1 Introduction

Let $G$ be a compact connected Lie Group with Lie algebra $\mathfrak{g}$. $T$ a maximal torus of $G$ with Lie Algebra $\mathfrak{t}$. Let $W = N_G(T)/T$ be the Weyl group of $T$ in $G$. $W$ acts on $\mathfrak{t}$ through the $\text{Ad}$ representations. $W$ is generated by reflections across kernels of roots of $\mathfrak{t}$ in $\mathfrak{g} \otimes \mathbb{C}$ or if you like the positive real roots.

The main result of these notes is that $H(G/T)$ vanishes in odd degrees. We will in fact provide a ring isomorphism $H(G/T)$ to a purely algebraic structure.

2 Background/Review

Let $(\langle , \rangle)$ be the $\text{Ad}$-invariant inner product on $\mathfrak{g}$ (average all inner products on $\mathfrak{g}$ or take the negative of the Killing Form). We then have an orthogonal decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{t}$. For $X, Y, Z \in \mathfrak{g}$, the inner product satisfies $\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$. Note that $\text{Ad}(T)$ has no nonzero invariant vectors in $\mathfrak{m}$ and no nonzero element of $\mathfrak{m}$ has zero bracket will all of $\mathfrak{t}$ (by maximally of $\mathfrak{t}$ as an abelian subalgebra).

A generic element $H_0 \in \mathfrak{t}$ is such that $\exp H_0$ is a regular element of $T$ (i.e. its powers are dense in $T$). Note, $H_0 \in \mathfrak{t}$ is regular iff $\text{Ad}(G)$ centralizer is precisely $\text{Ad}(T)$. For the remainder of this text, we choose some particular generic element $H_0 \in \mathfrak{t}$

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_v$ be an orthogonal decomposition given by the real irreducible representation of $T$, which are 2 dimensional. For $H \in \mathfrak{t}$, the eigenvalues of $\text{Ad} \exp H$ on $\mathfrak{m}_i$ are $\{\exp(\pm \sqrt{-1} \alpha_i(H))\}$, where $\alpha_i \in \mathfrak{t}^*$. We let the set of positive roots $\Delta^+ = \{\alpha_1, \ldots, \alpha_v\}$ be the set of roots that take positive values on our generic element $H_0$. Note that since $W$ acts faithfully on $\mathfrak{t}$, its image in $\text{GL}(\mathfrak{t})$ is generated by reflections about the kernels of elements in $\Delta^+$.

Since $\mathfrak{m}_i$ are preserved by $\text{ad}(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{i+v}\}$ for $\mathfrak{m}_i$ such that the matrix for $\text{ad}(H) |_{\mathfrak{m}_i}$, $H \in \mathfrak{t}$, is $\begin{bmatrix} 0 & \alpha_i(H) \\ -\alpha_i(H) & 0 \end{bmatrix}$. Then ad-invariance of the inner product gives us

$$\langle H, [X_i, X_j] \rangle = \langle [X_i, H], X_j \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+v}, X_j \rangle$$

for $1 \leq i \leq v$, $1 \leq j \leq 2v$. Above, the right hand can be nonzero only if $j = i + v$. Thus, if $j \neq i \pm v$, then $[X_i, X_j] \in \mathfrak{m}$.

For $1 \leq i \leq v$, we let $H_i = [X_i, X_{i+v}]$, which is $\text{Ad}(T)$-invariant so $H_i \in \mathfrak{m}$ and $\text{ad}(H_i) \mathfrak{m}_i \subset \mathfrak{m}_i$. The span of $X_i, X_{i+v}, H_i$ is a Lie subalgebra of $\mathfrak{g}$ that is actually isomorphic to $\mathfrak{su}(2)$.

