0}^{\infty} \frac{v^n}{n!}.$$

Let us compute this sum in closed form. As we did with $S^1$, we can write $v = \theta v_0$, where $v_0 \in S^3 \cap \mathbb{H}_0$, and $\theta = |v|$. Recall that $S^3 \cap \mathbb{H}_0 = C_0$ is the conjugacy class of elements of order four; these are the elements of $\mathbb{H}$ that behave like $\pm i$, in that they square to $-1$. It follows that $\exp(v)$ can be computed in the same way as $e^{\pm i\theta}$. For $v^2 = -\theta^2$, so for all $k \geq 0$ we have $v^{2k} = (-1)^k \theta^{2k}$ and $v^{2k+1} = (-1)^k \theta^{2k}v$. It follows that

$$\exp(v) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + v \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k+1)!} = \cos \theta + \frac{v \sin \theta}{\theta} = \cos \theta + v_0 \sin \theta. \quad (2)$$

as with Euler’s formula. This is consistent with our earlier definitions of $e^{i\theta}, e^{j\theta}, e^{k\theta}$; these were values of $\exp$ on the three lines $i\mathbb{R}, j\mathbb{R}, k\mathbb{R}$ in $\mathbb{H}_0$.

From (2) we observe that $\exp$ maps the sphere $\theta C_0 \subset \mathbb{H}_0$ of radius $\theta$ to the conjugacy-class $C_{\cos \theta}$.

**Proposition 1.8** The map $\exp : \mathbb{H}_0 \rightarrow S^3$ has the following properties:

1. $\exp(\mathbb{H}_0) = S^3$;

2. $\exp$ maps the open ball $\{ v \in \mathbb{H}_0 : |v| < \pi \} = [0, \pi) \cdot C_0$ bijectively onto $S^3 - \{-1\}$;

3. $\exp(\pi C_0) = \{-1\} \subset S^3$;
4. We have \( \exp(qvq^{-1}) = q \exp(v)q^{-1} \) for all \( q \in S^3 \) and \( v \in \mathbb{H}_0 \).

**Proof:** Item 1 is implied by items 2 and 3. If \( q \in C_{\cos \theta} \) then \( q = \cos \theta + q_0 \), where \( q_0 \in \mathbb{H}_0 \) has squared-length \( |q_0|^2 = 1 - \cos^2 \theta = \sin^2 \theta \). The vector \( v_0 = (\sin \theta)q_0 \) lies in \( C_0 \) and \( \exp(\theta v_0) = q \). Therefore \( \exp(\theta C_0) = C_{\cos \theta} \).

Suppose \( v, v' \in \mathbb{H}_0 \) have \( \exp(v) = \exp(v') \). Write \( v = \theta v_0, v' = \theta' v_0' \), with \( \theta, \theta' \in [0, \pi) \) and \( v_0, v_0' \in C_0 \). Since \( \cos \) is injective on \([0, \pi)\), it follows from (2) that \( \theta = \theta' \) and \( v_0 \sin \theta = v_0' \sin \theta \). If \( \theta = 0 \) then \( v = v' = 0 \). Otherwise \( \theta \in (0, \pi) \) and \( \sin \theta \neq 0 \), so \( v_0 = v_0' \), hence \( v = v' \). If \( v_0 \) is any point in \( C_0 \), then \( \exp(\pi v_0) = \cos \pi + v_0 \sin \pi = -1 \). This completes the proof of item 2.

Item 3 follows from the continuity of the map \( v \mapsto qvq^{-1} \).

\[ \square \]

## 2 Rotations of three-dimensional space

### 2.1 The orthogonal and special orthogonal groups

Let \( V = \mathbb{R}^n \) with the inner product

\[ \langle u, v \rangle = \sum_{i=1}^{n} u_i v_i, \]

where \( u_i, v_i \) are the coefficients of \( u, v \) with respect to the standard orthonormal basis \( \{e_i\} \) of \( \mathbb{R}^n \). This inner product is **positive-definite**, meaning that \( \langle u, u \rangle > 0 \) for all nonzero vectors \( u \in \mathbb{R}^n \). The **length** \( u \) is given by

\[ |u| = \langle u, u \rangle^{1/2}. \]

The **orthogonal group** of \( V \) is the subgroup \( O_n \subset \text{GL}_n(\mathbb{R}) \) preserving the the lengths of vectors:

\[ O_n = \{ g \in \text{GL}_n(\mathbb{R}) : |gu| = |u| \text{ for all } u \in \mathbb{R}^n \}. \]

It is useful to recognize when a matrix \( g \) belongs to \( O_n \) without having to check the condition \( |gu| = |u| \) for every vector \( u \in \mathbb{R}^n \).

**Proposition 2.1** On For a matrix \( g \in \text{GL}_n(\mathbb{R}) \), the following are equivalent.

1. \( g \in O_n \).
2. We have \( \langle gu, gv \rangle = \langle u, v \rangle \) for all \( u, v \in \mathbb{R}^n \).
3. The columns of \( g \) form an orthonormal basis of \( \mathbb{R}^n \).
4. The product of \( g \) with its transpose is the identity matrix: \( g \cdot ^t g = I \).
Proof: The equivalence of items 1 and 2 results from the formula
\[
\langle u, v \rangle = \frac{1}{2} \left( |u + v|^2 - |u|^2 - |v|^2 \right).
\]
Applying item 2 to the orthonormal basis \( \{e_i\} \), we get item 3. Conversely, item 3 implies item 2 by expanding \( u, v \) in terms of the basis \( \{e_i\} \). The entry in row \( i \) column \( j \) of \( g \cdot {}^t \! g \) is the inner product of columns \( i \) and \( j \) of \( g \), whence the equivalence of items 3 and 4.

The condition \( g \cdot {}^t \! g = I \) implies that \( \det(g) = \pm 1 \) for all \( g \in O_n \). The **special orthogonal group** is the subgroup of determinant \( = 1 \):
\[
SO_n = \{ g \in O_n : \det(g) = 1 \}.
\]
We give \( O_n \) and \( SO_n \) the topology inherited from the Euclidean space \( M_n(\mathbb{R}) = \mathbb{R}^{n^2} \) of \( n \times n \) real matrices.

**Proposition 2.2** The subsets \( SO_n, O_n \subset M_n(\mathbb{R}) \) are compact and \( SO_n \) is connected, while \( O_n \) has two connected components.

Proof: For \( 1 \leq i \leq j \leq n \), define functions \( f_{ij} : M_n(\mathbb{R}) \to \mathbb{R} \) by
\[
f_{ij}(g) = \langle g_i, g_j \rangle - \delta_{ij},
\]
where \( g_i, g_j \) are the \( i^{\text{th}} \) and \( j^{\text{th}} \) columns of \( g \in M_n(\mathbb{R}) \), and \( \delta_{ij} = 1 \) or \( 0 \) according as \( i = j \) or \( i \neq j \). Then \( O_n \) is the set of common zeros of all the functions \( f_{ij} \), and \( SO_n \) is the subset of \( O_n \) on which the additional function \( \det -1 \) is zero. All of these are polynomial, hence continous functions on \( M_n(\mathbb{R}) \), so \( O_n \) and \( SO_n \) are closed. Since the columns of any \( g \in O_n \) are orthonormal vectors, each entry of \( g \) belongs to \( [-1, 1] \), hence \( O_n \) is a bounded subset of \( M_n(\mathbb{R}) \). Since \( O_n \) and \( SO_n \) are closed and bounded subsets of \( M_n(\mathbb{R}) \), it follows from the Heine-Borel theorem that \( O_n \) and \( SO_n \) are compact.

To prove that \( SO_n \) is connected, we show that every element lies in a connected subgroup. We will use induction on \( n \). Since \( SO_1 = \{1\} \) and \( SO_2 = S^1 \) is a circle, we may assume \( n \geq 3 \) and that \( SO_m \) is connected for \( m < n \).

Let \( g \in SO_n \), and let \( G = \overline{\langle g \rangle} \) be the closure in \( SO_n \) of the subgroup generated by \( g \). As \( SO_n \) is compact, the group \( G \) is also compact. Let \( \lambda \in \mathbb{C}^\times \) be an eigenvalue of \( g \). If \( \lambda = \pm 1 \) then a corresponding eigenvector \( v \) lies in \( \mathbb{R}^n \). Scaling so that \( |v| = 1 \), and choosing an orthonormal basis of the orthogonal complement of the line \( \mathbb{R}v \), we obtain a matrix \( h \in O_n \) such that
\[
hgh^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \text{SO}_{n-1} \end{bmatrix},
\]
which is connected, by the induction hypothesis. The conjugate by \( h \) of this subgroup is also connected, so we have found a connected subgroup of \( SO_n \) containing \( g \).

Assume now that \( g \) has no eigenvalue equal to \( \pm 1 \). Let \( v = (v_1, \ldots, v_n) \in \mathbb{C}^n \) be an eigenvector of \( g \), with eigenvalue \( \lambda \in \mathbb{C}^\times \), and let \( L = \mathbb{C}v \) be the complex line spanned by \( v \). Since \( L \) is closed in \( \mathbb{C}^n \), it
is preserved by $G$, so we have a map $f : G \to L$ sending $\gamma \in G$ to $f(\gamma) = \gamma v$. The map $f$ has bounded image, since $G$ is compact. As $f(g^n) = \lambda^n v$ for all $n \in \mathbb{Z}$, it follows that $|\lambda| = 1$, so $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and $\theta \notin \pi \mathbb{Z}$ since $\lambda \neq \pm 1$. Let $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n)$. Since $g$ has real entries, we have $gu = cu + sv$, $gv = -su + cv$.

Since $g \in O_n$, we have

$$\langle u, u \rangle = \langle gu, gu \rangle = \langle cu + sv, cu + sv \rangle = c^2 \langle u, u \rangle + 2cs \langle u, v \rangle + s^2 \langle v, v \rangle.$$  
Likewise

$$\langle v, v \rangle = s^2 \langle u, u \rangle + 2cs \langle u, v \rangle + c^2 \langle v, v \rangle,$$
and

$$\langle u, v \rangle = \langle cu + sv, -su + cv \rangle = -cs \langle u, u \rangle + (c^2 - s^2) \langle u, v \rangle + cs \langle v, v \rangle.$$  
Adding, we find $\langle u, v \rangle = 0$. Hence $u' = |u|^{-1} u$ and $v' = |v|^{-1} v$ are an orthonormal basis of a two-dimensional plane $U \subset \mathbb{R}^n$.

Hence there exists $h \in O_n$ whose first two columns are $u', v'$ and whose last $n - 2$ columns are an orthonormal basis for the orthogonal complement $U^\perp$. We then have

$$hgh^{-1} \in \begin{bmatrix} \text{SO}_2 & 0 \\ 0 & \text{SO}_{n-2} \end{bmatrix},$$
which is connected, by the induction hypothesis. This completes the proof that $\text{SO}_n$ is connected.

Finally, $O_n$ consists of two cosets of $\text{SO}_n$, each of which is connected component.

\[\square\]

### 2.2 $\text{SO}_3$ and quaternions

Let us regard $\mathbb{R}^3$ as the space of the “pure” quaternions:

$$\mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \{ v \in \mathbb{H} : \tau(v) = 0 \}.$$  
The dot product may be expressed quaternionically as as

$$\langle u, v \rangle = \frac{1}{2} (u \bar{v} + v \bar{u}).$$  
(3)
For $q \in S^3$, let $R_q : \mathbb{H}_0 \to \mathbb{H}_0$ be the linear map given by

$$R_q(v) = qvq^{-1}.$$
A familiar calculation using (3) shows that
\[ \langle R_q(u), R_q(v) \rangle = \langle u, v \rangle, \]
for all \( u, v \in \mathbb{H}_0 \) and \( q \in S^3 \). Therefore \( R_q \in O_3 \) and we have a continuous homomorphism \( R : S^3 \rightarrow O_3 \), sending \( q \mapsto R_q \). Since \( S^3 \) is connected, the image of \( R \) is connected, and therefore lies in \( SO_3 \), by Prop. 2.2. Thus, we have a homomorphism
\[ R : S^3 \rightarrow SO_3, \quad \text{given by} \quad q \mapsto R_q, \]
where \( R_q(v) = qvq^{-1} \). To see this homomorphism explicitly, let \( q = a + bi + cj + dk \in S^3 \) and calculate
\[
qiq^{-1} = (a^2 + b^2 - c^2 - d^2)i + 2(bc + ad)j + 2(bd - ac)k \\
qqj^{-1} = 2(bc - ac)i + (a^2 - b^2 + c^2 - d^2)j + 2(cd + ab)k \\
qkq^{-1} = 2(bd + ac)i + 2(cd - ab)j + (a^2 - b^2 - c^2 + d^2)k,
\]
so the matrix of \( R_q \) with respect to the basis \( \{i, j, k\} \) is
\[
R_q = \begin{bmatrix}
a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd + ac) \\
2(bc + ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) \\
2(bd - ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2
\end{bmatrix}.
\]

**Proposition 2.3** The homomorphism \( R : S^3 \rightarrow SO_3 \) is surjective with \( \ker R = \{ \pm 1 \} \) equal to the center of \( S^3 \). In particular, every matrix in \( SO_3 \) is of the form (4) for some \( (a, b, c, d) \in \mathbb{R}^4 \) with \( a^2 + b^2 + c^2 + d^2 = 1 \).

**Proof:** We will use the following basic fact about group actions. Suppose a group \( G \) acts on a set \( X \), that \( H \) is another group, and that we have a homomorphism \( f : H \rightarrow G \). Assume that the subgroup \( f(H) \leq G \) acts transitively on \( X \) and that there exists \( x \in X \) such that \( f(H) \) contains the stabilizer \( G_x = \{ g \in G : g \cdot x = x \} \). Then \( f(H) = G \). For if \( g \in G \), there is \( h \in H \) such that \( f(h) \cdot x = g \cdot x \), by the transitivity assumption. Then \( g^{-1}f(h) \in G_x \), so \( g^{-1}f(h) = f(k) \) for some \( k \in H \), by the assumption that \( f(H) \supset G_x \). Thus we have \( g = f(hk^{-1}) \in f(H) \).

We apply this to the homomorphism \( R : S^3 \rightarrow SO_3 \), where \( SO_3 \) acts on the sphere \( C_0 = S^2 \). We have proved that \( R(S^3) \) acts transitively on \( C_0 \), and that
\[
R(T_i) = \begin{bmatrix} 1 & 0 \\ 0 & SO_2 \end{bmatrix}
\]
is the stabilizer of \( i \) in \( SO_3 \). It follows that \( R \) is surjective.

The kernel of \( R \) consists of those \( q \in S^3 \) commuting with every vector in \( \mathbb{H}_0 \). Since every quaternion commutes with \( \mathbb{R} \cdot 1 \) which is the center of \( \mathbb{H} \), it follows that \( \ker R \) is the intersection of \( S^3 \) with \( \mathbb{R} \cdot 1 \). This is the unit sphere in \( \mathbb{R} \), that is, \( \ker R = \{ \pm 1 \} \).

**Remark 1:** One can describe \( R \) more geometrically as follows. If \( q \in S^3 \), there is a unit vector \( u \in \mathbb{H}_0 \) and \( \theta \in [-\pi, \pi] \) such that
\[
q = \cos \theta + u \sin \theta.
\]
Since \( q \) commutes with \( u \), it follows that \( R_q(u) = u \) so \( R_q \) is a rotation about the axis through \( u \). To find the angle of rotation, we note that for \( u, v \in \mathbb{H}_0 \) the quaternionic product \( uv \) is given by

\[
uv = u \times v - u \cdot v \in \mathbb{H},
\]

where \( \times \) and \( \cdot \) are the cross and dot product on \( \mathbb{R}^3 \). Note that \( u \times v \in \mathbb{H}_0 \) and \( u \cdot v \in \mathbb{R} \). If \( u \cdot v = 0 \) this reduces to \( uv = u \times v \), and we compute that

\[
R_q(v) = (\cos \theta + u \sin \theta)(\cos \theta - u \sin \theta) = \cos(2\theta) + \sin(2\theta)(u \times v).
\]

This shows that \( R_q \) is rotation about \( u \) by \( 2\theta \) seen counterclockwise as \( u \) points towards you.

**Remark 2:** The quaternionic interpretation gives an explicit formula for the product of two rotations in \( \text{SO}_n \). Let \( S, T \in \text{SO}_3 \) be rotations by \( 2\theta, 2\phi \) about unit vectors \( u, v \in \mathbb{H}_0 \). Then \( S = R_p, T = R_q \), where

\[
p = \cos \theta + u \sin \theta, \quad q = \cos \phi + v \sin \phi.
\]

Then \( ST = R_{pq} \), and we compute

\[
pq = (\cos \theta \cos \phi) - (\sin \theta \sin \phi)u \cdot v + (\sin \theta \cos \phi)u + (\cos \theta \sin \phi)v + (\sin \theta \sin \phi)u \times v.
\]

Therefore \( ST \) is rotation by angle \( \psi \) about the axis through the vector \( w \), where

\[
\cos \psi = (\cos \theta \cos \phi) - (\sin \theta \sin \phi)u \cdot v,
\]

and

\[
w = (\sin \theta \cos \phi)u + (\cos \theta \sin \phi)v + (\sin \theta \sin \phi)u \times v.
\]

**Remark 3:** The image under \( R \) of the binary tetrahedral group \( N(Q_8) \) is the symmetry group of a regular tetrahedron, and is isomorphic to the alternating group \( A_4 \). Thus, we have an exact sequence

\[
1 \longrightarrow \{\pm 1\} \longrightarrow N(Q_8) \longrightarrow A_4 \longrightarrow 1.
\]

This sequence is non-split: there is no subgroup of \( N(Q_8) \) isomorphic to \( A_4 \). In particular, \( N(Q_8) \) and \( S_4 \) are non-isomorphic groups of order 24. Note that the latter fits into another exact sequence (which is now split)

\[
1 \longrightarrow A_4 \longrightarrow S_4 \longrightarrow \{\pm 1\} \longrightarrow 1.
\]

**Remark 4:** The cosets of \( \{\pm 1\} \) in \( S^3 \) are pairs of antipodal points. Each pair determines a line in \( \mathbb{R}^4 \), so the set of antipodal pairs is the real projective space \( \mathbb{RP}^3 \). Thus, Prop. 2.3 shows that \( \text{SO}_3 = \mathbb{RP}^3 \), as topological spaces. You can also regard \( \mathbb{RP}^3 \) as the quotient of a solid ball in \( \mathbb{R}^3 \) by identifying antipodal points on the boundary. Indeed, every element of \( \text{SO}_3 \) is rotation about some axis by some angle \( \theta \in [−\pi, \pi] \). The axis determines a line segment in the ball \( B_\pi \) of radius \( \pi \) in \( \mathbb{R}^3 \) and \( \theta \) determines a point on the axis. Each \( \theta \in (−\pi, \pi) \) gives a unique rotation about this axis, but the two values \( \theta = \pm \pi \), corresponding to antipodal points on the boundary of \( B_\pi \), give the same rotation. In the next section, the exponential map will make this latter interpretation more explicit.
2.3 The exponential map for $\text{SO}_3$

Let $A$ be an $n \times n$ real matrix. What conditions on $A$ ensure that the path $\theta \mapsto \exp(\theta A)$ lies in $\text{SO}_n$?

The condition $t(\exp(\theta A)) = (\exp(\theta A))^{-1}$ means that

$$I + \theta tA + \cdots = I - \theta A + \cdots,$$

so $\exp(\theta A) \in \text{O}_n$ iff $tA = -A$. Such matrices are called **skew-symmetric** and their diagonal entries are zero. In particular $\text{tr}(A) = 0$ so $\det \exp(\theta A) = 1$, so in fact $\exp(A)$ lies in $\text{SO}_n$ for any skew-symmetric $n \times n$ matrix $A$. Letting $\text{so}_n$ denote the set of such matrices, we therefore have an exponential map

$$\exp : \text{so}_n \longrightarrow \text{SO}_n.$$

We now take $n = 3$ and calculate $\exp$ explicitly. The matrices in $\text{so}_3$ are parametrized by vectors $v = (x, y, z) \in \mathbb{R}^3$, via

$$v = (x, y, z) \mapsto A_v = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

Note that $v \in \ker A_v$ and $\ker A = \mathbb{R}v$ as long as $v \neq (0, 0, 0)$. Let $|v| = \sqrt{x^2 + y^2 + z^2}$. Using the fact that $A_v^3 = -\theta^2 A_v$, we find that

$$\exp(A_v) = I + \left(\frac{\sin |v|}{|v|}\right) A_v + \left(\frac{1 - \cos |v|}{|v|^2}\right) A_v^2,$$

and that $\exp(A_v)$ is rotation by $|v|$ about the axis through $v$, where the direction is seen counterclockwise as $v$ points towards you. It follows that $\exp$ maps $\{A_v : |v| < \pi\}$ bijectively onto the complement in $\text{SO}_3$ of the conjugacy class $C$ of 180 degree rotations. If $|v| = \pi$ then $\exp(A_v) = \exp(A_{-v})$ so $\exp$ describes $C$ as the sphere of radius $\pi$ with antipodal points identified. That is, $C$ is the real projective plane.

