Proposition VI.20

Given: $ABCDE \sim FGHKL$
Claim: $ABCDE$ and $FGHKL$ are divided into triangles that are:
1) similar
2) equal in multitiude (equal number of triangles)
3) in the same ratios as the wholes
4) And $ABCDE : FGHKL = AB^2 : FG^2$
Proof:

Join $BE, EC, GL, LH$ (I.post.1)

Since $ABCDE \sim FGHKL$

$\angle BAE = \angle GFL$ and $AB : AE = FG : FL$ (VI. Def.1)

$\because \triangle ABE \sim \triangle FGL$ (VI.6, VI.4)

So $\angle ABE = \angle FGL$ (VI. Def.1)

But $\angle ABC = \angle FGH$ since $ABCDE \sim FGHKL$

So $\angle EBC = \angle LGH$

And $\because \triangle ABE \sim \triangle FGL$

$\therefore EB : AB = LG : FG$

$\therefore ABCDE \sim FGHKL$

$\therefore AB : BC = FG : GH$

$\therefore EB : BC = LG : GH$ (V.22)

\[\textbf{V.22}\]

If $A : B = D : E$ and $B : C = E : F$

then $A : C = D : F$
\[ \angle EBC = \angle LGH, \text{ and sides about the angle are proportional} \]
\[ \therefore \triangle EBC \sim \triangle LGH \text{ (VI.6, VI.4)} \]
Likewise, \( \triangle ECD \sim \triangle LHK \)

To prove 3), connect \( AC, BD, FH, GK \) (I.post.1)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}

\textbf{Something useful to know...}
\begin{itemize}
  \item Antecedents: \( \triangle ABE, \triangle EBC, \triangle ECD \)
  \item Consequents: \( \triangle FGL, \triangle LGH, \triangle LHK \)
\end{itemize}

We want to prove:
\( \triangle ABE : \triangle FGL = \triangle EBC : \triangle LGH = \triangle ECD : \triangle LHK \)
\[ \therefore \angle ABC = \angle FGH \text{ and } AB : BC = FG : GH \text{ (from } ABCDE \sim FGHKL) \]
\[ \therefore \triangle ABC \text{ and } \triangle FGH \text{ are equiangular (VI.6)} \]
\[ \therefore \angle BAC = \angle GHF \text{ and } \angle BCA = \angle GFH \]
\[ \therefore \angle BAM = \angle GFN \]
\[ \therefore \triangle ABE \sim \triangle FGL \text{ and } \angle ABM = \angle FGN \]
\[ \therefore \angle AMB = \angle FNG \text{ (I.32)} \]
\[ \triangle ABM \sim \triangle FGN \text{ (VI.4)} \]
\[ \triangle BMC \sim \triangle GNH \text{ since they are equiangular (VI.4)} \]

\[ AM : MB = FN : NG \]
\[ BM : MC = GN : NH \]
\[ \therefore AM : MC = FN : NH \text{ (V.22)} \]

\[ \triangle ABM : \triangle MBC = \triangle AME : \triangle EMC \text{ (VI.1)} \]

So \[ \triangle ABM : \triangle MBC = \triangle ABE : \triangle CBE \text{ (V.12)} \]

But \[ \triangle ABM : \triangle MBC = AM : MC \]
\[ \therefore AM : MC = \triangle ABE : \triangle CBE \text{ (V.11)} \]

Similarly, \[ FN : NG = \triangle FGL : \triangle GLH \]

And since \[ AM : MC = FN : NH \text{ and } \triangle ABE : \triangle CBE = \triangle FGL : \triangle GLH \]
\[ \text{ (V.11)} \]

So \[ \triangle ABE : \triangle FGL = \triangle CBE : \triangle GLH \]

Similarly, \[ \triangle CBE = \triangle GLH : \triangle ECD : \triangle LHK \]
\[ \therefore \triangle ABE : \triangle FGL = \triangle CBE : \triangle GLH = \triangle ECD : \triangle LHK \]

\[ \therefore \triangle ABE : \triangle FGL = ABCDE : FGHKL \text{ (V.12)} \]

But \[ \triangle ABE : \triangle FGL \text{ is duplicate } AB : FG \text{ (VI.19)} \]

Therefore \[ ABCDE : FGHKL = AB^2 : FG^2 \]

Q.E.D.