3 Invariant Theory

Let $\mathcal{P} = \bigoplus_{p=0}^{\infty} \mathcal{P}^p$ be the symmetric algebra on $\mathfrak{t}^*$ (i.e. $\mathcal{P}^p = (\mathfrak{t}^*)^p / \sim$ where $\lambda_1 \otimes \ldots \otimes \lambda_p \sim \lambda_{\sigma(1)} \otimes \ldots \otimes \lambda_{\sigma(p)}$ for $\sigma \in S_p$). One can think of $\mathcal{P}$ as polynomials over $\mathbb{R}$ where the monomials are products of functionals on $\mathfrak{t}$. The adjoint action of $W$ on $\mathfrak{t}$ induces an action/representations of $W$ on $\mathcal{P}$ by degree-preserving algebra automorphisms (for $\lambda \in \mathfrak{t}^*$ and $w \in W$, the action is $\lambda \mapsto \lambda \circ \text{Ad}(w^{-1})$).

We will be interested in the $W$-invariant polynomials $\mathcal{P}^W$.

**Example 1.** For $U(n)$, $\mathcal{P}^W$ is generated by elementary symmetric polynomials. For $U(n)$, $\mathfrak{t}$ is the set of diagonal complex matrices with $a_j \sqrt{-1}$ on the diagonal and $W$ acts as $S_n$, $\mathfrak{t}$ on by permuting $a_j$.

**Theorem 2.** (Chevalley) The ring $\mathcal{P}^W$ has algebraically independent homogeneous generators $F_1, \ldots, F_l$ with $\mathcal{P}^W = \mathbb{R}[F_1, \ldots, F_l]$ where $l = \dim \mathfrak{t}$. (Recall: algebraically independent means that the homomorphism $\mathbb{R}[X_1, \ldots, X_l] \to \mathbb{R}[F_1, \ldots, F_l]$ given by $X_i \mapsto F_i$ is an isomorphism)

The generators are numbered such that $\deg F_1, \ldots, \deg F_l$. We will call the numbers $m_i = \deg F_i - 1$ the exponents of $W$ acting on $\mathfrak{t}$. It is known that $m_1 + \ldots + m_l = v$ and $(1 + m_1) \ldots (1 + m_l) = |W|$.

**Example 3.** $SU(n)$ the $m_i$’s are $1, \ldots, n-1$ and for $G_2$ they are $1, 5$. Note that for $SU(n)$ you loose the generator it degree 1, which you had for $U(n)$, because of linear dependence. For $G_2$, the Lie algebra of $T$ is that of $SU(3)$ but the action of $W$ is extended by an inversion.
Let $\mathcal{D}$ be the ring of constant coefficient differential operators on $\mathcal{P}$. We can thing of $\mathcal{D}$ as the symmetric algebra $S(t)$, where $H \in t$ corresponds to the function on $t^*$ given by evaluation at $H$ (e.g. $H \cdot (\lambda_1 \lambda_2) = \lambda_1(H) \lambda_2 + \lambda_2(H) \lambda_1$ or the directional derivative for the vector $H$). We have that $W$ acts naturally on $\mathcal{D}$ (by it’s action on $S(t)$) and we define the “harmonic polynomials” in $\mathcal{D}$ to be those annihilated by the $W$-invariant differential operators

$$\mathcal{H} = \{ f \in \mathcal{P} : \mathcal{P} f = 0 \}.$$

One can think of $\mathcal{H}$ as the solution to a set of differential equations.

Let $\mathcal{H}^p = \mathcal{H} \cap \mathcal{D}^p$, then $\mathcal{H} = \bigoplus_p \mathcal{H}^p$ since a differential operator is $W$ invariant if and only if each homogeneous component in $W$ invariant (think the action of $W$ on $S(t)$). Note that the action of $W$ on $\mathcal{D}$ preserves $\mathcal{H}$ (for $g \in W$, $p \in \mathcal{P}$, $D \in \mathcal{D}$, we have that $D(g \cdot p) = (g^{-1} \cdot D)(p)$).

**Proposition 4.** If $\mathcal{J}$ is the ideal generated by the elements of $\mathcal{P}^W$ of positive degree, then $\mathcal{P} = \mathcal{H} \oplus \mathcal{J}$ and multiplication is a linear isomorphism $\mathcal{H} \otimes \mathcal{P}^W \rightarrow \mathcal{P}$.