2.4 The exponential diagram

We now have exponential maps

$$\exp : V \longrightarrow S^3, \quad \exp : \text{so}_3 \longrightarrow \text{SO}_3$$

and a homomorphism $R : S^3 \rightarrow \text{SO}_3$ given by $R_q(v) = qvq^{-1}$. The final piece is the derivative of $R$, which is a linear map

$$R' : V \longrightarrow \text{so}_3$$

defined as follows. For each $v \in V$, $R'_v \in \text{so}_3$ is the skew symmetric matrix acting on $V$ by

$$R'_v(u) = \frac{d}{d\theta} R_{\exp(\theta v)}(u)\big|_{\theta=0}.$$
Computing this explicitly using power series, we find the explicit formula

\[ R_v'(u) = vu - uv. \]

To see \( R_v' \) as a matrix, let \( v = (x, y, z) \) and compute \( R_v'(i), R_v'(j), R_v'(k) \) to find that

\[
R_v' = 2 \begin{bmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{bmatrix} = A_2v.
\]

Finally, one checks that

\[ R \circ \exp = \exp \circ R'. \]

That is, the following diagram is commutative.

\[
\begin{array}{ccc}
V & \xrightarrow{R'} & \mathfrak{so}_3 \\
\exp \downarrow & & \exp \downarrow \\
S^3 & \xrightarrow{R} & SO_3
\end{array}
\]

It follows that \( \exp : \mathfrak{so}_3 \to SO_3 \) is surjective.

Thus, the Lie groups \( S^3 \) and \( SO_3 \) are parametrized by \( \mathbb{R}^3 \) via the exponential maps, just as \( S^1 \) is parameterized by \( \mathbb{R} \) via the usual exponential map. Moreover, the group homomorphism \( R : S^3 \to SO_3 \) lifts to a linear map \( R' \) on the parameter spaces.

### 3 Definition and basic properties of Lie groups

Intuitively, a Lie group is a group which locally looks like \( \mathbb{R}^n \). We often imagine moving in a Lie group, where small motions seem like motion in \( \mathbb{R}^n \). For example, we can move along circles in \( S^3 \) or we can move in \( SO_3 \) by changing the axis and amount of rotation. To define all of this precisely requires the notion of a smooth manifold.

#### 3.1 Introduction to Manifolds

Let \( U \) be an open subset of \( \mathbb{R}^n \). A function \( f : U \to \mathbb{R}^m \) is **differentiable on** \( U \) if for each \( x \in U \) there exists a linear map \( f'_x : \mathbb{R}^n \to \mathbb{R}^m \) such that

\[
\lim_{h \to 0} \frac{1}{|h|} [f(x + h) - f(x) - f'_x(h)] = 0.
\]

If \( v \in \mathbb{R}^n \) has length \( |v| = 1 \) (with respect to the standard Euclidean metric, as we have been using) then we can take \( h = tv \) as \( t \to 0 \) and the formula for \( f'_x \) becomes

\[
f'_x(v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.
\]
If \( v = e_i \) is one of the standard basis vectors then \( f'_x(e_i) = \frac{\partial f}{\partial x_i}(x) \).

Now the function \( x \mapsto f'_x \) is a function \( f' : U \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{nm} \), called the derivative of \( f \). We say that \( f : U \to \mathbb{R}^m \) is smooth or \( C^\infty \) if each of \( f, f', f'', \ldots \) is differentiable on \( U \). The \( \mathbb{R} \)-vector space \( C^\infty(U) \) of all smooth functions on \( U \) is a ring containing all polynomial functions.

Now let \( M \) be a topological space. A local chart on \( M \) is a triple \((\varphi, U, M_\alpha)\) where \( U \subset \mathbb{R}^n \) and \( M_\alpha \subset M \) are open subsets and \( \varphi : U \to M_\alpha \) is a homeomorphism. Thus, \( \varphi_{\alpha l} \) parametrizes the open subset \( M_\alpha \) of \( M \). We call \( n \) the dimension of the chart. An atlas on \( M \) is a collection \( \{(\varphi, U, M_\alpha) : \alpha \in A\} \) of local charts indexed by some set \( A \) such that \( M = \bigcup_{\alpha \in A} M_\alpha \). We say the atlas has dimension \( n \) if \( U_\alpha \) is an open subset of \( \mathbb{R}^n \), with the same \( n \), for every \( \alpha \in A \).

Given two local charts \((\varphi_\alpha, U_\alpha, M_\alpha)\) and \((\varphi_\beta, U_\beta, M_\beta)\), we get a transition function

\[
\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(M_\alpha \cap M_\beta) \to \varphi_\beta^{-1}(M_\beta) \to \mathbb{R}^n
\]

defined on the open set \( U_{\alpha\beta} = \varphi_\alpha^{-1}(M_\alpha \cap M_\beta) \subset \mathbb{R}^n \). An atlas \( \{(\varphi, U, M_\alpha) : \alpha \in A\} \) is smooth if the transition functions \( \varphi_\beta^{-1} \circ \varphi_\alpha \) are smooth for each \( \alpha, \beta \in A \).

**Example 1:** Let \( U \) be an open subset of \( \mathbb{R}^n \). Then the identity map \( U \to U \) is a smooth atlas on \( U \), consisting of a single chart. \( \Box \)

**Example 2:** Let \( M = S^n = \{(x_0, x_1, \ldots, x_n) : x_0^2 + \sum x_i^2 = 1\} \). Take the two antipodal points \( z_\pm = (\pm 1, 0, \ldots, 0) \) and let \( M_\pm = S^n \setminus \{z_\pm\} \), and let \( U_\pm = \mathbb{R}^n \). For \( \alpha = \pm \), define \( \varphi_\alpha : \mathbb{R}^n \to S^n \) by

\[
\varphi_\alpha(u_1, \ldots, u_n) = \frac{1}{|u|^2}(\alpha(|u|^2 - 1), 2u_1, \ldots, 2u_n)
\]

Then \( \varphi_\alpha^{-1} : M_\alpha \to \mathbb{R}^n \) is given by

\[
\varphi_\alpha^{-1}(x_0, \ldots, x_n) = \frac{1}{1 - \alpha x_0}(x_1, \ldots, x_n),
\]

and the transition functions are given by

\[
\varphi_\alpha^{-1} \circ \varphi_\beta(u) = \frac{u}{|u|}.
\]

This is a smooth function on \( \varphi_\alpha^{-1}(M_\alpha \cap M_{\alpha'}) = \mathbb{R}^n - \{(0, \ldots, 0)\} \). Hence the collection \( \{(\varphi_\alpha, \mathbb{R}^n, M_\alpha) : \alpha = \pm\} \) is a smooth atlas. \( \Box \)

Let \( M \) and \( N \) be topological spaces with atlases \( \{(\varphi_\alpha, U_\alpha, M_\alpha) : \alpha \in A\} \) and \( \{(\psi_\beta, V_\beta, N_\beta) : \beta \in B\} \) respectively. A function \( f : M \to N \) is smooth if each composition

\[
\psi_\beta^{-1} \circ f \circ \varphi_\alpha : \varphi_\alpha^{-1}(M_\alpha \cap f^{-1}(N_\beta)) \to \mathbb{R}^m
\]

is smooth. For example, the identity \( M \to M \) is smooth, and the composition of smooth functions is smooth.
A given topological space $M$ may admit more than one atlas. Two atlases $\{(\varphi_\alpha, U_\alpha, M_\alpha) : \alpha \in A\}$ and $\{(\psi_\beta, V_\alpha, M_\beta) : \beta \in B\}$ on $M$ are considered equivalent if the identity map is smooth from one atlas to the other in the sense just defined. That is, the atlases are equivalent if each composition
\[
\psi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(M_\alpha \cap N_\beta) \to \mathbb{R}^n
\]
is smooth.

**Definition 3.1** An smooth manifold is a Hausdorff topological space $M$ with an equivalence class of smooth atlases. We say $M$ is $n$-dimensional if these atlases have dimension $n$.

The term “smooth manifold” is interchangeable with differentiable manifold. An equivalence class of smooth atlases is often called a smooth structure or a differentiable structure. Having a single smooth atlas on a Hausdorff topological space $M$ gives a smooth structure on $M$, via the equivalence class of the given atlas.

Two manifolds $M, N$ are diffeomorphic if there is a smooth map $f : M \to N$ having a smooth inverse $f^{-1} : N \to M$. It is possible for two manifolds to be homeomorphic but not diffeomorphic. That is, the topological space $M$ may admit more than one. For example, the 7-sphere $S^7$ admits exactly 28 smooth structures.

**Lemma 3.2** If $M$ and $N$ are smooth manifolds with atlases $\{(\varphi_\alpha, U_\alpha, M_\alpha) : \alpha \in A\}$ and $\{(\psi_\beta, V_\alpha, N_\beta) : \beta \in B\}$, then there is a canonical smooth structure on $M \times N$, given by the atlas $\{(\varphi_\alpha \times \psi_\beta, U_\alpha \times V_\beta, M_\alpha \times N_\beta) : \alpha \in A, \beta \in B\}$.

**Proof:** Left to the reader! ■

### 3.2 Manifolds defined by equations

Let $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function, where $n \geq m$, and write $n = k + m$. Let $M = f^{-1}(0) = \{p \in \mathbb{R}^n : f(p) = 0\}$. The Implicit Function Theorem gives a sufficient condition for $M$ to be a smooth manifold, in terms of the derivative $f'$ of $f$. Recall that $f'$ is the $m \times n$ matrix of functions
\[
f' = \left( \frac{\partial f_i}{\partial x_j} \right)
\]
which at each point $p \in \mathbb{R}^n$ gives a linear map
\[
f'_p = \left( \frac{\partial f_i}{\partial x_j}(p) \right) : \mathbb{R}^n \to \mathbb{R}^m.
\]

---

3 Recall $M$ is Hausdorff if given distinct points $x, y \in M$ there exist disjoint open subsets $U, V \subset M$ such that $x \in U$ and $y \in V$.

Theorem 3.3 (Implicit Function Theorem) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a smooth function, where \( n \geq m \), and let \( M = f^{-1}(0) \). Suppose \( f'_p(\mathbb{R}^n) = \mathbb{R}^m \). Then there exist open sets \( U \subset \mathbb{R}^k \) and \( V \subset \mathbb{R}^m \), and a unique smooth function \( g : U \to V \) such that \( p \in U \times V \) and \( (U \times V) \cap M = \{(u,g(u)) : u \in U\} \).

**Proof:** See [Rudin’s *Principles of Mathematical Analysis*].

Corollary 3.4 With notation as in Thm. 3.3 assume that \( f'_p \) has rank \( m \) for all \( p \in M \). Then \( M \) has the structure of a smooth manifold.

**Proof:** Our assumption means that for each \( p \in M \) we can choose \( m \) columns whose determinant is nonzero. Fix \( p \in M \) and order the variables so that these columns are the last \( m \) columns of \( f' \) and choose \( U, V, g \) as in Thm. 3.3. This gives a chart \( \varphi_p : U_p \to M_p \), where \( U_p = U, M_p = (U \times V) \cap M \), given by \( \varphi_p(u) = (u, g(u)) \). Doing this for each \( p \in M \) gives an atlas \( \{(\varphi, U_p, M_p) : p \in M\} \). One checks that for \( p, q \in M \) and \( u \in \varphi_p(M_p \cap M_q) \) we have \( \varphi_q^{-1} \circ \varphi_p(u) = u \). Hence the atlas is smooth.

### 3.3 Lie groups: definition and first examples

A **Lie group** is a smooth manifold \( G \) which is also a group whose multiplication and inverse maps

\[
\mu : G \times G \to G \quad \text{and} \quad \iota : G \to G
\]

are smooth. Here \( G \times G \) has the product smooth structure, as in Lemma 24.

Let \( G \) and \( H \) be Lie groups. A **Lie group homomorphism** is a smooth map \( f : G \to H \) which is also a group homomorphism. And \( f \) is an isomorphism of Lie groups if \( f \) is both a diffeomorphism of manifolds and an isomorphism of groups.

**Example 1:** The **General Linear Group** \( \text{GL}_n(\mathbb{R}) \) is the group of automorphisms of the vector space \( \mathbb{R}^n \). Concretely, \( \text{GL}_n(\mathbb{R}) \) is the group of \( n \times n \) invertible matrices, under matrix multiplication. To see that \( \text{GL}_n(\mathbb{R}) \) is a Lie group, let \( M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2} \) be the space of all \( n \times n \) matrices with entries in \( \mathbb{R} \). The determinant \( \det : M_n(\mathbb{R}) \to \mathbb{R} \) is a continuous function and \( \text{GL}_n(\mathbb{R}) \) is the open set of points in \( M_n(\mathbb{R}) \) on which \( \det \) is nonzero. Thus \( \text{GL}_n(\mathbb{R}) \) is an open subset of \( M_n(\mathbb{R}) \) and is a manifold. The group operations are polynomial functions (or polynomial divided by \( \det \)) of the matrix entries, hence are smooth functions on \( \text{GL}_n(\mathbb{R}) \). Therefore \( \text{GL}_n(\mathbb{R}) \) is a Lie group.

The determinant map sends \( \text{GL}_n(\mathbb{R}) \) onto \( \mathbb{R}^\times \), which has two components—the positive and negative real numbers. It follows that \( \text{GL}_n(\mathbb{R}) \) is disconnected. We will see that \( \text{GL}_n(\mathbb{R}) \) has two connected components,

\[
\text{GL}^+_n(\mathbb{R}) = \{g \in \text{GL}_n(\mathbb{R}) : \det(g) > 0\},
\]

and its complement. Two bases of \( \mathbb{R}^n \) have the same **orientation** if the matrix relating them lies in \( \text{GL}^+_n(\mathbb{R}) \). This means that one basis can be deformed continuously into the other. Thus \( \text{GL}^+_n(\mathbb{R}) \) is orientation-preserving automorphism group of \( \mathbb{R}^n \).
Example 2: If $U$ is a bounded open subset of $\mathbb{R}^n$ and $g \in \text{GL}_n(\mathbb{R})$ then $\text{vol}(gU) = \det(g) \text{vol}(U)$. The Special Linear Group $\text{SL}_n(\mathbb{R})$ is the group of volume-preserving automorphisms of $\mathbb{R}^n$. That is, 
$$\text{SL}_n(\mathbb{R}) = \{ g \in \text{GL}_n(\mathbb{R}) : \det(g) = 1 \}.$$ 
To see that $\text{SL}_n(\mathbb{R})$ is a Lie group we use Cor. 3.4, viewing $\text{SL}_n(\mathbb{R}) = f^{-1}(0)$ for the function $f : M_n(\mathbb{R}) \to \mathbb{R}$ defined by $f(A) = \det(A) - 1$. If $A = [a_{ij}]$ then 
$$f(A) = -1 + \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$ 
Let $A_j$ be the matrix obtained by deleting row 1 and column $j$ from $A$. If $\det(A) = 1$ then $\det(A_j) \neq 0$ for some $j$. And 
$$\frac{\partial f}{\partial a_{ij}} = \pm \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \pm \det(A_j) \neq 0,$$ 
so the matrix $f'(A)$ has rank $1$ for all $A \in \text{SL}_n(\mathbb{R})$ and the latter is indeed a manifold. Again, the group operations are polynomial, so $\text{SL}_n(\mathbb{R})$ is a Lie group.

Example 3: The Orthogonal Group $O_n$ is group of length-preserving automorphisms of the vector space $\mathbb{R}^n$. That is, 
$$O_n = \{ g \in \text{GL}_n(\mathbb{R}) : |gv| = |v| \quad \forall \ v \in \mathbb{R}^n \} = \{ g \in \text{GL}_n(\mathbb{R}) : ^tgg = I \}.$$ 
A matrix belongs to $O_n$ exactly when its columns form an orthonormal basis of $\mathbb{R}^n$. Thus, $O_n = f^{-1}(0)$, where $f = (f_{pq}) : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ has component functions 
$$f_{pq}(g) = -\delta_{pq} + \sum_i g_{ip}g_{iq},$$
where $\delta_{pq} = 1$ if $p = q$ and is zero otherwise. One checks that $f_g'$ is surjective if $g$ is invertible, in particular if $g \in O_n$. Hence $O_n$ is indeed a manifold. Again, the group operations are polynomial, so $O_n$ is a Lie group.

Since $O_n = f^{-1}(0)$ and each column of a matrix in $O_n$ is a unit vector, it follows that $O_n$ is a closed and bounded subset of $M_n(\mathbb{R})$, hence is compact. The determinant maps $O_n$ onto $\{ \pm 1 \}$, so $O_n$ is disconnected. We will see that it has exactly two connected components.

Example 4: The Special Orthogonal Group $\text{SO}_n$ is the subgroup of $\text{GL}_n(\mathbb{R})$ preserving orientation, volume and length. That is, 
$$\text{SO}_n = \{ g \in \text{SL}_n(\mathbb{R}) : |gv| = |v| \quad \forall \ v \in \mathbb{R}^n \} = \{ g \in \text{SL}_n(\mathbb{R}) : ^tgg = I \}.$$ 
A matrix lies in $\text{SO}_n$ exactly when its columns form an orthonormal basis having the same orientation as the standard orthonormal basis $\{ e_1, \ldots, e_n \}$. We will see that $\text{SO}_n$ is connected; it is the component of $O_n$ containing the identity matrix.

Example 5: The group $\mathbb{R}^3$ is an abelian Lie group under the operation of addition. But the same manifold $\mathbb{R}^3$ has another Lie group structure. The Heisenberg group 
$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : (x, y, z) \in \mathbb{R}^3 \right\}$$ 
is diffeomorphic to $\mathbb{R}^3$ but is not isomorphic to $\mathbb{R}^3$ as a Lie group. Indeed, $H_3$ is non-abelian.
4 The Lie algebra of a Lie group

Let $G$ be a Lie group. Imagine moving along paths in $G$. Each smooth path $\gamma : \mathbb{R} \to G$ has a position $\gamma(t)$ and velocity vector $\gamma'(t)$ which is “tangent” to $G$. The pair $(\gamma(t), \gamma'(t))$ tells us where we are and where we are going. Where does $\gamma'(t)$ live? If $G$ were a subset of some Euclidean space $\mathbb{R}^n$ then we could imagine $\gamma'(t)$ as a vector in $\mathbb{R}^n$. But $G$ may embed in many Euclidean spaces in many ways, and we want a home for $\gamma'(t)$ that is intrinsic to $G$ and independent of any particular realization of $G$ in a Euclidean space. We will see that $(\gamma(t), \gamma'(t))$ is a path in a new manifold.

4.1 The tangent bundle of a manifold

First let $U$ be an open subset of $\mathbb{R}^n$. The tangent bundle to $U$ is simply

$$TU := U \times \mathbb{R}^n.$$ 

If we have a smooth path $\gamma : (a, b) \to U$ defined on an open interval $(a, b) \subset \mathbb{R}$, then we are interested in both the position and velocity of the path. This pair of data is a new path in the tangent bundle $TU$, given by $t \mapsto (\gamma(t), \gamma'(t))$. The first projection $\pi : TU \to U$ shows the position, and the fiber $\pi^{-1}(u) = \mathbb{R}^n$ is the vector space of all possible velocities at $u$. Note that $TU$ is a union

$$TU = \bigsqcup_{p \in U} \{p\} \times \mathbb{R}^n.$$ 

of fibers of $\pi$. These fibers are called tangent spaces. Each tangent space is canonically identified with the ambient vector space $\mathbb{R}^n$ containing $U$ as an open subset.

Let $M$ be a $n$-dimensional smooth manifold with atlas $\{\varphi_{\alpha}, U_{\alpha}, M_{\alpha} : \alpha \in A\}$. For each $\alpha \in A$ let $\iota_{\alpha} : M_{\alpha} \hookrightarrow M$ be the inclusion map. The tangent bundle of $M$ is a new manifold $TM$ equipped with a projection map

$$TM \xrightarrow{\pi} M$$

whose fibers $\pi^{-1}(p)$ are $n$-dimensional vector spaces which vary smoothly. More precisely, $TM$ is the set of equivalence classes

$$TM = \bigsqcup_{\alpha \in A} (M_{\alpha} \times \mathbb{R}^n)/\sim,$$

where $(x, v) \sim (y, w)$ if

$$\iota_{\alpha}(x) = \iota_{\beta}(y) \quad \text{and} \quad w = (\varphi_{\beta}^{-1}(\varphi_{\alpha})'x(v)).$$

Thus, points in $TM$ are equivalence classes $[x, v]_{\alpha} \in TM$, for $(x, v) \in M_{\alpha} \times \mathbb{R}^n$, with the understanding that if $x \in M_{\alpha} \cap M_{\beta}$ we have

$$[x, v]_{\alpha} = [x, (\varphi_{\beta}^{-1}(\varphi_{\alpha})'x(v)]_{\beta}.$$ 

Each $M_{\alpha} \times \mathbb{R}^n = TM_{\alpha}$ injects into $TM$. Via the quotient topology, $TM$ has an open cover

$$TM = \bigcup_{\alpha \in A} TM_{\alpha}.$$
For each $\alpha \in A$ define
\[
\Phi_\alpha : U_\alpha \times \mathbb{R}^n \rightarrow TM_\alpha \quad \text{by} \quad \Phi_\alpha(u,v) = [\varphi_\alpha(u),v]_\alpha.
\]
One checks that
\[
\Phi_\alpha^{-1}(TM_\alpha \cap TM_\beta) = \varphi_\alpha^{-1}(U_{\alpha\beta}) \times \mathbb{R}^n,
\]
and that
\[
\Phi_\beta^{-1}\Phi_\alpha : \varphi_\alpha^{-1}(U_{\alpha\beta}) \times \mathbb{R}^n \rightarrow \varphi_\beta^{-1}(U_{\alpha\beta}) \times \mathbb{R}^n
\]
is given by $\varphi_\beta^{-1}\varphi_\alpha \times (\varphi_\beta^{-1}\varphi_\alpha)'$. Thus $TM$ is a smooth $2n$-dimensional manifold with atlas $\{(\Phi_\alpha, TU_\alpha, TM_\alpha) : \alpha \in A\}$.

The projection $\pi_M : TM \rightarrow M$ is given by $\pi_M([x,v]_\alpha) = x$. For $\alpha, \beta \in A$, one checks that $\varphi_\beta^{-1} \circ \pi_M \circ \Phi_\alpha$ is the composition
\[
\varphi_\alpha^{-1}(U_{\alpha\beta}) \times \mathbb{R}^n \xrightarrow{\text{proj}} \varphi_\alpha^{-1}(U_{\alpha\beta}) \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(U_{\alpha\beta}),
\]
which is smooth, so that $\pi_M : TM \rightarrow M$ is smooth.