The former gives us that $\mathcal{P}/\mathcal{J}$ is isomorphic to $\mathcal{H}$ as $W$ modules (Note: they are in fact isomorphic to the regular representation of $W$). The isomorphism $\mathcal{H} \otimes \mathcal{P}^W \sim \mathcal{P}$ implies

$$\sum_{p \geq 0} \dim \mathcal{H}^p \mathcal{P}^p = \prod_{i=1}^l (1 + t + t^2 + \ldots + t^{m_i}) \text{ (where } l = \dim t)$$

which shows that $\dim \mathcal{H}^v = 1$ and $\mathcal{H}^p = 0$ for $p > v$. This formula is deduced from

$$\sum_p \dim \mathcal{P}^p \mathcal{P}^p = \left( \sum_p \dim \mathcal{H}^p \mathcal{P}^p \right) \left( \sum_p \dim (\mathcal{P}^W \cap \mathcal{P}^p) \mathcal{P}^p \right),$$

$$\sum_p \dim \mathcal{P}^p \mathcal{P}^p = (1 + t + t^2 + \ldots)^l = \frac{1}{(1-t)^l} \text{ and } \sum_p \dim (\mathcal{P}^W \cap \mathcal{P}^p) \mathcal{P}^p = \prod_{i=1}^l \frac{1}{(1-t^{m_i})^l}.$$

The primordial harmonic polynomial is $\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathcal{H}^v$. For $U(n)$ this is the Vandermonde determinant $\prod_{i<j} (x_i - x_j)$, which is transformed by the sign character by the action of $S_n$. In general, $W$ act like the sign character on the span of $\Pi$, where the sign character $\varepsilon : W \rightarrow \{ \pm 1 \}$ gives the parity of the number of reflections for each $g \in W$. Any other polynomial whose span is preserved by the action of the sign character vanishes on all root hyperplanes and so is divisible by $\Pi$. Thus $\Pi$ generates $\mathcal{H}^v$ as dim $\mathcal{H}^v = 1$.

We may now state the theorem we will discuss at the end of this talk

**Theorem 5.** (Borel) There is a degree-doubling $W$-equivariant ring isomorphism

$$c : \mathcal{P}/\mathcal{J} \rightarrow H(G/T).$$

Consequently, $\mathcal{H}_2 \simeq H(G/T)$, where the subscript indicated degree doubling.

### 4 Invariant Differential Forms

Let $G$ act transitively on a manifold $M$ (think $M = G/T$). If $\tau_g$ is the diffeomorphism given by $g \in G$, then a differential $p$-form $\omega \in \Omega^p(M)$ is $G$-invariant if $\tau_g^* \omega = \omega$ for all $g \in G$. Since $G$ acts transitively, such a form is determined by its value at one point on $M$.

**Lemma 6.** Every de Rham cohomology class of $M$ is represented by a $G$-invariant form and the complex of $G$-invariant forms is preserved by the exterior derivative.

**Definition 7.** We define $\Lambda^n$ as the set of all skew-symmetric multilinear maps $\omega : n \times \ldots \times n \rightarrow \mathbb{R}$ where the domain has $p$ terms.

**Proposition 8.** The complex $\{(\Lambda^n)^K, \delta\}$ computes $H^*(M)$, where $K$ is the stabilizer of a point $o \in M$, $g = t \oplus n$ with $t$ the Lie algebra of $K$, and $\delta$ is defined below.
Proof. Identify $M = G/K$ and note that $T_o(M)$ is naturally identified with $n$. Thus, an invariant form $\tilde{\omega}$ is determined by a skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o : n \times \cdots \times n \to \mathbb{R},$$

that is $\omega \in \Lambda^p n^*$. The invariance of $\tilde{\omega}$ under $K$ implies that $\omega$ is $\text{Ad}(K)$ invariant. Conversely, any element $\omega \in (\Lambda^p n^* )^K$ determines a $G$ invariant form $\tilde{\omega}$ by

$$\tilde{\omega}_g o((d\tau_g)X_1, \ldots, (d\tau_g)X_p) = \omega(X_1, \ldots, X_p),$$

for $X_1, \ldots, X_p \in n \simeq T_o(M)$ and $g \in G$. Thus, we may identify the $G$-invariant $p$-forms with $(\Lambda^p n^* )^K$. The exterior derivative then become $\delta : (\Lambda^p n^* )^K \to (\Lambda^{p+1} n^* )^K$ given by

$$\delta \omega(X_0, \ldots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_n, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