The tangent space to $M$ at $x \in M$ is the fiber
\[
T_xM := \pi_M^{-1}(x).
\]
If $x \in M$, then $T_xM = \{[x,v]_\alpha : v \in \mathbb{R}^n\} \cong \mathbb{R}^n$, but this isomorphism is non-canonical, as it depends on $\alpha$.

In practice, the tangent bundle usually has a more concrete realization than the rather abstract general definition just given.

**Example:** Consider again the $n$-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Then we may realize $TS^n$ as
\[
TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} : x \cdot v = 0\},
\]
where $x \cdot v$ is the dot product on $\mathbb{R}^{n+1}$. To see that this realization is indeed a manifold, we use Cor. 3.4 to express the right side of (5) as $f^{-1}(0)$, where $f = (f_1, f_2) = (|x|^2 - 1, x \cdot v)$. We find that
\[
f' = \begin{pmatrix} 2x_1 & \cdots & 2x_n & 0 & \cdots & 0 \\ v_1 & \cdots & v_n & x_1 & \cdots & x_n \end{pmatrix}
\]
has rank two for all $x \in S^n$, so that $f^{-1}(0)$ is indeed a manifold. □

**Proposition 4.1** If $M$ and $N$ are smooth manifolds and $f : M \rightarrow N$ is a smooth map, then there is a unique smooth map $f' : TM \rightarrow TN$ making the following diagram commute:
\[
\begin{array}{ccc}
TM & \xrightarrow{f'} & TN \\
\pi_M \downarrow & & \downarrow \pi_N \\
M & \xrightarrow{f} & N
\end{array}
\]
If $L$ is another manifold and $g : N \rightarrow L$ is another smooth map then
\[
(g \circ f)' = g' \circ f'. \quad \text{chain rule.}
\]
Proof: Let \( \{(\varphi_{\alpha}, U_{\alpha}, M_{\alpha}) : \alpha \in A\} \) and \( \{(\psi_{\beta}, V_{\beta}, N_{\beta}) : \beta \in B\} \) be atlases on \( M \) and \( N \). Let \( x \in M \) and choose \( \alpha \in A \) such that \( x \in M_{\alpha} \) and \( \beta \in B \) such that \( f(x) \in N_{\beta} \). For \( v \in \mathbb{R}^n \), define \( f'(\[x, v\]_{\alpha}) = [f(x), (\psi_{\beta}^{-1}f\varphi_{\alpha})'(v)]_{\beta} \). One checks, using the chain rule for derivatives on \( \mathbb{R}^n \), that \( f'(\[x, v\]_{\alpha}) \) does not depend on the choice of \( \alpha \) such that \( x \in M_{\alpha} \), and that the resulting map \( f' \) is smooth. It is clear that \( f' \) commutes with the projections. The last assertion follows from the chain rule on \( \mathbb{R}^n \). \( \blacksquare \)

4.2 Vector fields

Let \( M \) be a smooth \( n \)-dimensional manifold. A vector field on \( M \) is a smooth map \( X : M \to TM \) such that \( \pi_M \circ X \) is the identity on \( M \). Intuitively, a vector field is choice of vector \( X(p) \in T_pM \) for each \( p \in M \), such that \( X(p) \) varies smoothly in \( TM \) as \( p \) varies smoothly in \( M \).

Example: Let \( M = S^3 \), viewed inside \( \mathbb{H} = \mathbb{R}^4 \) as before, and let \( \mathbb{H}_0 \) be the subspace of \( \mathbb{H} \) orthogonal to \( \mathbb{R} \). Then for any \( v \in \mathbb{H}_0 \), the function \( X_v(p) = pv \) is a vector field on \( S^3 \). \( \square \)

Let \( G \) be a Lie group. For any \( g \in G \) we have a map

\[
L_g : G \to G, \quad \text{given by } L_g(x) = gx,
\]
called left translation. This map has a derivative \( L'_g : TG \to TG \), mapping each tangent space \( T_eG \) to \( T_{gx}G \).

A vector field \( X \) on \( G \) is called left-invariant if for all \( g \in G \) the following diagram commutes:

\[
\begin{array}{ccc}
TG & \xrightarrow{L'_g} & TG \\
\uparrow X & & \uparrow X \\
G & \xrightarrow{L_g} & G
\end{array}
\]

Let \( g = T_eG \) denote the tangent space to \( G \) at the identity element \( e \in G \). For each \( v \in g \) there is a unique left-invariant vector field \( X_v : G \to TG \), given by

\[
X_v(g) = L'_g(v).
\]

Conversely, if \( X : G \to TG \) is any left-invariant vector field then

\[
X(g) = L'_eX(e),
\]

so \( X = X_v \), where \( v = X(e) \). Thus, the left-invariant vector fields are in canonical bijection with vectors in \( g \).

4.3 The tangent bundle of a Lie group

An \( n \)-dimensional manifold \( M \) is parallelizable if there is diffeomorphism

\[
f : M \times \mathbb{R}^n \sim \to TM
\]
restricting to a linear isomorphism \( \mathbb{R}^n = p \times \mathbb{R}^n \to T_p M \) for each \( p \in M \). Equivalently, \( M \) is parallelizable if there exist \( n \) vector fields \( X_1, \ldots, X_n \) which are linearly independent at each point in \( M \).

For example, \( S^n \) is parallelizable exactly when \( n = 1, 3, 7 \). This is easy to see for \( S^1 \) and for \( S^3 \) the vector fields \( X_i, X_j, X_k \) (see example above) are linearly independent at each point. For \( S^7 \) see section 8.3.

More generally, let \( G \) be any Lie group, and choose a basis \( e_1, \ldots, e_n \) of \( \mathfrak{g} \). Since \( L'_g : \mathfrak{g} \to T_g G \) is an isomorphism, the vector fields \( X_i(g) = L'_g(e_i) \) are linearly independent in \( T_g G \) for each \( g \in G \). This proves:

**Proposition 4.2** Any Lie group is parallelizable.

### 4.4 One-parameter-subgroups

Let \( M \) be an \( n \)-dimensional manifold. A **path** in \( M \) is a smooth map \( \gamma : I \to M \) defined on some open interval in \( I \subset \mathbb{R} \). The derivative of \( \gamma \) is then a map on tangent bundles

\[
\gamma' : TI \longrightarrow TM.
\]

Choose \( I \) small enough so that \( \gamma(I) \subset M_\alpha \) for some chart \( (\varphi_\alpha, U_\alpha, M_\alpha) \) on \( M \). Then \( TI = I \times \mathbb{R} \) and \( \gamma'(TI) \subset T M_\alpha = M_\alpha \times \mathbb{R}^n \cong U_\alpha \times \mathbb{R}^n \). Taking \( 1 \in \mathbb{R} \) as a basis vector, we have

\[
\gamma'(t, 1) = (\gamma(t), \dot{\gamma}(t)),
\]

where \( \dot{\gamma}(t) \) is the componentwise derivative of \( \gamma(t) \subset U_\alpha \subset \mathbb{R}^n \).

Given a smooth vector field \( X : M \to TM \) we seek a path \( \gamma \) in \( M \) such that \( \gamma'(t, 1) = X(\gamma(t)) \). This amounts to solving a differential equation, The following result is a consequence of the local existence and uniqueness theorems for differential equations of one variable.

**Theorem 4.3** Let \( X : M \to TM \) be a smooth vector field and let \( p \in M \). Then there exists \( \epsilon > 0 \) and a unique smooth path \( \gamma : (-\epsilon, \epsilon) \to M \) such that

\[
\gamma'(t, 1) = X(\gamma(t)) \quad \text{for} \quad |t| < \epsilon
\]

and \( \gamma(0) = p \).

We call \( \gamma \) a **local integral** of \( X \). As a corollary of this local uniqueness, it follows that global existence implies global uniqueness.

**Corollary 4.4** Suppose \( X : M \to TM \) is a smooth vector field and \( I \subset \mathbb{R} \) is an open interval containing a closed interval \([a, b]\). Suppose also that we have two smooth paths \( \gamma, \delta : I \to M \) such that

\[
\gamma'(t, 1) = X(\gamma(t)), \quad \delta'(t, 1) = X(\delta(t)), \quad \gamma(a) = \delta(a).
\]

Then \( \gamma(t) = \delta(t) \) for all \( t \in [a, b] \).
Proof: Let \( c = \sup \{ t \in [a, b] : \gamma(t) = \delta(t) \} \). By continuity, we have \( \gamma(c) = \delta(c) \). If \( c < b \) then for all \( \epsilon > 0 \) such that \( (c - \epsilon, c + \epsilon) \subset I \), the paths \( \gamma \) and \( \delta \) on are local integrals of \( X \) on \( (c - \epsilon, c + \epsilon) \), hence they agree here, by the local uniqueness of Thm. 4.3. This contradiction forces \( c = b \). \( \blacksquare \)

Now let \( G \) be a Lie group. A **one-parameter subgroup** of \( G \) is a Lie group homomorphism

\[
\gamma : \mathbb{R} \longrightarrow G.
\]

Note that \( \gamma(0) = e \) so \( \gamma'(0, 1) \in \mathfrak{g} = T_eG \).

**Lemma 4.5** Let \( \gamma : \mathbb{R} \rightarrow G \) be a one-parameter subgroup, with \( v := \gamma'(0, 1) \in \mathfrak{g} \), and let \( X_v : G \rightarrow TG \) be the corresponding left-invariant vector field on \( G \) given by \( X_v(g) = L'_g(v) \). Then for all \( t \in \mathbb{R} \) we have

\[
\gamma'(t, 1) = L'_{\gamma(t)}(v) = X_v(\gamma(t)).
\]

**Proof:** Regarding \( \mathbb{R} \) itself as a Lie group, we can extend the vector \( (0, 1) \in T_0\mathbb{R} \) to a left-invariant vector field \( X_{(0,1)} : \mathbb{R} \rightarrow T\mathbb{R} \), given by \( X_{(0,1)}(t) = L'_t(0,1) \), where \( L_t : \mathbb{R} \rightarrow \mathbb{R} \) is the translation map \( L_t(x) = t + x \). For all \( x \in \mathbb{R} \) we have \( L'_t(x,1) = (t + x,1) \).

Since \( \gamma \) is a homomorphism, the diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\gamma} & G \\
\downarrow{L_t} & & \downarrow{L_{\gamma(t)}} \\
\mathbb{R} & \xrightarrow{\gamma} & G
\end{array}
\]

is commutative. Taking derivatives, we get the commutative diagram

\[
\begin{array}{ccc}
T\mathbb{R} & \xrightarrow{\gamma'} & TG \\
\downarrow{L'_t} & & \downarrow{L'_{\gamma(t)}} \\
T\mathbb{R} & \xrightarrow{\gamma'} & TG
\end{array}
\]

We get

\[
\gamma'(t, 1) = \gamma' \circ L'_t(0,1) = L'_{\gamma(t)}(v) = X_v(\gamma(t)),
\]

proving the lemma. \( \blacksquare \)

Thus, a one-parameter subgroup \( \gamma \) is a global integral of the left-invariant vector field determined by \( \gamma'(0, 1) \in \mathfrak{g} \).

**Theorem 4.6** For each \( v \in \mathfrak{g} \) there is a unique one-parameter subgroup \( \gamma_v : \mathbb{R} \rightarrow G \) such that \( \gamma'(0, 1) = v \).

**Proof:** Suppose \( \gamma \) and \( \delta \) are two one-parameter subgroups such that \( \gamma'(0, 1) = v \). Then \( \gamma(0) = e = \delta(0) \) so \( \gamma(t) = \delta(t) \) for all \( t \geq 0 \) by Cor. 4.4. Since \( \gamma(-t) = \gamma(t)^{-1} \) and likewise for \( \delta \), we have \( \gamma(t) = \delta(t) \) for all \( t \). This proves the uniqueness part of Thm. thm:1psg.
Now let \( v \in \mathfrak{g} \). By the local existence part of Thm. 4.3, there exists \( \epsilon > 0 \) and a smooth path \( \gamma : (-\epsilon, \epsilon) \rightarrow G \) such that \( \gamma_0(0) = e \) and

\[
\gamma'_0(t, 1) = X_v(\gamma_0(t)) \quad \text{for} \quad |t| < \epsilon.
\]

The first step is to prove that \( \gamma_0 \) is a local homomorphism. That is,

\[
\gamma_0(s + t) = \gamma_0(s)\gamma_0(t), \quad \text{for} \quad |s|, |t| < \epsilon/2.
\]

Fix \( s \). Since both sides agree at \( t = 0 \), it is sufficient to show that both sides solve the equation \( f'(t, 1) = X_v(f(t)) \).

For the left side, we have \( \gamma_0(s + t) = \gamma_0 \circ L_s(t) \) and

\[
[\gamma_0 \circ L_s]'(t, 1) = \gamma'_0 \circ L'_s(t, 1) = \gamma'_0(t + s, 1) = X_v(\gamma_0(t + s)).
\]

For the right side, let \( g = \gamma_0(s) \). Then

\[
[\gamma_0(s)\gamma_0(t)]'(t, 1) = [L_g\gamma_0]'(t, 1) = L'_g\gamma'_0(t, 1) = L'_gX_v(\gamma_0(t)) = L'_gL'_{\gamma_0(t)}(v) \quad \text{(by Lemma 4.5)}
\]

\[
= L'_g(\gamma_0(t)) = X_v(g\gamma_0(t)) = X_v(\gamma_0(s)\gamma_0(t)).
\]

Thus \( \gamma_0 \) is indeed a local homomorphism.

The next step is to extend \( \gamma_0 \) to a homomorphism \( \gamma : \mathbb{R} \rightarrow G \). Given \( t \in \mathbb{R} \), choose a positive integer \( N \) such that \( |t/N| < \epsilon/2 \) and set

\[
\gamma(t) = \gamma_0(t/N)^N.
\]

If also \( |t/M| < \epsilon/2 \) then

\[
\gamma_0(t/M)^M = \gamma_0(t/MN)^MN = \gamma_0(t/N)^N,
\]

since \( \gamma_0 \) is a local homomorphism. Hence \( \gamma \) is well-defined. Similarly, \( \gamma \) is a homomorphism: Given \( t, s \in \mathbb{R} \), choose \( N \) such that \( |t/N| \) and \( |s/N| \) are both \( < \epsilon/2 \). Then

\[
\gamma(s + t) = \gamma_0 \left( \frac{s}{N} + \frac{t}{N} \right)^N = \gamma_0(s/N)^N\gamma_0(t/N)^N = \gamma(s)\gamma(t).
\]

Next, \( \gamma \) is smooth: For \( \gamma_0 \) is smooth on \( (-\epsilon, \epsilon) \), so \( \gamma(t) = \gamma_0(t/N) \) is smooth on \( (-\epsilon N, \epsilon N) \). Finally,

\[
\gamma'(0, 1) = \gamma'_0(0, 1) = X_v(\gamma_0(0)) = X_v(e) = v.
\]

This completes the proof the theorem.

For \( v \in \mathfrak{g} \), we let \( \gamma_v : \mathbb{R} \rightarrow G \) be the unique one-parameter subgroup such that \( \gamma'_v(0, 1) = v \).
4.5 The exponential map

Let $G$ be a Lie group. The **Exponential map** for $G$ is the map $\exp = \exp_G : \mathfrak{g} \to G$ defined by

$$\exp(v) = \gamma_v(1).$$

We wish to show that $\exp$ is a smooth map, where the vector space $\mathfrak{g}$ is regarded as a smooth manifold. We must examine how $\gamma_v$ varies with $v$. The simplest situation is when $v$ is scaled.

**Lemma 4.7** For all $s \in \mathbb{R}$, we have $\gamma_{sv}(t) = \gamma_v(st)$.

**Proof:** Let $M_s : \mathbb{R} \to \mathbb{R}$ be the map $M_s(t) = st$. We have $M'_s(0, 1) = (0, s)$, so

$$(\gamma_v \circ M_s)'(0, 1) = \gamma'_v \circ M'_s(0, 1) = \gamma'_v(0, s) = s \cdot \gamma'_v(0, 1) = sv.$$

Since $\gamma_{sv}$ is the unique one-parameter subgroup of $G$ such that $\gamma'_{sv}(0, 1) = sv$, we have $\gamma_v \circ M_s = \gamma_{sv}$ and the lemma follows. $lacklozenge$

Next we consider the smoothness of $v \mapsto \gamma_v$. This requires an extension of the uniqueness and existence theorem to families of vector fields.

**Theorem 4.8** Let $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open neighborhood of $(0, 0)$ and let $X : U \times V \to \mathbb{R}^n$ be a smooth map. Then there exists $\epsilon > 0$, a neighborhood $V_0$ of 0 in $V$ and a unique smooth map $f : (-\epsilon, \epsilon) \times V_0 \to U$ such that for all $v \in V_0$ the function $f_v(t) := f(t, v)$ satisfies the differential equation

$$f'_v(t) = X(f_v(t), v), \quad \text{for } |t| < \epsilon.$$

**Proof:** See references in Adams, page 8. $lacklozenge$

**Proposition 4.9** Let $G$ be a Lie group and let $\mathfrak{g} = T_eG$. The exponential map $\exp : \mathfrak{g} \to G$, defined by $\exp(v) = \gamma_v(1)$, is smooth.

**Proof:** Recall that $G$ is parallelizable, via the map

$$X : G \times \mathfrak{g} \to TG \quad \text{given by} \quad X(g, v) = L'_g(v).$$

The one-parameter subgroup $\gamma_v(t)$ solves the differential equation

$$\gamma'_v(t, 1) = X(\gamma_v(t), v).$$

By Thm. 4.8 the map $(t, v) \mapsto \gamma_v(t)$ is smooth on some neighborhood $(-\epsilon, \epsilon) \times V_0$ of $(0, 0)$ in $\mathbb{R} \times \mathfrak{g}$. Then

$$\gamma_N^{-1}v(t/N)^N = \gamma_v(t)$$
is smooth on \((-N\epsilon, N\epsilon) \times NV_0\), so \((t, v) \mapsto \gamma_v(t)\) is smooth on \(\mathbb{R} \times g\). Taking \(t = 1\) we see that \(\exp\) is smooth. ■

**Remark 1:** For each \(v \in g\), the image \(\exp(v) = \gamma_v(1) \in G\) is a value of the one parameter subgroup \(\gamma_v\). In fact all values of \(\gamma_v\) are obtained from \(\exp\). For Lemma 4.7 implies that

\[
\gamma_v(t) = \gamma_{tv}(1) = \exp(tv).
\]

**Remark 2:** Beware that \(\exp\) is not a group homomorphism if \(G\) is non-abelian.

We next show that the exponential map is functorial. Let \(\varphi : G \to H\) be a Lie group homomorphism and let \(g = T_eG\), \(h = T_eH\) be the respective tangent spaces at the identity elements of \(G\) and \(H\). Let

\[
\varphi' : g \to h
\]

be the derivative of \(\varphi\). This is a linear map obtained by restricting \(\varphi' : TG \to TH\) to \(g\).

**Proposition 4.10** The following diagram is commutative:

\[
\begin{array}{ccc}
g & \xrightarrow{\varphi'} & h \\
\exp_G \downarrow & & \downarrow \exp_H \\
G & \xrightarrow{\varphi} & H
\end{array}
\]

**Proof:** In the proof we write \(\varphi'\) instead of \(\varphi'_0\). For all \(v \in g\) we have

\[
(\varphi \circ \gamma_v)'(0, 1) = \varphi' \circ \gamma'_v(0, 1) = \varphi'(v),
\]

so \(\varphi \circ \gamma_v = \gamma_{\varphi(v)}\) by Thm. 4.6. Now

\[
\varphi(\exp_G(v)) = \varphi(\gamma_v(1)) = \gamma_{\varphi(v)}(1) = \exp_H(\varphi'(v)),
\]

as claimed. ■

This is the most useful result in the theory of Lie groups. 5

**Example:** Let \(G = \text{GL}_n(\mathbb{R})\). Since \(G\) is an open subset of \(M_n(\mathbb{R})\) we canonically identify \(g = M_n(\mathbb{R})\). Take \(A \in g\) and consider the path \(\gamma : \mathbb{R} \to G\) given by

\[
\gamma(t) = e^{tA} = I + tA + \frac{t^2}{2}A^2 + \cdots \in G.
\]

Since \(\gamma'(0) = A\), it follows that \(\gamma = \gamma_A\) and we have \(\exp_G(A) = e^A\). Thus, the exponential map for \(\text{GL}_n(\mathbb{R})\) is the familiar exponential map of matrices. □

**Proposition 4.11** Let \(G\) be a Lie group. Then there exists a neighborhood \(U\) of \(0\) in \(g\) mapped diffeomorphically by \(\exp_G\) onto a neighborhood of \(e\) in \(G\).

5This claim is really a challenge to find a result that is even more useful.
Proof: We need another result from analysis. \(^6\)

**Theorem 4.12 (Inverse Function Theorem)** Let \(U\) be an open subset of \(\mathbb{R}^n\) and let \(p \in U\). Let \(f : U \to \mathbb{R}^n\) be a smooth map whose derivative

\[ f'_p = \left[ \frac{\partial f_i}{\partial x_j}(p) \right] \]

is invertible at \(p\). Then there exists a neighborhood \(U_0\) of \(p\) mapped diffeomorphically by \(f\) onto an open neighborhood of \(f(p)\) in \(\mathbb{R}^n\).

The smooth map \(\exp : \mathfrak{g} \to G\) has derivative \(\exp'_G : T\mathfrak{g} \to TG\). Restricting to \(T_0\mathfrak{g} = \mathfrak{g}\), we get a linear map \(\exp'_0 : \mathfrak{g} \to \mathfrak{g}\). Let \(v \in \mathfrak{g}\), and let \(h : \mathbb{R} \to \mathfrak{g}\) be the map \(h(t) = tv\). Then \(\gamma_v = \exp \circ h\), so

\[ \exp'_0(v) = \exp'_0(h'(0)) = \gamma'_v(0) = v, \]

so \(\exp'_0\) is the identity map, which is invertible. Prop. 4.11 now follows from the Inverse Function Theorem. \(\blacksquare\)

**Example:** Again let \(G = \text{GL}_n(\mathbb{R})\) with \(\mathfrak{g} = M_n(\mathbb{R})\). A neighborhood of \(I\) in \(G\) is of the form \(I + U\), where \(U\) is a neighborhood of \(0\) in \(\mathfrak{g}\). The series

\[ \log(I + u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \cdots \]

converges for \(u\) near 0 (e.g., for \(|u| < 1\) when \(n = 1\)) and inverts the exponential map: \(\exp(\log(I + u)) = u\) for \(u\) near 0. \(\square\)

Prop. 4.11 shows that the exponential map parametrizes an open neighborhood of the identity in \(G\). It says nothing about how large this neighborhood is, but that does not matter, thanks to the following.

**Proposition 4.13** A connected Lie group \(G\) is generated by any open neighborhood of the identity. In particular \(G\) is generated by \(\exp(\mathfrak{g})\).

**Proof:** This result relies only on the continuity of the group laws, and not on the smooth structure. Let \(U\) be an open neighborhood of \(e\) in a connected Lie group \(G\) and let \(H\) be the subgroup of \(G\) generated by \(U\). Since the multiplication in \(G\) is continuous, \(gU\) is an open neighborhood of \(g\), for all \(g \in G\). Now

\[ H = \bigcup_{h \in H} hU \quad \text{and} \quad G - H = \bigcup_{g \in G - H} gU \]

are both open in \(G\), so \(H\) is both open and closed and nonempty (we have \(e \in H\)). As \(G\) is connected, it follows that \(H = G\).

From Prop. 4.11 we have that \(\exp(\mathfrak{g})\) contains an open neighborhood of \(e\), which generates \(G\), so \(\exp(\mathfrak{g})\) generates \(G\). \(\blacksquare\)

---

\(^6\)See Rudin *Principles of Mathematical Analysis.*
Remark: We will often find that $\exp(\mathfrak{g}) = G$, for example if $G$ is compact. However it is not always so, even if $G$ is connected. For example if $G = \text{SL}_2(\mathbb{R})$ then the image of $\exp$ consists of matrices $\exp(A)$ where $\text{tr}(A) = 0$. The eigenvalues $\lambda, \mu$ of $A \in \mathfrak{sl}_2(\mathbb{R})$ satisfy $\mu + \lambda = \text{tr}(A) = 0$ and $\lambda\mu = \det(A) \in \mathbb{R}$. Hence $\mu = -\lambda$ and $\lambda^2 \in \mathbb{R}$. It follows that $\lambda$ is either real or purely imaginary, and the eigenvalues $e^{\pm\lambda}$ of $A$ lie on the positive real axis or the unit circle, respectively. Thus, for example the matrix
\[
\begin{pmatrix}
-2 & 0 \\
0 & -1/2
\end{pmatrix} \in \text{SL}_2(\mathbb{R})
\]
is not in the image of $\exp : \mathfrak{sl}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{R})$.

**Corollary 4.14** Let $G$ and $H$ be Lie groups with $G$ connected. Then any Lie group homomorphism $\varphi : G \to H$ is determined by its derivative $\varphi'_0 : \mathfrak{g} \to \mathfrak{h}$.

**Proof:** If $\varphi$ and $\psi$ are two Lie group homomorphisms from $G \to H$ with $\varphi'_0 = \psi'_0$ then applying functorality Prop. 4.10, we have
\[
\varphi \circ \exp_G = \exp_H \circ \varphi'_0 = \exp_H \circ \psi'_0 = \psi \circ \exp_G.
\]
From Prop. 4.11 it follows that $\varphi$ and $\psi$ agree on a neighborhood of $e$ in $G$. From Prop. 4.13 it follows that $\varphi$ and $\psi$ agree everywhere. ■

**Corollary 4.15** Let $\varphi : G \to H$ be a Lie group homomorphism and assume $H$ is connected. If $\varphi' : \mathfrak{g} \to \mathfrak{h}$ is surjective then $\varphi$ is surjective.

**Proof:** Since $H$ is connected it is generated by $\exp_H(\mathfrak{h})$, by Prop. 4.13. Since $\varphi'$ is surjective we have $\exp_H(\mathfrak{h}) = \exp_H(\varphi'(\mathfrak{g})) = \varphi(\exp_G(\mathfrak{g}))$. Hence the image of $\varphi$ generates $H$, so $\varphi$ is surjective. ■

### 4.6 The Adjoint representation

A **representation** of a group $G$ is a homomorphism
\[
\rho : G \longrightarrow \text{GL}(V), \tag{6}
\]
where $V$ is a vector space and $\text{GL}(V) = \text{Aut}(V)$ is the group of linear automorphisms of $V$. If $V = \mathbb{R}^n$ for some $n$ then a choice of basis of $V$ gives an isomorphism $\text{GL}(V) \simeq \text{GL}_n(\mathbb{R})$, so a finite dimensional representation may be regarded as a homomorphism
\[
\rho : G \longrightarrow \text{GL}_n(\mathbb{R}).
\]
However in most situations there is no natural choice of basis and it is better to think of a representation in the form (6).

A Lie group $G$ has a canonical representation
\[
\text{Ad} : G \to \text{GL}(\mathfrak{g})
\]
called the **adjoint representation**, defined as follows. Take an element \( g \in G \) and let \( c_g : G \to G \) be the conjugation map:

\[
c_g(x) = gxg^{-1}.
\]

Since \( c_g(e) = e \), the derivative \( \text{Ad}(g) := c'_g \) maps \( g \to g \). And since \( c_g \) is a homomorphism, functorality Prop. 4.10 gives a commutative diagram

\[
\begin{array}{ccc}
g & \xrightarrow{\text{Ad}(g)} & g \\
\exp & \downarrow & \downarrow \exp \\
G & \xrightarrow{c_g} & G
\end{array}
\]

Since \( c_{gh} = c_g \circ c_h \), it follows that \( \text{Ad}(gh) = \text{Ad}(g) \circ \text{Ad}(h) \). Thus we have a representation

\[
\text{Ad}_G = \text{Ad} : G \longrightarrow \text{GL}(g), \quad g \mapsto \text{Ad}(g).
\]

For \( v \in g \), the image \( \text{Ad}(g)v \) is the initial tangent direction of the path \( t \mapsto g \exp(t)g^{-1} \) in \( G \). Thus, we have

\[
\text{Ad}(g)v = \frac{d}{dt} \left( g \exp(tvg^{-1}) \right) \big|_{t=0}.
\]

Now \( \text{Ad} \) itself is a Lie group homomorphism; its derivative at \( e \) is the linear map

\[
ad : g \to \text{End}(g), \quad u \mapsto \text{ad}(u) = \text{Ad}'(u).
\]

For \( u, v \in g \), the image \( \text{ad}(u)(v) \) is the initial tangent direction of the path \( t \mapsto \text{Ad}(\exp(tu))v \) in \( g \). Thus, we have

\[
ad(u)v = \frac{d}{dt} \left( \text{Ad}(\exp(tu))v \right) \big|_{t=0}.
\]

Functorality relates \( \text{Ad} \) and \( \text{ad} \) via the commutative diagram

\[
\begin{array}{ccc}
g & \xrightarrow{\text{ad}(g)} & \text{End}(g) \\
\exp_G & \downarrow & \downarrow \exp_{\text{GL}(g)} \\
G & \xrightarrow{\text{Ad}} & \text{GL}(g)
\end{array}
\]

Recall that \( \exp_{\text{GL}(g)}(A) = e^A = I + A + \frac{1}{2}A^2 + \cdots \). Thus, the diagram expresses the equality

\[
\text{Ad}(\exp(x)) = e^{\text{ad}(x)} \in \text{GL}(g).
\]

It is convenient to use the following alternative notation for \( \text{ad} \). The **Lie bracket** is the mapping

\[
[ , ] : g \times g \longrightarrow g, \quad \text{given by} \quad [u, v] = \text{ad}(u)(v).
\]

(7)

So the adjoint representation \( \text{Ad} : G \to \text{GL}(g) \) is given by

\[
\text{Ad}(\exp(u))(v) = e^{\text{ad}(u)}(v) = v + [u, v] + \frac{1}{2}[u, [u, v]] + \cdots.
\]

The adjoint representation \( \text{Ad} \) and its derivative \( \text{ad} \) are functorial.
Proposition 4.16 Let $\varphi : G \to H$ be a Lie group homomorphism. Then for every $g \in G$ the following diagram commutes

$$\begin{align*}
\begin{array}{c}
g \xrightarrow{\varphi'} h \\
\downarrow \text{Ad}_G(g) \quad \downarrow \text{Ad}_H(\varphi(g))
\end{array}
\end{align*}$$

and for all $u, v \in \mathfrak{g}$ we have

$$[\varphi'(u), \varphi'(v)] = \varphi'([u, v]).$$

Proof: For $\text{Ad}$ we compute

$$\varphi' \circ \text{Ad}_G(g) = \varphi' \circ c_g' = (\varphi \circ c_g)' = c_{\varphi(g)}' \circ \varphi' = \text{Ad}_H(\varphi(g)) \circ \varphi',$$

making the diagram commute as claimed. Next we compare two power series in $t$, namely

$$\varphi'(\text{Ad}_G(\exp(tu)v) = \varphi'(v + t[u, v] + \cdots) = \varphi'(v) + t\varphi'([u, v]) + \cdots,$$

which is equal to

$$\text{Ad}_H(\varphi(\exp(tu)))\varphi'(v) = \varphi'(v) + t[\varphi'(u), \varphi'(v)] + \cdots.$$

Comparing coefficients in $t$ we obtain $[\varphi'(u), \varphi'(v)] = \varphi'([u, v])$. ■

Finally, the Adjoint representation is as faithful as possible.

Proposition 4.17 If $G$ is a connected Lie group then the kernel of $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ is the center of $G$.

Proof: If $z$ is in the center of $G$ then $c_z$ is the trivial automorphism of $G$ so its derivative $\text{Ad}(z)$ is the identity map $I$. Conversely, for any $g \in G$ and $v \in \mathfrak{g}$ we have

$$g \exp(v)g^{-1} = \exp(\text{Ad}(g)v).$$

If $\text{Ad}(g) = I$ then $g$ centralizes $\exp(\mathfrak{g})$, which generates $G$, so $g$ is in the center of $G$. ■

4.6.1 The product rule for paths

At various points we will need to differentiate a product of two paths in a Lie group. We install the result now for later use.

Let $G$ be a Lie group and let $\gamma, \delta : \mathbb{R} \to G$ be two smooth paths in $G$. We then have a product path $\gamma \delta$, given by $(\gamma \delta)(t) = \gamma(t)\delta(t)$. Our goal is to express the derivative $(\gamma \delta)'$ in terms of $\gamma, \gamma', \delta, \delta'$. We will write $\gamma'(t)$ instead of $\gamma'(t, 1)$ etc.

For $g, h \in G$ let $L_g, R_h : G \to G$ be the maps $L_g(x) = gx$ and $R_h(x) = xh$. Associativity of the group law on $G$ means that the maps $L_g$ and $R_h$ commute. The product rule is as follows.
Proposition 4.18  For all $t \in \mathbb{R}$ we have

$$(\gamma \delta)'(t) = L'_{\gamma(t)} \delta'(t) + R'_{\delta(t)} \gamma'(t) \in G_{\gamma \delta(t)}$$

**Proof:** Let $\mu : G \times G \to G$ be the product map: $\mu(g, h) = gh$. We compute

$$\mu' : T_gG \oplus T_hG \to T_{gh}G$$

as follows. A typical element of $T_gG$ is of the form $L'_g(v)$, where $v \in g = T_eG$. Then

$$\mu'(L'_g(v), 0) = \frac{d}{dt}\mu(g \exp(tv), h)|_{t=0}$$

$$= \frac{d}{dt} [g \exp(tv)h]|_{t=0}$$

$$= \frac{d}{dt} [L_g R_h(\exp(tv))]|_{t=0}$$

$$= (L_g R_h)'(v)$$

$$= R'_h \circ L'_g(v)$$

since $R_h$ and $L_g$ commute. Likewise,

$$\mu'(0, R'_h u) = L'_g \circ R'_h(u).$$

It follows that for all $x \in T_gG$ and $y \in T_hG$ we have

$$\mu'(x, y) = \mu'(x, 0) + \mu'(0, y) = R'_h(x) + L'_g(y).$$

Now $\gamma \delta = \mu(\gamma, \delta)$ so the chain rule gives

$$(\gamma \delta)'(t) = \mu'(\gamma'(t), \delta'(t)) = L'_{\gamma(t)}(\delta'(t)) + R'_{\delta(t)}(\gamma'(t)),$$

as claimed. $\blacksquare$

We use this to express the bracket $[\cdot , \cdot]$ on $g$ as the derivative of a commutator of one-parameter subgroups in $G$. Let $[g, h] = ghg^{-1}h^{-1}$ be the commutator in $G$.

Proposition 4.19  For all $u, v \in g$ we have

$$[u, v] = \frac{d}{dt} \left( \frac{d}{ds} [\exp(tu), \exp(tv)]_{s=0} \right)_{t=0}.$$

**Proof:** Using the product rule for the paths $\gamma(s) = g(\exp(sv)g^{-1}$ and $\delta(s) = \exp(sv)^{-1} = \exp(-sv)$, we find that

$$\frac{d}{ds} [g, \exp(sv)]_{s=0} = \text{Ad}(g)v - v,$$

and the result follows. $\blacksquare$
Corollary 4.20 For all \( u, v \in \mathfrak{g} \) we have

\[
[u, v] = -[v, u].
\]

Proof: A similar argument shows that

\[
\frac{d}{dt} \{\exp(tu), h\}_{t=0} = u = \text{Ad}(h)u.
\]

Interchanging the order of differentiation gives the result.

4.7 The Lie algebra

A Lie algebra is a vector space \( L \) over a field \( F \) together with a map

\[
[,]: L \times L \to L
\]

called the bracket, satisfying the following three properties for all \( x, y, z, w \in L \) and \( a, b, c, d \in F \):

- (bilinearity) \( [ax + by, cz + dw] = ac[x, z] + ad[x, w] + bc[y, z] + bd[u, w] \);
- (skew-symmetry) \( [x, y] = -[y, x] \);
- (Jacobi-identity) \( [x, [y, z]] = [[x, y], z] + [y, [x, z]] \).

A homomorphism of Lie algebras \( L, M \) is a linear map \( \alpha : L \to M \) preserving the bracket:

\[
\alpha([x, y]) = [\alpha(x), \alpha(y)], \quad \text{for all} \quad x, y \in L.
\]

We let \( \text{Aut}(L) \) denote the group of automorphisms of \( L \).

Proposition 4.21 Let \( G \) be a Lie group. Then the tangent space \( \mathfrak{g} = T_eG \), with bracket \([ , ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) as defined in (7), is a Lie algebra.

Proof: Since \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is linear, it follows that the bracket is bilinear. Skew-symmetry was proved in Cor. 4.20. It remains to prove the Jacobi identity. Fix \( t \in \mathbb{R} \) and \( u \in \mathfrak{g} \) and consider the map \( \varphi : G \to G \) given by conjugation by \( \exp(tu) \). We find, for any \( v \in \mathfrak{g} \):

\[
\varphi'(v) = \text{Ad}(\exp(tu))v = v + t[u, v] + \cdots,
\]

so

\[
\varphi'([u, v]) = [v, w] + t[u, [v, w]] + \cdots,
\]

but by functoriality of the Lie bracket (Prop. 4.16) this is equal to

\[
[\varphi'(v), \varphi'(w)] = [v, w] + t([[u, v]w] + [v, [u, w]]) + \cdots.
\]

Comparing coefficients of \( t \) gives the Jacobi identity.

From now on we call \( \mathfrak{g} = T_eG \), with bracket defined in (7), the Lie algebra of the Lie group \( G \).
Proposition 4.22 Let $G, H$ be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, and let $\varphi : G \to H$ be a Lie group homomorphism. Then $\varphi' : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. In particular, $\text{Ad}(G)$ is contained in the automorphism group $\text{Aut}(\mathfrak{g})$ of $\mathfrak{g}$.

Proof: The first assertion was proved in Prop. 4.16 and the second is an immediate consequence. ■

If $\varphi$ is injective (or more generally has discrete kernel) then by functorality Prop. 4.10 we have $\varphi'$ injective and we can identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{h}$. In fact, we have

$$\mathfrak{g} = \{ x \in \mathfrak{h} : \exp_{H}(tx) \in \varphi(G) \text{ for all } t \in \mathbb{R} \}. \quad (8)$$

The containment $\subseteq$ follows from functorality. We will prove the other containment in Cor. 6.5 below, under the additional assumption that $\varphi(G)$ is closed in $H$. (This is no additional assumption if $G$ is compact. For the general case see [Warner, 3.33]). For now, let us use (8) to compute the Lie algebras of the groups in section 3.3.

Example 1: Let $G = \text{GL}_n(\mathbb{R})$. We have seen that $\mathfrak{g} = M_n(\mathbb{R})$ and that the exponential map is the matrix exponential $\exp(A) = I + A + \frac{1}{2}A^2 + \cdots$. To compute the Lie bracket we can use Prop. 4.19. This says that for $A, B \in \mathfrak{g}$, the bracket $[A, B]$ is the coefficient of $st$ in the group commutator

$$[\exp(tA), \exp(sB)] = (I + tA + \cdots)(I + sB + \cdots)(I - tA + \cdots)(I - sB + \cdots),$$

namely,

$$[A, B] = AB - BA.$$

Example 2: Let $G = \text{SL}_n(\mathbb{R})$, a subgroup of $H = \text{GL}_n(\mathbb{R})$. By (8) the Lie algebra $\mathfrak{g} = \text{sl}_n(\mathbb{R})$ consists of matrices $A \in M_n(\mathbb{R})$ for which $\det(\exp(tA)) = 1$. As $\det(\exp(tA)) = e^{t \text{tr}(A)}$, it follows that $\text{sl}_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \text{tr}(A) = 0 \}$.

Examples 3,4: Let $G = \text{O}_n$ or $\text{SO}_n$. Since $\exp(\mathfrak{g})$ is connected, it must lie in $\text{SO}_n$ and both groups have the same Lie algebra. $\mathfrak{g} = \text{so}_n$ consisting of matrices $A \in M_n(\mathbb{R})$ such that $\exp(-sA)$ is the transpose of $\exp(sA)$ for all $s \in \mathbb{R}$. The transpose of an exponential matrix is the exponential of the transpose: $^t \exp(sA) = \exp(s \cdot ^t A)$. It follows that

$$\text{so}_n = \{ A \in M_n(\mathbb{C}) : A + ^t A = 0 \}.$$

Example 5: Let $G = H_3$, the Heisenberg group in $\text{GL}_3(\mathbb{R})$. Here it is easiest to directly compute $\mathfrak{g}$ as the set of all tangent vectors to paths $\gamma : \mathbb{R} \to G$. Any path is of the form

$$\gamma(t) = \begin{pmatrix} 1 & x(t) & z(t) \\ 0 & 1 & y(t) \\ 0 & 0 & 1 \end{pmatrix}$$

and has derivative

$$\gamma'(0) = \begin{pmatrix} 0 & x'(0) & z'(0) \\ 0 & 0 & y'(0) \\ 0 & 0 & 0 \end{pmatrix}.$$
Hence the Lie algebra is
\[ h_3 = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \]

We note that \( A^3 = 0 \) for all matrices \( A \in h_3 \), so the exponential map is given by
\[ \exp(A) = I + A + \frac{1}{2} A^2, \]
with inverse
\[ \log(I + A) = A - \frac{1}{2} A^2. \]

### 5 Abelian Lie groups

If a Lie group \( G \) is abelian, then the adjoint representation is trivial and the Lie bracket on \( \mathfrak{g} \) is zero. Nevertheless, abelian Lie groups are important for understanding the structure of non-abelian Lie groups.

**Lemma 5.1** If \( G \) is an abelian Lie group then the map \( \exp_G : \mathfrak{g} \to G \) is a group homomorphism.

Two examples of abelian Lie groups are \( \mathbb{R} \) and \( S^1 \). The main result says that these two account for all connected abelian Lie groups.

**Proposition 5.2** Let \( G \) be a connected abelian Lie group of dimension \( n \). Then there is \( 0 \leq k \leq n \) such that \( G \) is isomorphic to \((S^1)^k \times \mathbb{R}^{n-k}\) as Lie groups.