Where $[X_i, X_j]_n$ is the projection of $[X_i, X_j]$ on $n$ along $r$ and $\hat{\cdot}$ means the term is omitted. By the Lemma, the complex $\{ (\Lambda^p n^* )^K, \delta \}$ computes $H^*(M)$.

\[ \square \]

Example 9. Define $\omega(X, Y, Z) = \langle X, [Y, Z] \rangle$ then $[\omega] \neq 0 \in H^3(G)$. In particular, $S^n$ is not a Lie group for $n > 3$.

5 Cohomology of Flag Manifolds

We will use Morse Theory to show that the odd dimensional cohomology of $G/T$ vanishes. We can further use this approach to decompose the flag manifold $G/T$ into cells. This is called the Bruhat Decomposition. This process will be the generalization of decomposing the $S^2 = SU(2)/T$ into a 0-cell and a 2-cell.

We will find a Morse function $f$ on $G/T$. For a smooth manifold $M$, a morse function $f : M \to \mathbb{R}$ is a smooth function with non-singular Hessian $H_x f$ at each critical point $x$. The function we will be the analogue of the dot product of vectors on a 2-sphere with the vector pointing to the north pole. The span of the gradient flow lines emanating from a critical point will provide us with a cell decomposition. For the sphere the flow lines from the south pole give us the 2-cell and the north pole, which has no flow lines emanating, gives us the 0-cell.

If $f$ is a Morse function and $x$ is critical point, let $\lambda(x)$ be the number of negative eigenvalues of $H_x f$. Then the Morse polynomial is $M_t(f) = \sum \lambda(x) t^x$ over the critical points $x$ of $f$.

Theorem 10. For a morse function $f : M \to \mathbb{R}$, we have that $M_t(f) \geq \sum_i \dim H^i(M)t^i$. Moreover, if the morse polynomial has no consecutive exponents, equality holds.

To construct a Morse function on $G/T$, we take the regular element $H_0 \in t$ that we chose for the positive roots. Recall that the $\text{Ad}(G)$ centralizer of $H_0$ is exactly $\text{Ad}(T)$, so we may view $G/T \subset \mathfrak{g}$ as the $\text{Ad}(G)$ orbit of $H_0$ (analogous to $S^2 \subset \mathbb{R}^3$). We define $f : G/T \to \mathbb{R}$ by

$$f(gT) = \langle \text{Ad}(g) H_0, H_0 \rangle.$$ 

For $X \in \mathfrak{g}$, we can compute the vector field

$$\tilde{X}f(gT) = \frac{d}{ds} f(\exp(sX)gT) |_{s=0} = \langle \text{Ad}(g) H_0, [H_0, X] \rangle,$$

where the last equality is given by $\text{ad}$ invariance of the inner product. Since the centralizer of $H_0$ in $\mathfrak{g}$ is exactly $t$ as $H_0$ is regular, it follows that the image of $\text{ad}(H_0)$ is $\mathfrak{m}$. So $gT$ is a critical point of $f$ if and only if $\langle \text{Ad}(g) H_0, \mathfrak{m} \rangle = 0$. Therefore, $\text{Ad}(g) H_0 \in t$ by the orthogonal decomposition of $\mathfrak{g}$. It follows that $\text{Ad}(g) H_0 = \text{Ad}(w) H_0$ for some $w \in W$ and that $wT$, for $w \in W$, are precisely the critical points of $f$.