**Proof:** Let \( u, v \in \mathfrak{g} \) and define \( \gamma : \mathbb{R} \to G \) by
\[ \gamma(t) = \exp(tu) \exp(tv). \]

Then
\[ \gamma(t + s) = \exp((t + s)u) \exp((t + s)v) = \exp(tu) \exp(su) \exp(tv) \exp(sv). \]

Since \( G \) is abelian, this is equal to
\[ \exp(tu) \exp(tv) \exp(su) \exp(sv) = \gamma(t) \gamma(s). \]

Therefore \( \gamma \) is a one-parameter subgroup. From the product formula we have \( \gamma'(0) = \gamma_u'(0) + \gamma_v'(0) = u + v \), so in fact \( \gamma(t) = \exp(t(u + v)) \). Therefore, we have
\[ \exp(tu) \exp(tv) = \exp(tu + tv) \]
so \( \exp \) is a group homomorphism, as claimed.  \( \blacksquare \)
Let \( L = \ker \exp \). Since \( \exp \) is injective on a neighborhood of \( 0 \in \mathfrak{g} \), it follows that \( L \) is a discrete subgroup of \( \mathfrak{g} \). We may write \( L = \mathbb{Z}e_1 \oplus \cdots \mathbb{Z}e_k \) for some \( k \leq n \), and extend to a basis \( \{e_1, \ldots, e_n\} \) of \( \mathfrak{g} \). This gives an isomorphism \( \mathfrak{g} \cong \mathbb{R}^n \) sending \( L \to \mathbb{Z}^k \oplus 0^{n-k} \). Hence

\[
G \cong \mathfrak{g}/L \cong \mathbb{R}^k \oplus \mathbb{R}^{n-k} \cong (S^1)^k \times \mathbb{R}^{n-k}.
\]

A torus is a Lie group which is isomorphic to a product \((S^1)^n\) for some integer \( n \). From Prop. 5.2 we have the immediate

**Corollary 5.3** Every compact abelian Lie group is a torus.

Since \( S^1 = \mathbb{R}/\mathbb{Z} \), a torus \( T \) of dimension \( n \) is isomorphic to \( \mathbb{R}^n/\mathbb{Z}^n \). The Lie algebra \( \mathfrak{t} \) of \( T \) is \( \mathbb{R}^n \) with Lie bracket identically zero, and the exponential map is the projection \( \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \).

A key property of tori is that they are **topologically cyclic**. That is, a torus contains a dense cyclic subgroup.

**Proposition 5.4** Let \( T \) be a torus. Then there exists \( t \in T \) whose powers \( \{t^n : n \in \mathbb{Z}\} \) are dense in \( T \).

**Proof:** Let \( T = \mathbb{R}^n/\mathbb{Z}^n \) with projection map \( \pi : \mathbb{R}^n \to T \) and choose a countable basis \( \{U_1, U_2, \ldots\} \) for the topology on \( T \). A **cube** in \( \mathbb{R}^n \) is a product of closed intervals \( [a_1, b_1] \times \cdots \times [a_n, b_n] \) of constant side length \( |a_i - b_i| \).

Let \( C_0 \subseteq \mathbb{R}^n \) be any cube. Inductively define a nested sequence of cubes \( C_0 \supset C_1 \supset \cdots \) as follows. If \( C_0 \supset \cdots \supset C_{k-1} \) has been defined, let \( \epsilon \) be the side length of \( C_{k-1} \) and choose an integer \( N = N(m) > 1/\epsilon \). Then \( NC_{k-1} \) is a cube of side > 1, so \( \pi(NC_{k-1}) = T \). The map \( \pi N \), given by the composition

\[
C_{k-1} \xrightarrow{N} NC_{k-1} \xrightarrow{\pi} T
\]

is continuous, so \((\pi N)^{-1}(U_k)\) is an open subset of \( \mathbb{R}^n \) contained in \( C_{k-1} \). Then \( C_k \) is chosen to be any cube contained in \((\pi N)^{-1}(U_k)\).

Take \( x \in \bigcap_k C_k \) and let \( t = \pi(x) \in T \). For all \( k > 1 \) we have

\[
t^{N(k)} = \pi N_k(x) \in \pi N_k(C_k) \subset U_k.
\]

Hence the powers \( \{t, t^2, \ldots\} \) meet every open set in \( T \).

An element of a torus \( T \) whose powers are dense in \( T \) is called a **topological generator** of \( T \).
6 Subgroups of Lie groups

6.1 Closed subgroups

Let $M$ be an $n$-dimensional manifold and let $S$ be a subset of $M$. We say that $S$ is a submanifold of $M$ if there exists a subspace $W \subset \mathbb{R}^n$ and an atlas $\{ (\varphi_\alpha, U_\alpha, M_\alpha) : \alpha \in A \}$ in the smooth structure on $M$ such that for all $\alpha \in A$ we have either

- $S \cap M_\alpha = \emptyset$, or
- $\varphi_\alpha(W \cap U_\alpha) = S \cap M_\alpha$.

We say that such an atlas is good for $S$.

Remark 1: If $S$ is a submanifold of $M$ then $S$ is a manifold, with charts $(\varphi_\alpha, W \cap U_\alpha, S \cap M_\alpha)$ where $S \cap M_\alpha \neq \emptyset$.

Remark 2: If $G$ is a Lie group and $H \subset G$ is a submanifold and a subgroup then $H$ is a Lie group. However, there are subgroups of Lie groups which are Lie groups but not submanifolds. An example is $G = \mathbb{R}^2 \mathbb{Z}^2$ and $H = \{ [t\sqrt{2}, t] : t \in \mathbb{R} \}$. Here $H \simeq \mathbb{R}$ is dense in $G$.

Theorem 6.1 A closed subgroup of a Lie group is a submanifold.

Proof: Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Put a metric $|x|$ on $\mathfrak{g}$, say by choosing a basis of $\mathfrak{g}$ and declaring it to be orthonormal.

Let $H$ be a subgroup of $G$ which is closed in $G$. The proof requires three lemmas.

We say a vector $v \in \mathfrak{g}$ is “known by $H$” if there exists a sequence of nonzero vectors $(x_n) \subset \mathfrak{g}$ such that $\exp(x_n) \in H$ for all $n$, $x_n \to 0$ and $|x_n|^{-1}x_n \to v$. Define

$$\mathfrak{h} := \{ tv : t \in \mathbb{R} \text{ and } v \text{ is known by } H \}.$$ 

Lemma 6.2 We have $\exp(\mathfrak{h}) \subset H$.

Proof: Let $tv \in \mathfrak{h}$ and let $(x_n)$ be a sequence for $v$ as above. Choose integers $m_n$ such that $m_n = t/|x_n| + r_n$, where $0 \leq r_n < 1$. Then $m_n|x_n| \to t m_n x_n = m_n|x_n| \cdot |x_n|^{-1}x_n \to tv$. We have $\exp(m_n x_n) = \exp(x_n)^{m_n} \in H$ for all $n$, so $\exp(m_n x_n) \to \exp(tv) \in H$ since $H$ is closed.

Lemma 6.3 $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$.

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There are various definitions of submanifold. The one used here is usually called “embedded submanifold”. 

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We have $f(0) = 0$ and $\exp(f(t)) = \exp(tx) \exp(ty)$. Differentiating at $t = 0$, we get

$$f'(0) = x + y.$$ 

Let $x_n = f(1/n)$, for $n > 1/\epsilon$. Then $\exp(x_n) = \exp(x/n) \exp(y/n) \in H$ by Lemma 6.2, $x_n \to f(0) = 0$ and

$$|x_n|^{-1} x_n = \left| \frac{1/n}{f(1/n)} \right| \cdot \frac{1}{1/n} f(1/n) \to \frac{1}{|x+y|} \cdot (x+y).$$

Hence the latter is known by $H$ so $x + y \in \mathfrak{h}$. 

Choose a vector space complement $m$ to $\mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{h} \oplus m$. Define $\varphi : \mathfrak{g} \to G$ by $\varphi(x,y) = \exp(x) \exp(y)$ where $x \in \mathfrak{h}$ and $y \in m$. Since $\varphi = \exp$ on each summand $\mathfrak{h}$ and $m$, it follows that $\varphi' = I$. Hence there is a neighborhood $0 \in U_0 \subset \mathfrak{g}$ such that $\varphi : U_0 \to \varphi(U_0)$ is a diffeomorphism onto an open neighborhood $\varphi(U_0)$ of $e$ in $G$.

**Lemma 6.4** There exists a neighborhood $e \in V \subset \varphi(U_0)$ in $G$ such that $H \cap V \subset \varphi(\mathfrak{h} \cap U_0)$.

**Proof:** If no such $V$ exists then there is a sequence $h_n \to e$ in $H \cap \varphi(U_0)$ such that $\varphi^{-1}(h_n) \not\subset n\mathfrak{h}$ for all $n$. We have $\varphi^{-1}(h_n) = (x_n, y_n)$ with $x_n \in \mathfrak{h}$ and $y_n \in m$ with $y_n \neq 0$ for all $n$. However, $h_n = \exp(x_n) \exp(y_n)$, so $\exp(y_n) \in H$ for all $n$. Since $h_n \to e$ we have $(x_n, y_n) \to (0,0)$. The sequence $|y_n|^{-1} y_n$ is bounded, hence has a convergent subsequence $(u_{nk})$, whose limit $u \in m$ satisfies $|u| = 1$. But since $\exp(u_{nk}) \in H$ and $u_{nk} \neq 0$, it follows that the subsequence $y_{nk}$ is known by $H$ and therefore $u \in \mathfrak{h}$. Hence $u \in \mathfrak{h} \cap m = \{0\}$, so $u = 0$, a contradiction.

We can now prove Thm. 6.1. Let $U = \varphi^{-1}(V) \subset U_0$. Then $H \cap V = \varphi(\mathfrak{h} \cap U)$. On $G$ define an atlas $\{(\varphi_g, U_g, G_g) : g \in G\}$ as follows.

For $h \in H$, set

$$U_h = U, \quad \varphi_h(u) = h \cdot \varphi(u), \quad G_h = hV.$$ 

Then $\varphi_h(\mathfrak{h} \cap U_h) = h \cdot \varphi(\mathfrak{h} \cap U) = H \cap G_h$. 

For $g \notin H$, choose a neighborhood $U_g$ of $0$ in $U$ such that $g\varphi(U_g) \cap H = \varnothing$, using again the assumption that $H$ is closed. Then set $G_g = g \cdot \varphi(U_g)$ and define $\varphi_g : U_g \to G_g$ by $\varphi_g(u) = g \cdot \varphi(u)$. Then $G_g \cap H = \varnothing$. This shows that $\{(\varphi_g, U_g, G_g) : g \in G\}$ is a good atlas for $H$, so $H$ is a submanifold of $G$. 

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Corollary 6.5 If $H$ is a closed subgroup of $G$ then $H$ is a Lie group with Lie algebra

$$\mathfrak{h} = \{ v \in \mathfrak{g} : \exp(tv) \in H \text{ for all } t \in \mathbb{R} \},$$

and the inclusion map $H \hookrightarrow G$ is a diffeomorphism.

Proof: The proof above constructed an atlas on $H$ given by $\{(\varphi_h, U_h, H \cap G_h) : h \in H\}$. As this came from an atlas on $G$, the inclusion $H \hookrightarrow G$ is a diffeomorphism. We showed in Lemma 6.2 that $\mathfrak{h} \subset \{ v \in \mathfrak{g} : \exp(tv) \in H \text{ for all } t \in \mathbb{R} \}$. For the reverse containment, we may assume $|v| = 1$. Then the sequence $x_n = (1/n)v$ shows that $v$ is known by $H$. Since $\exp : \mathfrak{h} \rightarrow H$ is a diffeomorphism near $0$, it follows that $\mathfrak{h} = T_e(H)$ is the Lie algebra of $H$. ■

Example: Let $G = \text{GL}_n(\mathbb{C})$ be the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. Regarding $\mathbb{C}^n$ as a real vector space $\mathbb{R}^{2n}$, we have $\text{GL}_n(\mathbb{C}) \subset \text{GL}_{2n}(\mathbb{R})$. Scalar multiplication by $\sqrt{-1}$ gives an element $J \in \text{GL}_{2n}(\mathbb{R})$ whose square $J^2 = -I$, and $\text{GL}_n(\mathbb{C})$ is precisely the centralizer of $J$ in $\text{GL}_{2n}(\mathbb{R})$. Since centralizing is a closed condition, it follows that $\text{GL}_n(\mathbb{C})$ is a Lie group. Likewise $\text{SL}_n(\mathbb{C}) = \{ g \in \text{GL}_n(\mathbb{C}) : \det(g) = 1 \}$ is a closed subgroup of $\text{GL}_n(\mathbb{C})$ hence is a Lie group.

The **Unitary Group** $U_n$ is the subgroup of $\text{GL}_n(\mathbb{C})$ preserving the hermitian form

$$\langle u, v \rangle = \sum_i u_i \bar{v}_i.$$ 

More explicitly,

$$U_n = \{ g \in \text{GL}_n(\mathbb{C}) : g^{-1} = {}^t \bar{g} \}.$$ 

These are closed conditions, so $U_n$ is a Lie group. In fact, each column of a matrix in $U_n$ lies in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ and $U_n$ is compact. Note that $U_1 = S^1$.

The **Special Unitary Group** is the Lie group $\text{SU}_n = U_n \cap \text{SL}_n(\mathbb{C})$. For $n = 2$ we have $\text{SU}_2 = S^3$.

### 6.2 Homogeneous spaces

Let $G$ be a Lie group and let $H$ be a closed subgroup, and let $\pi : G \rightarrow G/H$ be the projection of $G$ onto the set $G/H = \{ gH : g \in G \}$ of left cosets of $H$ in $G$. Declare a subset $U \subset G/H$ to be open iff $\pi^{-1}(U)$ is open in $G$. This makes $\pi$ an open mapping, and $H$ being closed is equivalent to $G/H$ being a Hausdorff topological space.

Choose a neighborhoods $U, V$ of $0 \in \mathfrak{g}$ and $e \in G$ such that $\exp$ maps $U$ diffeomorphically onto $V$. Recall that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a vector space complement to $\mathfrak{h}$ in $\mathfrak{g}$. Define

$$\psi : \mathfrak{m} \cap U \longrightarrow G/H, \quad \text{by} \quad \varphi(u) = \pi \exp(u).$$

From Lemma 6.4 it follows that $\varphi$ is a homeomorphism onto an open neighborhood of $eH$ in $G/H$. Left-translating as above, we obtain an atlas on $G/H$ making $G/H$ into a smooth manifold such that the projection $\pi : G \rightarrow G/H$ is smooth and $\pi'$ maps $\mathfrak{m}$ isomorphically onto $T_eH(G/H)$.  

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Such manifolds arise as follows. A **homogeneous space** for \( G \) is a manifold \( M \) on which \( G \) acts transitively, such that the action map
\[
G \times M \rightarrow M
\]
is smooth. Recall from the theory of group actions that for any \( p \in M \) with stabilizer \( G_p \) in \( G \) the map \( g \mapsto g \cdot p \) gives a \( G \)-equivariant bijection \( f_p : G/G_p \rightarrow M \). Since the action is smooth, it follows that \( f_p \) is actually a diffeomorphism. Hence every homogeneous space for \( G \) is of the form \( G/H \) for some closed subgroup \( H \) of \( G \).

### 6.3 Compact subgroups

Let \( G \) be a Lie group and let \( H \) be a compact subgroup. Then \( H \) is closed, hence is itself a Lie group whose Lie algebra
\[
\mathfrak{h} = \{ v \in \mathfrak{g} : \exp(tv) \in H \text{ for all } t \in \mathbb{R} \}
\]
is a subalgebra of \( \mathfrak{g} \). In the proof of Thm. 6.1, where \( H \) was only assumed to be close, we made use of an arbitrary vector space complement to \( \mathfrak{h} \) in \( G \). Since now \( H \) is compact, we can do better.

We first need a general property of representations of compact groups. Let \( V \) be a finite dimensional real vector space. A **positive definite inner product** \( \langle \ , \ \rangle \) on \( V \) is a bilinear map
\[
V \times V \rightarrow \mathbb{R}, \quad (u, v) \mapsto \langle u, v \rangle
\]
which is symmetric: \( \langle u, v \rangle = \langle v, u \rangle \) and satisfies \( \langle v, v \rangle > 0 \) for all nonzero \( v \in V \). It is a fact (which we will not use) that for any positive definite inner product there exists a basis \( \{ v_i \} \) of \( V \) such that \( \langle v_i, v_i \rangle = 1 \) for all \( i \) and \( \langle v_i, v_j \rangle = 0 \) if \( i \neq j \).

**Proposition 6.6** Let \( \rho : K \rightarrow \text{GL}(V) \) be a continuous representation of a compact Lie group \( K \) on a finite dimensional real vector space \( V \). Then there exists a positive definite inner product \( \langle \ , \ \rangle \) on \( V \) such that
\[
\langle \rho(k)u, \rho(k)v \rangle = \langle u, v \rangle \quad \text{for all } k \in K, \ u, v \in V.
\]

Such an inner product is called **\( K \)-invariant**.

**Proof:** We prove this assuming the existence of Haar measure. This is a measure \( dk \) on \( K \) such that for all continuous functions \( f : K \rightarrow \mathbb{R} \) and all \( h \in K \), we have
\[
\int_K f(hk) \, dk = \int_K f(k) \, dk. \tag{9}
\]
This measure \( dk \) is unique up to scalar multiple; we normalize it so that \( \int_K 1 \, dk = 1 \).

Now on \( V \) we take any inner product \( \langle \ , \ \rangle \) which is positive-definite and define a new inner-product by averaging:
\[
\langle u, v \rangle = \int_K \langle \rho(k)u, \rho(k)v \rangle \, dk.
\]
This new inner product is still positive definite, since

$$\langle v, v \rangle = \int_K (\rho(k)v, \rho(k)v) \, dk > 0,$$

and is also invariant under $K$, using (9).

**Remark:** In the case where $G$ is compact with finite center and $V = \mathfrak{g}$ is the adjoint representation, there is a canonical positive-definite inner product on $\mathfrak{g}$ which is $G$-invariant, namely

$$\langle u, v \rangle = -\text{tr}(\text{ad}(u) \circ \text{ad}(v)).$$

This form (rather its negative) is called the **Killing form** on $\mathfrak{g}$.

**Proposition 6.7** Let $U \subset V$ be a subspace such that $\rho(K)U = U$. Then there exists a vector space complement $W$ to $U$ in $V$ such that $\rho(K)W = W$.

**Proof:** Define $W := \{v \in V : \langle u, v \rangle = 0 \text{ for all } u \in U\}$. From (??) it follows that $\rho(k)W = W$ for all $k \in K$. ■

A subspace $U \subset V$ is **$K$-invariant** if $\rho(K)U = U$. A representation $(\rho, V)$ is **irreducible** if there are no proper invariant subspaces. Applying Prop. 6.7 repeatedly, we obtain:

**Corollary 6.8** Every finite dimensional representation of a compact group is a direct sum of irreducible representations.

Applying Prop. 6.7 to the compact subgroup $H \subset G$ and its representation $\rho = \text{Ad}_G|_H$ obtained by restricting $\text{Ad}_G$ to $H$, we obtain:

**Corollary 6.9** If $H$ is a compact subgroup of a Lie group $G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, then there is a vector space complement $\mathfrak{m} \subset \mathfrak{g}$ such that $\text{Ad}_G(H)\mathfrak{m} = \mathfrak{m}$.

Thus the restriction of $\text{Ad}_G$ to $H$ gives a representation

$$\text{Ad}_G : H \to \text{GL}(\mathfrak{m}).$$

This is called the **isotropy representation** of the subgroup $H \subset G$ because the projection $G \to G/H$ identifies $\mathfrak{m}$ with the tangent space to $G/H$ at $eH$, where $H$ is the isotropy group. More generally $\text{Ad}(g)\mathfrak{m}$ is the tangent space to $T(G/H)$ at $gH$ and we may regard the entire tangent bundle of $G/H$ as a set of pairs:

$$T(G/H) = \{(gH, v) : gH \in G/H, \ v \in \text{Ad}(g)\mathfrak{m}\}.$$

In contrast to the manifold $G$ (where $H = \{1\}$) the manifold $G/H$ is generally not parallelizable. For example, if $G = \text{SO}_3$ and $H = \text{SO}_2$ we have $G/H = S^2$, on which every vector field vanishes somewhere.

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Example 1: Take \( G = \text{GL}_n(\mathbb{R}) \) and \( H = \text{O}_n(\mathbb{R}) \). Then \( \mathfrak{g} = M_n(\mathbb{R}) \) decomposes as \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \), where
\[
\mathfrak{h} = \text{so}_n = \{ A \in M_n(\mathbb{R}) : {}^tA = -A \}, \quad \mathfrak{m} = \{ B \in M_n(\mathbb{R}) : {}^tB = B \},
\]
are the spaces of skew-symmetric and symmetric matrices, respectively. Note that if \( h \in \text{O}_n \) and \( B \in \mathfrak{m} \) then \( \text{Ad}(h)B = hBh^{-1} \) is again in \( \mathfrak{m} \). The homogeneous space \( M = G/H \) is the set of positive definite inner products on \( \mathbb{R}^n \).

Example 2: Take \( G = \text{SO}_{n+1} \) and let \( H = \text{SO}_n \) be the subgroup fixing \( e_{n+1} \). Then
\[
\mathfrak{m} = \left\{ A_x = \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\}.
\]
Identifying \( \mathfrak{m} = \mathbb{R}^n \), \( \text{Ad}_G \mid_H \) becomes the natural representation of \( \text{SO}_n \) on \( \mathbb{R}^n \). The homogeneous space \( M = G/H \) is the sphere \( S^n \) and the map \( \mathfrak{m} \to S^n \) given by \( A_x \mapsto \exp(A_x) \cdot e_{n+1} \) is spherical coordinates on \( S^n \). If we change this example slightly and take \( H = \text{O}_n \) fixing \( e_{n+1} \) up to sign, then \( \mathfrak{h} \) and \( \mathfrak{m} \) are unchanged and \( G/H \) is the real projective space \( \mathbb{P}^n(\mathbb{R}) \).

Example 3: The previous example generalizes as follows. Take \( G = \text{SO}_n \) and let \( H = G \cap (\text{O}_k \times \text{O}_\ell) \) be the subgroup preserving a \( k \)-dimensional subspace \( U \subset \mathbb{R}^n \) (and therefore also preserving the \( \ell = n-k \)-dimensional orthogonal complement of \( U \)). Then
\[
\mathfrak{m} = \left\{ \begin{pmatrix} 0 & x \\ -{}^t x & 0 \end{pmatrix} : x = k \times \ell \text{ matrix} \right\}
\]
and \( (h_1, h_2) \in H \) acts by \( x \mapsto h_1 x h_2^{-1} \). The homogeneous space is the Grassmannian of \( k \)-planes in \( \mathbb{R}^n \). If we take instead \( H = \text{SO}_k \times \text{SO}_\ell \) then \( \mathfrak{h} \) and \( \mathfrak{m} \) are unchanged but now \( G/H \) is the manifold of oriented \( k \)-planes in \( \mathbb{R} \). When \( k = 1 \) an oriented line is a ray which may be identified with a point on the sphere.

Example 4: Take \( G = \text{U}_{n+1} \) and \( H = \text{U}_1 \times \text{U}_n \). Then \( \mathfrak{m} = \mathbb{C}^n \) with the natural action of \( \text{U}_n \) and scalar multiplication by \( \text{U}_1 \), and \( \text{U}_{n+1} / \text{U}_1 \times \text{U}_n \) is the complex projective space \( \mathbb{P}^n(\mathbb{C}) \). Since \( \mathbb{C}^\times \) is connected and complex manifolds have a natural orientation, this example has only one version, unlike the previous examples.

7  Maximal Tori

Let \( G \) be a compact Lie group. A **maximal torus** in \( G \) is a subgroup \( T \subset G \) which is a torus, and is contained in no larger torus.

**Lemma 7.1** A conjugate of a maximal torus is a maximal torus.

**Proof:** Let \( T \) be a maximal torus in \( G \) and let \( g \in G \). The conjugation map \( c_g : G \to G \) is a diffeomorphism of the manifold \( G \) and an automorphism of the group \( G \), so \( gTg^{-1} \) is a torus. If \( U \) is
a torus containing \( gTg^{-1} \) then \( g^{-1}Ug \) is a torus containing \( T \), which equals \( T \) by maximality of \( T \), so \( gTg^{-1} = U \) and so \( gTg^{-1} \) is a maximal torus.

Let \( \mathfrak{t} \) denote the Lie algebra of a maximal torus \( T \) in \( G \). As \( G \) is compact, the adjoint representation \( \text{Ad}_G \) decomposes as

\[ \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}, \]

where \( \mathfrak{m} \) is the orthogonal complement of \( \mathfrak{t} \) with respect to some \( \text{Ad}_G \)-invariant inner product on \( \mathfrak{g} \) (for example, the negative of the Killing form). We will study the representation of \( T \) on \( \mathfrak{m} \), which is the tangent space to \( G/T \) at \( eT \).

**Lemma 7.2** Every continuous finite dimensional irreducible real representation of a torus \( T \) is either trivial (one-dimensional with trivial \( T \)-action) or is two-dimensional, of the form

\[ \rho_\lambda(\exp(x)) = \begin{bmatrix} \cos 2\pi \lambda(x) & -\sin 2\pi \lambda(x) \\ \sin 2\pi \lambda(x) & \cos 2\pi \lambda(x) \end{bmatrix} \]

for a unique linear functional \( \lambda : \mathfrak{t} \to \mathbb{R} \).

**Proof:** Let \( \rho : T \to \text{GL}(V) \) be a non-trivial continuous irreducible real representation. Let \( t \in T \) be a topological generator. Since \( T \) is connected and compact, the eigenvalues of \( \rho(t) \) lie on the unit circle and at least one has infinite order. Since \( \rho(t) \) is a real matrix, the latter come in complex-conjugate pairs \( e^{\pm i\theta} \). Let \( V_\theta \subset V \otimes \mathbb{C} \) be the span of two eigenvectors with eigenvalues \( e^{i\theta} \) and \( e^{-i\theta} \). Then \( V_\theta \) is a four-dimensional real vector space whose fixed-points under complex conjugation are two-dimensional. Since \( V \) is irreducible, this two-dimensional space is all of \( V \). Again, since \( T \) is compact and connected, \( \rho(T) \subset \text{SO}_2 \). The functional \( \lambda \) is the derivative \( \rho' : \mathfrak{t} \to \mathfrak{so}_2 = \mathbb{R} \). Since \( \lambda \) determines \( \rho \), this functional is unique. \( \square \)

The linear functionals \( \lambda \) which can arise in Lemma 7.2 are precisely those which take integer values on the lattice \( L = \ker[\exp : \mathfrak{t} \to T] \). Thus, the irreducible representations of \( T \) are parametrized by the lattice

\[ L^* = \{ \lambda \in \mathfrak{t}^* : \lambda(L) \subset \mathbb{Z} \}. \]

**Lemma 7.3** A maximal torus \( T \) in \( G \) is maximal connected abelian.

**Proof:** Let \( A \) be a connected abelian subgroup of \( G \) containing \( T \). Since multiplication is continuous, the closure \( \overline{A} \) is also abelian and is still connected. Hence \( \overline{A} \) is a torus, so \( T = \overline{A} \) and therefore \( T = A \).

**Lemma 7.4** The vectors in \( \mathfrak{g} \) fixed by \( \text{Ad}_G(T) \) are precisely those in \( \mathfrak{t} \).

**Proof:** The vectors in \( \mathfrak{t} \) are fixed by \( \text{Ad}_G(T) \) since \( T \) is abelian and \( \text{Ad}_G(T) = \text{Ad}_T(T) \) on \( \mathfrak{t} \). Conversely, suppose \( u \in \mathfrak{g} \) is fixed by \( \text{Ad}_G(T) \). Let \( H = \{ \exp(su) : s \in \mathbb{R} \} \). For all \( t \in T \) we have

\[ t \cdot \exp(su) \cdot t^{-1} = \exp(s \text{Ad}(t)u) = \exp(su), \]

for all \( s, t \in \mathbb{R} \).
so \( HT \) centralizes \( T \). The group \( HT \) is abelian and connected, as it is the image of the map \( H \times T \to G \), sending \((h, t) \mapsto ht\). From the previous lemma, \( HT = T \), so \( \exp(su) \in T \) for all \( s \in \mathbb{R} \). By Cor. 6.5, we have \( u \in t \).

From Lemmas 7.2 and 7.4, it follows that there is a finite set

\[
R \subset t^*\
\]

of linear functionals \( \alpha \) on \( t \), closed under \( \alpha \mapsto -\alpha \), such that the representation \( \text{Ad}_G : T \to \text{GL}(m) \) decomposes as

\[
m = \bigoplus_{\alpha \in R/\pm} m_\alpha
\]

where \( m_\alpha \) is the two dimensional representation \( \rho_\alpha \) of \( T \) on which \( \exp(x) \) (for \( x \in t \)) acts via the matrix

\[
\begin{bmatrix}
\cos 2\pi \alpha(x) & -\sin 2\pi \alpha(x) \\
\sin 2\pi \alpha(x) & \cos 2\pi \alpha(x)
\end{bmatrix}.
\]

The elements of \( R \) are called the roots of \( T \) in \( G \), because they determine the roots of the characteristic polynomial \( \det(I - \text{Ad}_G(t)) \), for \( t \in T \). The roots can be seen more directly via the derivative \( \text{ad} \) of \( \text{Ad} \). Indeed for \( x \in t \) the matrix of \( \text{ad}(x) \) on \( m_\alpha \) is given by

\[
\text{ad}(x) = 2\pi \alpha(x) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

**Example 1:** Take \( G = U_n \), with Lie algebra

\[
g = u_n = \{ X \in M_n(\mathbb{C}) : \; ^t X = -X \}.
\]

The diagonal matrices in \( G \) form a maximal torus \( T \simeq U_1^n \). The Lie algebra of \( T \) consists of diagonal matrices with imaginary entries \( u = \sqrt{-1}(u_1, \ldots, u_n) \), so \( t^* \) has basis \( \{ x_i \} \) where \( x_i(u) = u_i \). For \( 1 \leq i < j \leq n \) and \( z \in \mathbb{C} \) let \( X_{ij}(z) \) be the matrix with \( ij \)-entry equal to \( z \), \( ji \)-entry equal to \( -\bar{z} \), and all other entries zero. Let

\[
m_{ij} = \{ X_{ij}(z) : \; z \in \mathbb{Z} \}.
\]

Then \( m_{ij} \) is preserved by \( \text{Ad}_G(T) \) with roots \( x_i - x_j, x_j - x_i \) and the decomposition of \( su_n \) with respect to \( \text{Ad}_G(T) \) is

\[
su_n = t \oplus \sum_{i < j} m_{ij}
\]

and the roots of \( U_n \) are given by

\[
R = \{ x_i - x_j : \; 1 \leq i \neq j \leq n \}.
\]

For \( SU_n \) a maximal torus is the \( \det = 1 \) subgroup of the above \( T \), and \( su_n \) is the \( \text{tr} = 0 \) subspace of \( u_n \). The subspaces \( m, m_{ij} \) and the roots are unchanged.

**Example 2:** Take \( G = SO_{2n} \) (or \( O_{2n} \)), with Lie algebra

\[
g = so_{2n} = \{ X \in M_{2n}(\mathbb{R}) : \; ^t X = -X \}.
\]
Think of matrices in $\mathfrak{so}_{2n}$ as $n \times n$ matrices where each entry is a $2 \times 2$ matrix. For $1 \leq i < j \leq n$ and a $2 \times 2$ matrix $z$, let $X_{ij}(z)$ be the matrix with $ij$-entry equal to $z$, $ji$-entry equal to $-t_z$, and all other entries zero. For $1 \leq i < j \leq n$ let

$$m_{x_i-x_j} = \{X_{ij} \left( \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right) : x, y \in \mathbb{R} \},$$

$$m_{x_i+x_j} = \{X_{ij} \left( \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \right) : x, y \in \mathbb{R} \}.$$

Then $m_{x_i-x_j}$ and $m_{x_i+x_j}$ is preserved by $\text{Ad}_G(T)$ with roots $\pm(x_i - x_j)$ and $\pm(x_i + x_j)$ respectively and the decomposition of $\mathfrak{so}_{2n}$ with respect to $\text{Ad}_G(T)$ is

$$\mathfrak{so}_{2n} = t \oplus \sum_{i<j} (m_{x_i-x_j} \oplus m_{x_i+x_j})$$

and the roots of $SO_{2n}$ are given by

$$R = \{x_i \pm x_j : 1 \leq i \neq j \leq n\}.$$

**Example 3:** Take $G = SO_{2n+1}$ (or $O_{2n+1}$), with Lie algebra

$$\mathfrak{g} = \mathfrak{so}_{2n+1} = \{X \in M_{2n+1}(\mathbb{R}) : t X = -X \}.$$

Then $G$ contains $SO_{2n}$ and a maximal torus $T$ of $SO_{2n}$ is also a maximal torus of $SO_{2n+1}$. We have

$$\mathfrak{so}_{2n+1} = \mathfrak{so}_{2n} \oplus \mathbb{R}^{2n},$$

as in Example 2 of section 6.3. The action of $T = SO_2^\mathbb{R}$ on $\mathbb{R}^{2n}$ is a direct sum of the $n$ irreducible representations obtained by projecting onto each factor of $T$. Thus, the roots of $SO_{2n+1}$ are given by

$$R = \{x_i \pm x_j : 1 \leq i \neq j \leq n\} \cup \{x_i : 1 \leq i \leq n\}.$$

### 7.1 The Weyl group

Let $G$ be a compact Lie group with maximal torus $T$. Let $N(T) = \{n \in G : nTn^{-1} = T\}$ be the normalizer of $T$ in $G$. The **Weyl group** is the quotient

$$W = N(T)/T.$$

**Proposition 7.5** The Weyl group $W$ is finite.
Proof: We first show that $N(T)$ is compact. For all $t \in T$, define maps $f_t, \hat{f}_t : G \to G$ by $f_t(g) = gt g^{-1}$ and $\hat{f}_t(g) = g^{-1}tg$. Then

$$N(T) = \left[ \bigcap_{t \in T} f_t^{-1}(T) \right] \cap \left[ \bigcap_{t \in T} \hat{f}_t^{-1}(T) \right],$$

so $N(T)$ is closed in the compact group $G$ hence is compact. For each $n \in N(T)$ we have a commutative diagram

$$\begin{array}{ccc}
\mathfrak{t} & \xrightarrow{\text{Ad}(n)} & \mathfrak{t} \\
\exp_T & | & | \exp_T \\
T & \xrightarrow{e_n} & T.
\end{array}$$

It follows that $\text{Ad}(n)L = L$, where $L = \ker \exp_T$. This gives a map $\text{Ad} : N(T) \to \text{Aut}(L) \simeq \text{GL}_n(\mathbb{Z})$, where $n = \dim \mathfrak{t}$. Here $\text{GL}_n(\mathbb{Z})$ is the discrete group of $n \times n$ matrices whose inverse is also integral. Since $N(T)$ is compact, it follows that $\text{Ad}(N(T))$ is compact and discrete, hence is finite.

The kernel of $\text{Ad} : N(T) \to \text{Aut}(L)$ is the centralizer $C(T)$ of $T$ in $G$. Indeed, if $\text{Ad}(n)$ is trivial on $L$ then $\text{Ad}(n)$ is trivial on $\mathfrak{t}$, since $L$ spans $\mathfrak{t}$. For all $x \in \mathfrak{t}$ we then have $\text{Ad}(n)x = x$ so $n \exp(x)n^{-1} = \exp(\text{Ad}(n)x) = \exp(x)$ so $n \in C(T)$. The other containment is clear.

Let $H = N(T)^0$ be the identity component of $N(T)$. Since $N(T)/C(T)$ is the image of $N(T)$ in $\text{Aut}(L)$, it is finite. It follows that $H \subset C(T)$. Since $N(T)$ is compact and $H$ is open in $N(T)$, the index $[N(T) : H]$ is still finite.

I claim that $H = T$. Since $T$ is abelian and connected, we have $T \subset H$. Let $\mathfrak{h}$ be the Lie algebra of $H$ and let $x \in \mathfrak{h}$. Let $S = \{ \exp(tx) : t \in \mathbb{R} \}$. Since $S \subset H \subset C(T)$, the group generated by $S$ and $T$ is abelian and connected. But $T$ is maximal abelian connected, by Lemma 7.3. It follows that $S \subset T$, so $x \in \mathfrak{t}$ by Cor. 6.5. Hence $\mathfrak{h} = \mathfrak{t}$. By Cor. 4.15, the inclusion map $T \hookrightarrow H$ is surjective, so $H = T$, as claimed. It follows that $|W| = [N(T) : T] = [N(T) : H]$ is finite.

### 7.2 The flag manifold

Let $G$ be a compact Lie group with maximal torus $T$. The **flag manifold** of $G$ is the homogeneous space $G/T$. The group $G$ acts on $G/T$ by left translations: $L_g(xT) = gxT$. And the Weyl group acts on $G/T$ by right translations: $R_w(xT) = xn^{-1}T$, where $w = nT$. Note that $R_w$ is well-defined and commutes with the operators $L_g$ for $g \in G$.

Example: Take $G = S^3$ and $T = S^1 \subset \mathbb{C}$. Then $N(T) = \langle T, j \rangle$ and $|W| = 2$ and $j$ represents the nontrivial element of $W$. The flag manifold is $G/T = S^2$. The left action of $G$ on $G/T$ factors through the homomorphism $R : S^3 \to SO_3$ and the natural action of $SO_3$ on $S^2$. In particular $T$ acts by rotation with axis through $\pm i$.

The right action of $W$, given by $R_j$, is the antipodal map on $S^2$. This action does not come from any element of $SO_3$. Note that $R_j$ interchanges the two fixed-points $\pm i$ of $T$ in $S^2$. ■
In general a normalizer preserves the fixed-point set of the normalizee. The fixed-point set of $T$ in $G/T$ is

$$[G/T]^T = \{ nT : n \in N(T)/T \} = W.$$ 

Note that $G/T$ is not a group, but $W$ is a subset of $G/T$ and coincides with the fixed-point set of $T$ in $G/T$.

### 7.3 Conjugacy of maximal tori

Always $G$ is a compact Lie group. Our goal is to prove the following.

**Theorem 7.6** Let $G$ be a compact Lie group with maximal torus $T$. Then $T$ meets every conjugacy class in $G$. Equivalently,

$$G = \bigcup_{xT \in G/T} xT x^{-1}.$$ 

This is the deepest result in our course. We will prove it in the next section using cohomology. Here we give derive some consequences of Thm. 7.6.

**Corollary 7.7** Every element of $G$ lies in a maximal torus.

**Proof:** Thm. 7.6 implies that every element of $G$ lies in the maximal torus $xT x^{-1}$ (see Lemma 7.1). ■

**Corollary 7.8** All maximal tori are conjugate in $G$.

**Proof:** Let $T, U$ be a maximal tori in $G$. Let $u \in U$ be a topological generator. By Thm. 7.6 some conjugate $gu^g^{-1}$ lies in $T$. Then

$$gUg^{-1} = \overline{g(u)}g^{-1} = \overline{g(u)}g^{-1} \subseteq T.$$ 

But $gUg^{-1}$ is also a maximal torus, so $gUg^{-1} = T$. ■

**Corollary 7.9** The exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective.

**Proof:** Let $g \in G$. Then $g \in T$ for some maximal torus $T$ in $G$, by Cor. 7.7 Since $\exp_T : \mathfrak{t} \rightarrow T$ is surjective there is $x \in \mathfrak{t}$ such that $\exp_T(x) = g$. By functorality, $\exp_G$ agrees with $\exp_T$ on $T$, so $g = \exp_G(x)$ as well. ■

**Corollary 7.10** A maximal torus in $G$ is its own centralizer in $G$. 

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**Proof:** Let \( T \) be a maximal torus in \( G \) and let \( C(T) \) be the centralizer of \( T \) in \( G \). Since \( T \) is abelian, it is clear that \( C(T) \supset T \) and we must show the reverse containment.

Let \( g \in C(T) \). The group \( H = \langle g, T \rangle \) is abelian and is compact since \( G \) is compact. The identity component \( H^o \) is connected abelian and contains \( T \), so \( H^o = T \). Hence \( T \) is open in the compact group \( H \). Therefore \( H/T \) is finite. By construction, the union of cosets

\[
\bigcup_{n \in \mathbb{Z}} g^nT
\]

is dense in \( H \), hence meets every open set in \( H \). Each coset \( hT \) is open in \( H \), so the union meets every coset of \( T \) in \( H \). It follows that \( H \) is a finite union of cosets of the form \( g^T \). That is, \( H/T \) is finite cyclic, generated by \( gT \). This means that \( g^m \in T \) for some \( m \in \mathbb{Z} \).

Let \( t \in T \) be a topological generator, and consider the element \( t \cdot g^{-m} \in T \). Every element in \( T \) is an \( m \)th power, since \( e^{i\theta} = (e^{i\theta/m})^m \). Hence there is \( s \in T \) such that \( t \cdot g^{-m} = s^m \). Since \( T \) is abelian and \( g \in C(T) \), we have \( t_0 = (sg)^m \).

By Cor. 7.7, there exists a maximal torus \( U \) containing \( sg \). Then \( t = (sg)^m \in U \) so \( T \subset U \), hence \( T = U \) by maximality. We therefore have \( sg \in T \). Since \( s \in T \), it follows that \( g \in T \) and we have proved that \( C(T) \subset T \).

**Remark:** Cor. 7.10 is equivalent to asserting that a maximal torus is a maximal abelian subgroup of \( G \). However, not all maximal abelian subgroups are tori. For example, the diagonal matrices in \( \text{SO}_n \) are isomorphic to \( C_2^{n-1} \), but the 2-torsion in a maximal torus has rank equal to the largest integer \( \leq n/2 \). Likewise in \( G_2 \) there is a copy of \( C_2^3 \), but the maximal torus has rank two. Maximal abelian subgroups which lie in no torus of \( G \) are very interesting subgroups; they play an important role in the topology of \( G \) and in connections between Lie groups and local Galois groups.

### 7.4 Fixed-points in flag manifolds

In this section we prove Thm. 7.6. Let \( G \) be a compact Lie group with maximal torus \( T \). We wish to show that every element \( g \in G \) has a conjugate in \( T \). Equivalently, we wish to show that \( gxT = xT \) for some \( x \in G \). In other words, we wish to show that the left-translation map \( L_g \) on \( G/T \) has a fixed-point. This is not obvious. If we were considering left-translation on a circle, there would be no fixed-point. However, on \( S^2 \), which is the flag manifold of \( \text{SO}_3 \), every nontrivial element of \( \text{SO}_3 \) fixes the points in \( S^2 \) on its rotation axis.

#### 7.4.1 de Rham cohomology

\(^8\) Let \( M \) be a compact connected oriented smooth manifold of dimension \( n \). The cohomology of \( M \) is a sequence of finite dimensional real vector spaces

\[
H^i(M), \quad i = 0, 1, \ldots
\]

\(^8\)For more details on cohomology of manifolds see Bott and Tu, *Differential forms in algebraic topology.*
Satisfying (among others) the following properties

i) $H^i(M) = 0$ for $i > n$ and $H^0 \simeq H^n \simeq \mathbb{R}$.

ii) $H^{n-i}(M) \simeq H^i(M)$ for all $i$.

iii) $H^i(S^n) = 0$ for $0 < i < n$.

iii) A smooth map $f : M \to M$ induces a linear map $f^* : H^i(M) \to H^i(M)$ for all $i$ such that if $g : M \to M$ is another smooth map we have $(f \circ g)^* = g^* \circ f^*$.

iv) If $f : M \times [0, 1] \to M$ is a continuous map such that $f_t := f(\cdot, t)$ is smooth for each $t \in [0, 1]$ then $f_0^* = f_1^*$.

v) If $f : M \to M$ is a smooth map with only finite many fixed-points then

$$\sum_{i=0}^{n} (-1)^i \text{tr}(f^*, H^i(M)) = \sum_{p \in M} \text{sgn det}(I - f'_p),$$

where the right side is understood to be zero if $f$ has no fixed-points.