Let $X_1, \ldots, X_{2n}$ be the orthonormal basis for $\mathfrak{m}$ we discussed earlier. Note that the differential of $\pi : G \to G/T$ maps $\text{Ad}(w) \mathfrak{m} = \mathfrak{m}$ isomorphically onto $T_{\pi(T)}(G/T)$, so we may use our basis to compute
the Hessian at each point $wT$. If $h_{ij}$ is the $ij$ entry in $H_{wT}f$, then using our identities for the inner product

$$h_{ij}(wT) = X_i X_j f(wT) = \langle [X_i, \text{Ad}(w)H_0], [H_0, X_j] \rangle = -\alpha_i(\text{Ad}(w)H_0)\alpha_j(H_0)\langle X_{i\pm v}, X_{j\pm v} \rangle.$$  

Note that it follows that $h_{ij} = 0$ for $i \neq j$ and $h_{ii}(wT) = -\alpha_i(\text{Ad}(w)H_0)\alpha_i(H_0)$. Since $H_0$ is regular, then so is $\text{Ad}(w)H_0$ and therefore $h_{ii}(w) \neq 0$ and $H_{wT}f$ is non singular. Thus, as $\dim m = 2\nu$, the index $\lambda(wT)$ is twice the number $m(w)$ of positive roots $\alpha$ such that $H \mapsto \alpha(\text{Ad}(w)H)$ (i.e. $w^{-1} \cdot \alpha$) is again a positive root.

The Morse polynomial of $f$ is then $M_t(f) = \sum_{w \in W} t^{2m(w)}$. Since all the exponents of $M_t(f)$ are odd, $M_t(f) = \sum_i H^i(M)t^i$ and it follows that $H^i(M) = 0$ for $i$ odd. In particular, $\sum_i \dim H^{2i}(G/T) = |W|$.

The Schubert cell $X_w$ in the Bruhat Decomposition is the cell spanned by the flow lines of the gradient of $f$ emanating from $wT$. The dimension of this cell is then the number of positive eigenvalues of the $H_{wT}f$, or, equivalently, twice the number of positive roots that become negative under $w^{-1} \cdot \alpha$.

Note that $W$ acts on $G/T$ by $w \cdot gT = gw^{-1}T$, which gives us an action of $W$ on $H(G/T)$. Since $H(G/T)$ vanishes in odd degrees, the Lefschetz number associated to $wT$ in the Bruhat Decomposition is the cell spanned by the flow lines of the gradient emanating from $wT$. The dimension of this cell is then the number of positive eigenvalues of the $H_{wT}f$, or, equivalently, twice the number of positive roots that become negative under $w^{-1} \cdot \alpha$.

We can now give the proof of our final result, which we restate here.

**Theorem 11.** (Borel) There is a degree-doubling $W$-equivariant ring isomorphism

$$c : \mathcal{P}/\mathcal{J} \rightarrow H(G/T).$$

Consequently, $\mathcal{H}(2) \simeq H(G/T)$, where the subscript indicated degree doubling.

**Proof.** The idea is to describe $H(G/T)$ in terms of $G$-invariant differential forms. For each $\lambda \in \mathfrak{t}^*$, we extend $\lambda$ to all of $\mathfrak{g}$ by making it zero on $m$ and define an $\text{Ad}(T)$-invariant 2-form on $m$ by

$$\omega_{\lambda}(X, Y) = \lambda([X, Y]).$$

We can identify $\omega_{\lambda}$ with an honest $G$-invariant differential form $\tilde{\omega}_{\lambda}$ as before. The action of $W$ on $G$-invariant forms is given by its action on $G/T$. One can compute that $w \cdot \omega_{\lambda} = \omega_{\tilde{w}\lambda}$. Further, the Jacobi identity implies that $\delta \omega_{\lambda}(X, Y, Z) = \frac{1}{3}([[[X, Z]_m, Y] - [[X, Y]_m, Z] - [[Y, Z]_m, X]) = 0$. We let $c(\lambda) = [\tilde{\omega}_{\lambda}] \in H^2(G/T)$ and extend it to degree-doubling map

$$c : \mathcal{P} \rightarrow H(G/T)$$

which preserves the $W$-action on both sides. Since $H(G/T)$ is the regular representation of $W$, its $W$-invariants are 1-dimensional and can therefore only occur in $H^0(G/T)$. Since $c$ is $W$-equivariant, it follows that the kernel of $c$ contains the ideal $\mathcal{J}$. The rest of the proof deals with showing that $\mathcal{J}$ is exactly the kernel of $c$.