Here, if $f(p) = p$ then $f'_p : TM \to TM$ restricts to a map $f'_p : T_p M \to T_p M$ and $\text{det}(I - f'_p)$ is the determinant of the linear map $I - f'_p$ on $T_p M$ and $\text{sgn det}(I - f'_p) \in \{+1, -1, 0\}$ is the sign of this determinant. The formula v) is the Lefschetz fixed-point formula, and the alternating sum

$$\mathcal{L}(f) := \sum_{i=0}^{n} (-1)^i \text{tr}(f^*, H^i(M))$$

is the Lefschetz number of $f$. In the situation of item iv) we have $\mathcal{L}(f_1) = \mathcal{L}(f_0)$. It may happen that $f_1$ has infinitely many fixed-points but $f_0$ has only finitely many, so we can apply v) to $f_0$ to compute $\mathcal{L}(f_1)$.

If $f : M \to M$ is the identity map then $f^*$ is the identity map on $M$ and

$$\mathcal{L}(f) = \sum_{i=0}^{n} (-1)^i \dim H^i(M)$$

is the Euler characteristic $\chi(M)$ of $M$, which can be computed by triangulating $M$. For example, any polyhedron drawn on the sphere $S^2$ with $v$ vertices, $e$ edges and $f$ faces has $v - e + f = 2$ by Euler’s theorem. Here $\chi(S^2) = 2$ and we also have

$$\dim H^0(S^2) + \dim H^2(S^2) = 2.$$

We can also compute this using the Lefschetz fixed-point formula, even though the identity map has infinitely many fixed-points. Let $f \in \text{SO}_3$ be a nontrivial rotation by angle $\theta$. Then $f$ has two fixed-points on $S^2$, and $f'$ acts by a rotation by $\theta$ in the tangent space at each fixed-point $p$, where we have

$$\det(I - f'_p) = \text{det} \begin{bmatrix} 1 - \cos \theta & \sin \theta \\ -\sin \theta & 1 - \cos \theta \end{bmatrix} = 2(1 - \cos \theta) > 0.$$
So
\[ \sum_{p \in S^2} \det(I - f'(p)) = 1 + 1 = 2. \]

On the other hand, let \( f_t \) be rotation about the same axis of \( f \) by \( t \theta \). Then \( f_1 = f \) and \( f_0 = I \), so \( f^* = f_0^* \) is the identity map on \( H^*(S^2) \) and we have
\[ \mathcal{L}(f) = \chi(S^2). \]

We generalize this example to prove Thm. 7.6 as follows. Let \( G \) be a compact connected Lie group with maximal torus \( T \). Each \( g \in G \) acts on \( G/T \) by \( L_g(xT) = gxT \). Since \( g \) is connected to the identity \( e \in G \) by a path in \( G \), we have
\[ L(L_g) = L(L_e) = \chi(G/T), \]

independent of \( g \).

Let \( t \in T \) be a topological generator. We have \( txT = xT \) if and only if \( t \in xT x^{-1} \), which means \( T = xT x^{-1} \), so \( x \in N(T) \). Hence the fixed-point set of \( L_t \) in \( G/T \) is exactly
\[ W = \{ nT : n \in N(T) \}. \]

For each \( n \in N(T) \), the derivative map \( L_n' \) gives a commutative diagram
\[ \begin{array}{ccc}
T_eT(G/T) & \xrightarrow{L_n'} & T_nT(G/T) \\
L_{n^{-1}t} & & \downarrow L_t' \\
T_eT(G/T) & \xrightarrow{L_n'} & T_nT(G/T),
\end{array} \]

so that
\[ \det(I - L_t')_{nT} = \det(I - L_{n^{-1}t} eT). \]

Recall the derivative of the projection \( \pi : G \to G/T \) restricts to an isomorphism
\[ \pi' : m \sim T_eT(G/T). \]

For any \( s \in T \) this isomorphism transforms \( \text{Ad}_G(s) \) on \( m \) to the map \( L_n' \) on \( T_eT(G/T) \). Let \( s = n^{-1}tn \) as above and write \( s = \exp(u) \) for \( u \in t \). Then
\[ \det(I - L_{n^{-1}t} eT) = \prod_{\alpha \in R/\pm} \det(I - \text{Ad}_G(s))_{m_{\alpha}} \]
\[ = \prod_{\alpha \in R/\pm} \det \begin{bmatrix} 1 - \cos 2\pi\alpha(u) & \sin 2\pi\alpha(u) \\ -\sin 2\pi\alpha(u) & 1 - \cos 2\pi\alpha(u) \end{bmatrix} \]
\[ = \prod_{\alpha \in R/\pm} 2(1 - \cos 2\pi\alpha(u)) > 0, \]

by Lemma 7.4. It follows that
\[ \text{sgn} \det(I - L_t')_p = +1 \]

for every fixed-point \( p \) of \( L_t \) in \( G/T \). For any \( g \in G \) we therefore have
\[ \mathcal{L}(L_g) = \mathcal{L}(L_t) = |W| \neq 0 \]

so \( L_g \) has a fixed-point in \( G/T \). This completes the proof of Thm. 7.6.
8 Octonions and $G_2$

In this section we construct the 14-dimensional compact Lie group of type $G_2$, using the non-associative algebra $O$ of octonions. The quaternions will play an important role, but unlike $\mathbb{H}$, the unit sphere in $O$ is not a group. Instead, $G_2$ arises as the automorphism group of $O$. Thus, $G_2$ is analogous to $\text{Aut}(\mathbb{H}) = \text{SO}_3$.

8.1 Composition algebras

In this section and the next we follow Conway-Smith \(^9\) on the basic equations in composition algebras.

A quadratic form on a real vector space $A$ is a function $N : A \to \mathbb{R}$ such that

- $\text{for all } r \in \mathbb{R} \text{ and } x \in A.$
- $N(rx) = r^2 N(x)$
- $N(x) > 0$ for all nonzero $x \in A$.

The function $(x, y) = \frac{1}{2} [N(x + y) - N(x) - N(y)]$ is symmetric and bilinear.

A quadratic form $N$ is positive definite if $N(x) > 0$ for all nonzero $x \in A$.

Let $A$ be an algebra with unit, not necessarily associative, containing $\mathbb{R}$ as a subalgebra. We say $A$ is a composition algebra if there is a quadratic form $N : A \to \mathbb{R}$ which is multiplicative:

\[N(xy) = N(x)N(y) \quad \text{for all } x, y \in A.\]  \hfill (10)

In these notes we always assume without further mention that $N$ is positive definite and call it the norm on $A$. Since $N$ takes positive values on $A - \{0\}$, we have $N(1) = N(1 \cdot 1) = N(1)^2$ so $N(1) = 1$ and likewise $N(-1) = 1$. The associated bilinear form is given as above by

\[(x, y) = \frac{1}{2} [N(x + y) - N(x) - N(y)].\]

This form is symmetric by definition and satisfies $(x, x) = N(x) > 0$ if $x \neq 0$. This implies that if $(x, t) = 0$ for all $t \in A$ iff $x = 0$. Replacing $x$ by $x - y$, we have $x = y$ iff $(x, t) = (y, t)$ for all $t \in A$.

We often derive equations in $A$ using $(\ , \ )$ in this way.

Define the conjugation $x \mapsto \bar{x}$, by

$$\bar{x} = 2(1, x) - x.$$  \hfill (11)

We now have the following equations:

\[(xy, xz) = N(x) \cdot (y, z) = (yx, zx).\]  \hfill (12)

That is, left and right multiplication by $x$ preserve the bilinear form, up to the scalar $N(x)$. Equation (12) is derived from (10) upon replacing $y$ by $y + z$.

\[(xy, zw) + (xw, zy) = 2(x, y)(z, w).\]  \hfill (13)

This follows from (12) upon replacing \( z \to w \) and \( x \to x + z \). We only use this when \((z, w) = 0\), when we get

\[
(xy, zw) = -(xw, zy) \quad \text{if} \quad (z, w) = 0.
\]

(14)

\[
(y, \bar{x}z) = (xy, z) = (x, z\bar{y}).
\]

Thus, the adjoint of right multiplication by \( x \) is right multiplication by \( \bar{x} \), and similarly for left multiplication. Equation (15) is proved by setting \( z = 1 \) in (13).

(15)

\[
\bar{x} = x.
\]

This is proved by setting \( y = 1 \) and \( z = t \) in (13).

(16)

\[
\bar{xy} = \bar{y}x.
\]

Proof: use (15) repeatedly to get \((\bar{y}\bar{x}, t) = (x, yt) = (\bar{x}t, y) = (\bar{t}, xy), 1) = (\bar{xy}, t)\).

(17)

\[
N(x) = x\bar{x} = xx \quad \text{and} \quad (x, y) = \frac{1}{2}(xy + x\bar{y}).
\]

Proof: Using (15) we have \( N(x) = (x, x) = (1, xx) \). From (17) we have \( x\bar{x} = xx \), so \((1, xx) = xx \) from (11). Now \( 2(x, y) = N(x + y) - N(x) - N(y) = (x + y)(\bar{x} + \bar{y}) - x\bar{x} - y\bar{y} = xy + y\bar{x} \).

(18)

\[
x^2 = -1 \iff (x, x) = 1 \quad \text{and} \quad (x, 1) = 0.
\]

Proof: If \( x^2 = -1 \) then \( N(x)^2 = N(x^2) = N(-1) = 1 \) so \((x, x) = N(x) = 1 \) and \((x, 1) = (x^2, x) = (-1, x) = -(x, 1) \), so \((x, 1) = 0 \). Conversely, \((x, 1) = 0 \) implies \( \bar{x} = -x \) by (11) and \((x, x) = 1 \) implies \( 1 = x\bar{x} = -x^2 \) by (18).

This implies that any pair of vectors orthogonal to each other and to 1 must anti-commute:

\[
xy = -yx \quad \text{whenever} \quad (x, y) = (x, 1) = (1, y) = 0.
\]

(20)

Proof: Since \((x, y) = 0 \) we have \( x\bar{y} + y\bar{x} = 0 \) from (18) and \( \bar{x} = -x, \bar{y} = -1 \) from (15).

Finally, we have the alternative law, which is a form of associativity for two elements.

\[
x(xy) = (x^2)y \quad \text{and} \quad (xy)y = x(y^2).
\]

(21)

Proof: From \((\bar{x}(xy), t) \overset{\text{def}}{=} (xy, xt) \overset{\text{norm}}{=} N(x)(y, t) \overset{\text{norm}}{=} ((\bar{x}x)y, t) \) we have \( \bar{x}(xy) = (\bar{x}x)y \). Replacing \( \bar{x} \) by \( 2(x, 1) - x \) and cancelling gives (21).

\[
8.2 \quad \text{The product on the double of a subalgebra}
\]

Let \( H \) be any proper subalgebra of \( A \). We let \( a, b, c, \ldots \) denote typical elements of \( H \). The restriction of \( N \) to \( H \) makes \( H \) into a composition algebra. From (11) it follows that \( H \) is closed under the conjugation operation \( a \mapsto \bar{a} \).
Choose any unit vector \( x \in A \) orthogonal to \( H \). From (19) it follows that \( x^2 = -1 \). The entire subspace \( xH \) is orthogonal to \( H \), for \( (xa, b) = (x, b\bar{a}) = 0 \). Since \( H \) contains 1, we have
\[
\bar{x}a = -xa. \tag{22}
\]
It follows that \( xH = Hx \) and
\[
xa = \bar{a}x \quad \text{for all} \quad a \in H. \tag{23}
\]

The **double** of \( H \) is the subspace \( H + xH \subset A \). Since the summands are orthogonal, the dimension of \( H + xH \) is twice that of \( H \). In fact since \( (xa, xb) = (a, b) \), it follows that left multiplication \( L_x : H \rightarrow xH \) is an isometry.

If the double \( H + xH \) is not all of \( A \), we may take the double of \( H + xH \) etc. This proves:

**Proposition 8.1** Any composition algebra has dimension equal to a power of two.

We now come to the essential result.

**Proposition 8.2** The double \( H + xH \) is closed under the product in \( A \) and we have
\[
(a + xb)(c + xd) = (ac - d\bar{b}) + x(cb + \bar{a}d), \tag{24}
\]
for all \( a, b, c, d \in H \).

**Proof:** Equation (24) is equivalent to the following three relations:

i) \( a(xd) = x(\bar{a}d) \).

ii) \( (xb)c = x(cb) \).

iii) \( (xb)(xd) = -d\bar{b} \).

Let \( t \in A \) be arbitrary.

For i) we have
\[
(a(xd), t) \overset{(15)}{=} (xd, \bar{a}t) \overset{(14)}{=} -(\bar{a}d, xt) \overset{(15)}{=} (x(\bar{a}d), t).
\]

For ii) we have
\[
((xb)c, t) \overset{(15)}{=} (xb, \bar{t}c) \overset{(23)}{=} (\bar{b}x, \bar{t}c) \overset{(14)}{=} -(\bar{b}c, tx) \overset{(17)}{=} -((cb), tx) \overset{(15)}{=} (cb, x, t) \overset{(23)}{=} (x(cb), t)
\]

For iii) we have
\[
((xb)(xd), t) \overset{(15)}{=} (xb, t\bar{xd}) \overset{(22)}{=} -(xb, t(xd)) \overset{(14)}{=} (x(xd), tb) \overset{(15)}{=} -(xd, x(tb)) \overset{(12)}{=} -(d, tb) \overset{(15)}{=} (-d\bar{b}, t).
\]
Equation (24) is not a definition of a product; it is a consequence of the existence of the norm \( N \) on \( A \).

However, we can reverse the process: Starting with a composition algebra \( H \) with norm \( N \), let \( x \) be a symbol and let \( H + xH \) be the set of pairs \( a + xb \) with \( a, b \in H \) and multiplication given by (24). Extending the norm to \( H + xH \) by

\[
N(a + xb) = N(a) + N(b)
\]

makes \( H \) and \( xH \) orthogonal. Now define the product on \( H + xH \) via (24):

\[
(a + xb)(c + xd) \overset{\text{def}}{=} (ac - db) + x(cb + \bar{a}d).
\]

Now \( H + xH \) will be a composition algebra exactly when

\[
[N(a) + N(b)][N(c) + N(d)] = N(ac - db) + N(cb + \bar{a}d).
\]

Expanding and cancelling terms, this will hold exactly when

\[
(cb, \bar{a}d) = (ac, d\bar{b}),
\]

which by (15) is equivalent to

\[
(a(cb), d) = ((ac)b, d)
\]

for all \( a, b, c, d \in H \), which is equivalent to \( H \) being associative. This proves:

**Proposition 8.3** Given a composition algebra \((H, N)\), the double \( H + xH \), with product as above and norm \( N(a + xb) = N(a) + N(b) \), is a composition algebra exactly when \( H \) is associative.

Starting with \( \mathbb{R} \) and the norm \( N(x) = x^2 \), the doubling process produces composition algebras \( \mathbb{C}, \mathbb{H}, \mathbb{O} \), the complex numbers, quaternions and octonions, respectively. There the process ends, because \( \mathbb{O} \) is not associative. So these are the only composition algebras, a theorem first proved by Hurwitz in 1898.

### 8.3 Parallelizable spheres

We remarked in section 4.3 that the spheres \( S^n \) are parallelizable exactly for \( n = 1, 3, 7 \). The proof that \( S^3 \) is parallelizable works in all cases, using multiplication in composition algebras.

Let \( A \) be composition algebra of dimension \( 2^k \). Since \( N \) is positive-definite, the set \( S = \{ x \in A : N(x) = 1 \} \) of all unit vectors in \( A \) is a sphere of dimension \( 2^k - 1 \). The tangent bundle to \( S \) is the manifold

\[
TS = \{(s, v) \in S \times A : (s, v) = 0 \}.
\]

Let \( V \) be the subspace of \( A \) orthogonal to \( 1 \). Then \( V = T_1 S \) is the tangent space to \( S \) at \( 1 \). More generally, for any \( x \in S \), we have \( (x, vx) = (x\bar{x}, v) = (1, v) = 0 \), so \( Vx = T_x S \) is the tangent space to \( S \) at \( s \). Hence the function

\[
X_v : S \to S \times A, \quad \text{given by} \quad X_v(x) = vx
\]

takes values in \( TS \) and is a smooth vector-field on \( S \). Since \( v \mapsto X_v(x) \) is an isomorphism \( V \cong T_x S \), it follows that \( S \) is parallelizable.
8.4 Automorphisms of composition algebras

An automorphism of a composition algebra $A$ is an $\mathbb{R}$-linear isomorphism $\sigma : A \to A$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in A$. The automorphisms of $A$ form a group $\text{Aut}(A)$, under composition.

**Lemma 8.4** If $\sigma \in \text{Aut}(A)$ then for all $x, y \in A$ we have

1. $N(\sigma(x)) = N(x)$;
2. $(\sigma(x), \sigma(y)) = (x, y)$;
3. $\sigma(x) = \sigma(\overline{x})$.

**Proof:** Each item is a consequence of the one before it, so we need only prove item 1.

Let $A_0$ be the subspace of $A$ orthogonal to $\mathbb{R}$, and let $S = \{x \in A_0 : N(x) = 1\}$ be the unit sphere in $A_0$. From (19) we have $x \in S$ if and only if $x^2 = -1$. It follows that every $\sigma \in \text{Aut}(A)$ preserves $S$.

Now let $x \in A_0$ is an arbitrary nonzero element, and let $n = N(x)^{1/2}$. Then $y := (n^{-1}x \in S$ so $\sigma(y) \in S$ so $\sigma(x) = ny \in nS \subset A_0$. Hence $\sigma$ preserves $A_0$ and $N(\sigma(x)) = n^2 = N(x)$. And since $\sigma$ is an $\mathbb{R}$-linear algebra automorphism we have $\sigma(r) = r$ for all $r \in \mathbb{R}$. Since $\mathbb{R}$ and $A_0$ are orthogonal, it follows that $\sigma$ preserves $N$ on all of $A$.

It follows that $\text{Aut}(A)$ is a subgroup of the orthogonal group $O(A, N) = \{g \in \text{GL}(A) : N(g(x)) = N(x) \text{ for all } x \in A\}$. For each $x, y \in A$ let $f_{x,y} : O(A, N) \to A$ be the function $f_{x,y}(g) = g(x)g(y) - g(xy)$. Then $G$ is the common zeros of the continuous functions $f_{x,y}$, so $G$ is closed in $O(A, N)$. It follows that $\text{Aut}(A)$ is a compact Lie group. All cases are tabulated below.

<table>
<thead>
<tr>
<th>$\mathbb{R}$</th>
<th>$\text{Aut}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$\mathbb{H}$</td>
<td>$\text{SO}_3$</td>
</tr>
<tr>
<td>$\mathbb{O}$</td>
<td>$G_2$</td>
</tr>
</tbody>
</table>

Let $H$ be a subalgebra of $A$ of dimension half that of $A$, and let $x \in A$ be orthogonal to $A$ with $N(x) = 1$. Then we have $A = H + xH$, an orthogonal sum, and the product in $A$ is derived from the product in $H$ via the formula (24).

The following result shows how to construct automorphisms of $A$.

**Proposition 8.5** Suppose we are given a subalgebra $J \subset A$, an algebra isomorphism $\tau : H \to J$, and an element $y \in A$ orthogonal to $J$. Then $\tau$ extends uniquely to an automorphism $\sigma$ of $A$ such that $\sigma(x) = y$. Conversely, any automorphism of $A$ arises in this way.
Proof: We define $\sigma$ by

$$\sigma(a + xb) = \tau(a) + y\tau(b).$$

This is the only possible extension of $\tau$ to $A$. It is clearly $\mathbb{R}$-linear and is bijective, since $\tau$ is bijective and the sums $H + xH = J + yJ$ are both direct. We must show that $\sigma$ preserves the product in $A$.

The key point is that we can apply Prop. 8.2 to both $(H, x)$ and $(J, y)$. Thus, we get

$$\sigma \left( ((a + xb) \cdot (c + xd)) \right) \overset{(24)}{=} \sigma \left( ((ac - db) + x(cb + âd)) \right)$$

$$= \tau(ac - db) + y\tau(cb + âd)$$

$$= \tau(a)\tau(c) - \tau(d)\tau(b) + y(\tau(c)\tau(b) + \tau(a)\tau(d))$$

$$\overset{(8.4)}{=} \tau(a)\tau(c) - \tau(d)\tau(b) + y \left( \tau(c)\tau(b) + \overline{\tau(a)\tau(d)} \right)$$

$$\overset{(24)}{=} (\tau(a) + y\tau(b)) \cdot (\tau(c) + y\tau(d))$$

$$= \sigma(a + xb) \cdot \sigma(c + xd).$$

Conversely, given any automorphism $\sigma$ of $A$, let $J = \sigma(H), y = \sigma(x)$. Then $N(y) = N(x) = 1$ and $(y, J) = (x, H) = 0$, by Lemma autoconj. Then $\sigma$ arises as in the first part of the Proposition from its restriction $\tau$ to $H$. ■

8.5 The Octonions $\mathbb{O}$

The octonions are the double of the quaternions. Thus we have

$$\mathbb{O} = \mathbb{H} + \mathbb{H}\ell,$$

where $\mathbb{H}$ is the quaternion algebra with basis $\{1, i, j, k\}$ and $\ell$ is orthogonal to $\mathbb{H}$ with $\ell^2 = -1$. Thus, $\mathbb{O}$ has $\mathbb{R}$-basis

$$1, i, j, k, \ell, i\ell, j\ell, k\ell.$$

Since the quaternions are associative the octonions are a composition algebra, with norm

$$N(x_0 + x_1i + x_2j + x_3k + x_4\ell + x_5i\ell + x_6j\ell + x_7k\ell) = \sum_{i=0}^{7} x_i^2.$$

For all $q \in \mathbb{H}$ we have $\ell q = \overline{q}\ell$ by (23), so from (24) the multiplication in $\mathbb{O}$ is given by

$$(p + q\ell)(r + s\ell) = (pr - s\overline{q}) + (q\overline{r} + sp)\ell.$$  

In particular, we have

$$j(i\ell) = k\ell, \quad i(k\ell) = j\ell, \quad k(i\ell) = j\ell,$$

and orthogonal elements $u, v$ anticommute: $uv = -vu$.

Let $V$ be the orthogonal complement of 1 in $\mathbb{O}$ and let $S$ be the six-dimensional sphere in $V$:

$$S = \{x \in V : N(x) = 1\}.$$  

We next show that any pair of orthogonal vectors in $S$ generates a quaternion subalgebra of $\mathbb{O}$. More precisely, we have:
Recall that automorphisms preserve the norm, so Prop. 8.7

\[ N(u) = N(v) = 1 \text{ and } (u, 1) = (v, 1) = 0, \text{ we have } u^2 = v^2 = -1 \text{ by (19). Likewise } \]

\[ N(uv) = N(u)N(v) = 1 \text{ and } (1, uv) = (u, v) = (-u, v) = 0, \text{ so } (uv)^2 = -1. \]

From (15) we have \((uv, u) = (v, uuv) = (v, 1) = 0\) and likewise \((uv, v) = 0\). Hence the vectors 
\(1, u, v, uv\) are orthonormal and \(uv\) anticommutes with \(u\) and \(v\) by (20). From the alternative law (21) 
we have \(v(uv) = -v(vu) = -(v^2)u = u\) and \((uv)u = -(vu)u = -v(u^2) = v\). This shows that 
\(v, uv\) satisfy the same relations as \(i, j, k\) so the lemma is proved. \[\blacksquare\]

The automorphism group \(G := \text{Aut}(\mathbb{O})\) is a compact subgroup of \(O(V) \cong O_7\). Every element of \(G\) may be constructed as follows. Choose orthogonal vectors \(u, v \in S\). By Lemma 8.6 these generate a quaternion subalgebra \(J\), spanned by \(1, u, v, uv\). Then choose \(x \in S\) orthogonal to \(u, v, uv\). Then by 
Prop. 8.5 there is a unique automorphism \(\sigma \in G\) such that \(\sigma(i) = u, \sigma(j) = v\) and \(\sigma(k) = x\). Thus, we may identify \(G\) with the manifold of triples 

\[ T = \{(u, v, x) \in S \times S \times S : (u, v) = (x, u) = (x, v) = (x, uv) = 0\}. \]

Let \(\pi : T \to S\) be the projection \(\pi(u, v, x) = u\), and let \(S_u\) be the five-dimensional sphere in \(S\) orthogonal to \(u\). Then \(\pi^{-1}(u)\) may be identified with the set of pairs 

\[ T_u = \{(v, x) \in S_u \times S_u : (x, v) = (x, uv) = 0\}, \]

and we have \(\dim T = \dim S + \dim T_u\). Let \(\pi_u : T_u \to S_u\) be the projection \(\pi_u(v, x) = v\). Then \(\pi_u^{-1}(v)\) is the three-dimensional sphere \(S_{u,v,uv}\) in \(S_u\) orthogonal to \(v\) and \(uv\), and we have \(\dim T_u = \dim S_u + S_{u,v,uv}\). Therefore 

\[ \dim G = \dim T = \dim S + \dim S_u + \dim S_{u,v,uv} = 6 + 5 + 3 = 14. \]

8.6 The SU3 in Aut(\(\mathbb{O}\))

Recall that automorphisms preserve the norm, so \(G = \text{Aut}(\mathbb{O})\) acts on the 6-sphere \(S = \{x \in V : N(x) = 1\}\).

Proposition 8.7 \(G\) acts transitively on \(S\).

Proof: Let \(x \in S\). Choose \(u \in S\) orthogonal to \(x\) and then choose \(v \in S\) orthogonal to each of \(x, u, uv\). Then \((x, uv) = -(ux, v) = 0\), so \(x\) is orthogonal to each of \(u, v, uv\) hence there is an automorphism \(\sigma \in G\) sending \(\ell\) to \(x\). \[\blacksquare\]

Let \(G_\ell\) be the stabilizer of \(\ell \in G\), and let \(W\) be the orthogonal complement of \(\ell\) in \(V\). Then if \(w \in W\) we have \((\ell w, 1) = -(w, \ell) = 0\) and \((\ell w, \ell) = (w, \ell \ell) = (w, 1) = 0\), so \(\ell W = W\). Since \(\ell^2 = -1\), we

\[10\text{The definite article in the title of this and later sections indicates that the subgroup is unique up to conjugacy in } \text{Aut}(\mathbb{O}).\]
can identify \( \mathbb{C} \) with the subalgebra of \( \mathbb{O} \) generated by \( \{1, \ell\} \) and multiplication by \( \ell \) makes \( W \) into a \( \mathbb{C} \)-vector space with basis \( \{i, j, k\} \). Define a hermitian form on \( W \) by
\[
(v, w) = (v, w) + \ell(v, w).
\]
For all \( \sigma \in G_\ell \) and \( v \in W \) we have \( \sigma(\ell v) = \ell \sigma(v) \), so
\[
\langle \sigma(v), \sigma(w) \rangle = (\sigma(v), \sigma(w)) + \ell(\sigma(v), \sigma(w)) = (\sigma(v), \sigma(w)) + \ell(\sigma(v), \sigma(w)) = (v, w) + \ell(v, w) = (v, w).
\]
It follows that \( G_\ell \) is contained in the unitary group \( U(W) \).

**Proposition 8.8** \( G_\ell = SU(W) \).

Since \( \dim G_\ell = \dim G - \dim S = 14 - 6 = 8 = \dim SU(W) \), it suffices to show that \( G_\ell \subset SU(W) \).

Let \( \sigma \in G_\ell \). Since \( \sigma \in U(W) \), there exist eigenvectors \( u, v, w \) for \( \sigma \) in \( W \) which are orthonormal with respect to the hermitian form \( (\ , \ ) \). Since \( uv \) is a unit vector orthogonal to both \( u \) and \( v \), we may take \( w = uv \). In fact, if \( \sigma(u) = e^{\theta \ell} u \) and \( \sigma(v) = e^{\phi \ell} v \), then (letting \( c = \cos \theta, c' = \cos \phi \), etc.) we have
\[
\sigma(uv) = \sigma(u)\sigma(v) = (e^{\theta \ell} u)(e^{\phi \ell} v) = (cu + s\ell u)(c'v + s'\ell v) = cc'uv + ss'(\ell u)(\ell v) + cs'\ell(vu) + sc'\ell(uv) = cc'uv - ss'uv - cs'\ell(uv) - sc'\ell(uv) = [\cos(\theta + \phi) - \sin(\theta + \phi)\ell]uv = e^{-(\theta + \phi)\ell}uv.
\]

Thus, the eigenvalues of \( \sigma \) are \( e^{\theta \ell}, e^{\phi \ell} \) and \( e^{-(\theta + \phi)\ell} \), so \( \det \sigma = 1 \) and \( \sigma \in SU(W) \).

Therefore the homogeneous space \( G_2 / SU_3 \) is equivariantly diffeomorphic to the 6-sphere \( S = S^6 \) and we have

**Corollary 8.9** Every automorphism of \( \mathbb{O} \) fixes a point on \( S \) and acts on the orthogonal 5-sphere via an element of \( SU_3 \).

### 8.7 The maximal torus in \( \text{Aut}(\mathbb{O}) \)

The above proof (with \( i, j, k \) instead of \( u, v, uv \)) shows that we have a torus \( T_\ell \subset G_\ell \) consisting of automorphisms which fix \( \ell \) and send
\[
i \mapsto e^{\alpha \ell} i, \quad j \mapsto e^{\beta \ell} j, \quad k \mapsto e^{\gamma \ell} k,
\]
where \( \alpha + \beta + \gamma = 0 \). This torus \( T_\ell \) is a maximal torus in \( G_\ell \).
In fact, $T$ is a maximal torus in $G$ as well. For suppose $U$ is a maximal torus of $G$ containing $T$. Then $U$ preserves the fixed-point set of $T$ in $\mathbb{O}$, which is the complex plane $\mathbb{R} \oplus \mathbb{R} \ell$. But $U$ fixes $\mathbb{R}$, hence $U$ must preserve the orthogonal complement $\mathbb{R} \ell$ of $\mathbb{R}$ in this plane. Since $U$ is connected and preserves length, it follows that $U$ acts trivially on $\mathbb{R} \ell$, so $U \subset G_\ell$. But $T$ is a maximal torus in $G_\ell$, so $U = T$.

The normalizer $N(T)$ must also preserve the fixed-point set of $T$ in $\mathbb{O}$, and hence must preserve $\{\pm \ell\}$. Now $N(T)$ is not connected, so it can change the sign of $\ell$. For example, the element $\sigma_2 \in \text{Aut}(\mathbb{O})$ acting by $+I$ on $\mathbb{H}$ and $-I$ on $\mathbb{H} \ell$ lies in $N(T)$ and acts on $T$ by $\sigma_2 t \sigma_2^{-1} = t^{-1}$. The subgroup of $N(T)$ fixing $\ell$ lies in $G_\ell$, and has index two in $N(T)$. Since the Weyl group of $SU(3)$ is $S_3$, it follows that the Weyl group $W = N(T)/T$ is $S_3 \times \langle -1 \rangle \simeq D_6$.

In general, the extension

$$1 \longrightarrow T \longrightarrow N(T) \longrightarrow W \longrightarrow 1$$

is not split (eg. in $S^3$). In this case however, there is a copy of $S_3$ inside $SU(3)$ projecting isomorphically onto the Weyl group of $SU(3)$. This subgroup is generated by the involutions

$$\tau = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \tau' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$  

Hence the group generated by $\langle \sigma_2, \tau, \tau' \rangle$ is isomorphic to $D_6$ and projects isomorphically onto $W$.

8.8 The $SO_4$ in $\text{Aut}(\mathbb{O})$

Let $G_\mathbb{H} = \{ \sigma \in G : \sigma(\mathbb{H}) = \mathbb{H} \}$ be the stabilizer of the quaternion subalgebra $\mathbb{H}$ spanned by $\{1, i, j, k\}$.

**Proposition 8.10** $G_\mathbb{H} = SO_4$.

**Proof:** Since $G_\mathbb{H}$ preserves $\mathbb{H}$, it also preserves the orthogonal complement $\ell \mathbb{H}$. If $\sigma \in G_\mathbb{H}$ acts trivially on $\ell \mathbb{H}$ then $G_\mathbb{H}$ fixes $\ell$ so $\sigma(\ell u) = \sigma(u) \ell = u \ell$ so $\sigma$ acts trivially on $\mathbb{H}$ as well, so $\sigma = 1$. Thus $G_\mathbb{H}$ acts faithfully on $\ell \mathbb{H}$ (though not on $\mathbb{H}$) and we have an injective homomorphism $G_\mathbb{H} \hookrightarrow O(\mathbb{H}) \simeq O_4$. To show that $G_\mathbb{H} \simeq SO_4$ it remains to show $G_\mathbb{H}$ is connected of dimension six.

By restriction we get another map $r : G_\mathbb{H} \rightarrow \text{Aut}(\mathbb{H}) = SO_3$, which is seen to be surjective by taking $H = J = \mathbb{H}$ in Prop. 8.5. Set $K = \ker r = \{ \sigma \in G : \sigma(u) = u \forall u \in \mathbb{H} \}$. An automorphism $\sigma \in K$ is completely determined by its effect on $\ell$, and

$$\sigma(\ell) = u_\sigma \ell,$$

for some $u_\sigma \in \mathbb{H}$. Since $N(\ell) = N(\sigma(\ell)) = 1$ we have $N(u_\sigma) = 1$, so $u_\sigma \in S^3 \subset \mathbb{H}$. If $\tau \in K$ is another automorphism then since $\tau$ fixes $u_\sigma$ we have

$$u_{\tau \sigma} \ell = \tau \sigma(\ell) = \tau(u_\sigma \ell) = u_\sigma \tau(\ell) = u_\sigma (u_\tau \ell) = (u_\tau u_\sigma) \ell,$$

60
so $\sigma \mapsto u_\sigma$ is an injective homomorphism $K \hookrightarrow S^3$, which is also surjective by another application of Prop. 8.5. Thus, $G_H$ fits into an exact sequence

$$1 \longrightarrow S^3 \longrightarrow G_H \longrightarrow SO_3 \longrightarrow 1.$$ 

In particular $G_H$ is connected and $\dim G_H = 6$. Hence $G_H \simeq SO_4$ as claimed.

**Remark 1:** The proof shows that the intersection

$$G_\ell \cap G_H = SO_3 \subset SU_3,$$

is the subgroup of $SU_3$ consisting of real matrices.

**Remark 2:** From Prop. 8.5, it follows that $G_2$ acts transitively on quaternion subalgebras of $\mathbb{O}$. Hence $G_2/ SO_4$ is the eight-dimensional manifold of quaternion subalgebras of $\mathbb{O}$ and every automorphism of $\mathbb{O}$ stabilizes a quaternion subalgebra.

### 8.9 The Lie algebra of $\text{Aut}(\mathbb{O})$

The automorphism group $G$ of any finite dimensional $\mathbb{R}$-algebra $A$ is a closed subgroup of $GL(A)$. Hence the Lie algebra $\mathfrak{g}$ of $G$ is the Lie subalgebra of $\text{End}(A)$ given by

$$\mathfrak{g} = \{D \in \text{End}(A) : e^{tD} \in G \text{ for all } t \in \mathbb{R}\}.$$ 

Expanding and comparing both sides of the identity (for all $a, b \in A$)

$$e^{tD}(ab) = [e^{tD}a][e^{tD}b]$$

we find that $D \in \mathfrak{g}$ if and only

$$D(ab) = D(a)b + aD(b).$$

Such an operator is called a **derivation** of $A$, and

$$\mathfrak{g} = \text{Der}(A)$$

is the Lie algebra of derivations on $A$ with the Lie bracket $[D, D'] = DD' - D'D$ inherited from $\text{End}(A)$.

If $A$ is a composition algebra then $\text{Aut}(A) \subset O(A)$, the orthogonal group of the norm $N : A \rightarrow \mathbb{R}$, so $\text{Der}(A) \subset \mathfrak{so}(A)$ is represented by skew-symmetric matrices with respect to an orthonormal basis of $A$. Since $D(1) = D(1 \cdot 1) = 2D(1)$, it follows that $D(1) = 0$ for any derivation. Hence we in fact have $A \subset \mathfrak{so}(A_0)$.

Returning to the octonions, we see that the Lie group $G = \text{Aut}(\mathbb{O})$ has Lie algebra $\mathfrak{g} = \text{Der}(\mathbb{O}) \subset \mathfrak{so}_7$. We will use this representation to find the roots $R$ and the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R/\pm} \mathfrak{m}_\alpha.$$
As a subalgebra of $\mathfrak{g}$, the Lie algebra of $G_\ell$ is
\[
\{D \in \mathfrak{g} : D(\ell) = 0\}.
\]
These derivations are linear with respect to the subalgebra $C_\ell = \mathbb{R} \oplus \mathbb{R} \ell$, hence they may be represented as $3 \times 3$ matrices with respect to the $C_\ell$-basis $i, j, k$ of the hermitian space $W$. Our maximal torus $T \subset G_\ell$ consists of the automorphisms $t$ given by
\[
t(i) = e^{a \ell} i, \quad t(j) = e^{b \ell} j, \quad t(k) = e^{c \ell} k, \quad t(\ell) = \ell
\]
with $a, b, c \in \mathbb{R}$ such that $a + b + c = 0$. Its Lie algebra $t$ consists of the derivations $D \in \mathfrak{g}_\ell$ given by
\[
D(i) = a i, \quad D(j) = b j, \quad D(k) = c k, \quad D(\ell) = 0,
\]
and we have
\[
\mathfrak{g}_\ell = t \oplus m_{b-c} \oplus m_{a-c} \oplus m_{a-b},
\]
where
\[
m_{b-c} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -z \\ 0 & z & 0 \end{bmatrix} : z \in \mathbb{C}_\ell \right\}, \quad m_{a-c} = \left\{ \begin{bmatrix} 0 & 0 & -\bar{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{bmatrix} : z \in \mathbb{C}_\ell \right\}, \quad m_{b-c} = \left\{ \begin{bmatrix} 0 & -\bar{z} & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : z \in \mathbb{C}_\ell \right\}.
\]
The remaining root spaces consist of derivations which are not $C_\ell$-linear, so we have to use $7 \times 7$ real matrices in $\mathfrak{so}_7$. We order the basis as $(i, j, k, i\ell, j\ell, k\ell, \ell)$. The missing root spaces are each spread out in two root spaces of $SO_7$, whose maximal torus is like $T$ but without the condition $a + b + c = 0$. For example, $\pm (b + c)$ and $\pm a$ for $SO_7$ both restrict to $\pm a$ in $G$. This allows us to find the derivations comprising the remaining root spaces, as follows:
We therefore have 12 roots
\[ R = \pm \{ a - b, b - c, a - c, a, b, c \}. \]

These are linear functionals on
\[ t = \{(a, b, c) : a + b + c = 0\}. \]

To see \( R \) more explicitly, identify \( t \) with its dual space \( t^* \), using the dot product. The three linear functionals
\[ (a, b, c) \mapsto a, b, c \]
become dot product with
\[ \frac{1}{3}(2, -1, -1), \quad \frac{1}{3}(-1, 2, -1), \quad \frac{1}{3}(-1, -1, 2), \]
and we get
\[ R = \pm \{(1, -1, 0), (0, 1, -1), (1, 0, -1), \frac{1}{3}(2, -1, -1), \frac{1}{3}(-1, 2, -1), \frac{1}{3}(-1, -1, 2)\}. \]

Setting \( \alpha = \frac{1}{3}(-1, -1, 2), \beta = (0, 1, -1) \), we have
\[ R = \pm\{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}, \]
the root-system of type \( G_2 \). The action of the Weyl group \( W \) on \( t \) is generated by reflections about the lines perpendicular to the roots.\(^{11}\)

\(^{11}\)Picture taken from Wikipedia.
8.10 The nonsplit extension $2^3 \cdot \text{GL}_3(2)$ in $\text{Aut}(\mathcal{O})$ and seven Cartan subalgebras

Let $\Gamma$ be the subgroup of $\text{Aut}(\mathcal{O})$ stabilizing the set

$$\mathcal{B} = \{\pm i, \pm j, \pm k, \pm \ell, \pm i\ell, \pm j\ell, \pm k\ell\}.$$ 

Letting $O_7(\mathbb{Z}) \cong C_2^7 \rtimes S_7$ denote the stabilizer in $O_7$ of the lattice spanned by $\mathcal{B}$ in $\mathcal{O}_0$, we have

$$\Gamma = \text{Aut}(\mathcal{O}) \cap O_7(\mathbb{Z}).$$

Since $\Gamma$ permutes the seven lines $\mathbb{R}i, \mathbb{R}j, \ldots$ and preserves the structure of the projective plane $\mathbb{P}^2(\mathbb{F}_2)$ whose points are identified with these lines, it follows that this permutation action gives a homomorphism $\Gamma \to \text{GL}_3(2)$, whose kernel is the subgroup of sign changes of the vectors in $\mathcal{B}$. We find the order of $\Gamma$ as follows. Every element $\gamma \in \Gamma$ is determined by $\gamma(i) = u, \gamma(j) = v, \gamma(\ell) = w$. We have 14 choices for $u \in \mathcal{B}$, then 12 choices for $v$, and then $w$ can be any vector in $\mathcal{B}$ orthogonal to $u, v, uv$, which leaves 8 choices. Hence

$$\Gamma = 14 \cdot 12 \cdot 8 = 2^3 \cdot 168.$$ 

The count also shows there are $2^3$ sign changes. Hence the map $\Gamma \to \text{GL}_3(2)$ is surjective.

It is not possible to permute the seven vectors $\{i, j, k, i\ell, j\ell, k\ell, \ell\}$ in 168 ways by automorphisms of $\mathcal{O}$ without changing some signs. This means the extension

$$1 \longrightarrow C_2^3 \longrightarrow \Gamma \longrightarrow \text{GL}_3(2) \longrightarrow 1$$

is nonsplit. However it splits over the subgroup $F_{21}$ corresponding to a Borel subgroup of $\text{PSL}_2(7) \cong \text{GL}_3(2)$. In particular, there is an element $\sigma_7 \in \text{Aut}(\mathcal{O})$ permuting the seven lines $\mathbb{R}i, \mathbb{R}j, \ldots$ in a cycle.

A Cartan subalgebra in $\mathfrak{g}$ is the Lie algebra of a maximal torus in $G$. Each line $\lambda \in \{\mathbb{R}i, \mathbb{R}j, \ldots\}$ generates a copy of $\mathbb{C}$ in $\mathcal{O}$. The subalgebra of $\mathfrak{g}$ killing $\lambda$ and preserving the remaining six lines is a Cartan subalgebra $\mathfrak{t}_\lambda$. For example, if $\lambda = \ell$, we get the Cartan subalgebra $\mathfrak{t}$ in (25). It follows that the Lie algebra $\mathfrak{g} = \text{Der}(\mathcal{O})$ is a direct sum

$$\mathfrak{g} = \mathfrak{t}_i \oplus \mathfrak{t}_j \oplus \mathfrak{t}_k \oplus \mathfrak{t}_{i\ell} \oplus \mathfrak{t}_{j\ell} \oplus \mathfrak{t}_{k\ell} \oplus \mathfrak{t}_\ell$$

of seven Cartan subalgebras, permuted simply-transitively by the element $\sigma_7$. This is completely analogous to the decomposition

$$\mathfrak{so}_3 = \mathfrak{t}_i \oplus \mathfrak{t}_j \oplus \mathfrak{t}_k,$$

arising from the coordinate lines $\mathbb{R}i$, $\mathbb{R}j$, $\mathbb{R}k$ in the quaternion algebra $\mathbb{H}$. 