To prove that $\ker c = \mathcal{J}$, it suffices to show that $c$ is injective on $\mathcal{H}$ as $\mathcal{P} = \mathcal{H} \oplus \mathcal{J}$. This is done by induction starting at the highest degree of $2\nu$ and descending down. For degree $2\nu$ it suffices to show that $c(\Pi)$, where $\Pi$ is the primordial harmonic polynomial, is non zero in $H^{2\nu}(G/T)$.

For each root $\alpha_i \in \Delta^+$, we have element $X_i, X_{i+\nu}$ that form a basis for $m$ such that $[X_i, X_{i+\nu}] = H_i \mathfrak{t} \mathfrak{t}$. Recall that $[X_i, X_j] \in m$ is $j \neq i + \nu$ where $1 \leq i \leq \nu$. For each $i$, write $\omega_i = \omega_{\alpha_i}$. Then by definition $c(\Pi) = [\tilde{\omega}_{\alpha_1} \wedge \cdots \wedge \tilde{\omega}_{\alpha_\nu}]$ and we can evaluate

$$\omega_1 \wedge \cdots \wedge \omega_\nu(X_1, X_{1+\nu}, \ldots, X_\nu, X_{2\nu}) = \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} \text{sgn}(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+\nu)}) \cdots \omega_\nu(X_{\sigma(\nu)}, X_{\sigma(2\nu)}) =$$

$$= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} \text{sgn}(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+\nu)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(2\nu)}])$$

\]
Since \( \alpha_i([X_{\sigma(i)}, X_{\sigma(i+\nu)}]) = 0 \) unless \([X_{\sigma(i)}, X_{\sigma(i+\nu)}] \in m\), the term for \( \sigma \) is zero unless \( \sigma \) permutes the pairs \( \{i, i+\nu\} \), and possibly switches the order of members. Note that \( \sigma(\sigma) \) is minus one the number of switches, so it follows that

\[
\omega_1 \wedge \ldots \wedge \omega_\nu(X_1, X_{1+\nu}, \ldots, X_\nu, X_{2\nu}) = \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1([X_{\sigma(1)}, X_{\sigma(1)+\nu}] \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(\nu)+\nu}]) =
\]

\[
= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1(H_{\sigma(1)}) \cdots \alpha_\nu(H_{\sigma(\nu)}) = \frac{2^\nu}{(2\nu)!} \partial_1 \cdots \partial_\nu \Pi
\]

where \( \partial_i \) is the derivation of \( \mathcal{D} \) extending \( \lambda \mapsto \lambda(H_i) \). Since the pairing \( \mathcal{D} \otimes \mathcal{D} \to \mathbb{R} \) given by \( (D, f) \mapsto (Df)(0) \) is perfect, it follows that there is a degree \( \nu \) differential operator that pairs non trivially with \( \Pi \). Further, since an irreducible \( W \)-module can only pair non trivially with its dual, and the self-dual character \( \varepsilon \) occurs with multiplicity one in \( \mathcal{D} \), affored by \( \partial_1 \cdots \partial_\nu \), it follows that \( \partial_1 \cdots \partial_\nu \Pi \neq 0 \) and \( c(\Pi) \neq 0 \).

We may now inductively assume that \( c : \mathcal{H}^k \to H^{2k}(G/T) \) is injective for some \( k \leq \nu \). Let \( V = \mathcal{H}^{k-1} \cap \ker c \). Note that \( V \) is preserved by \( W \) since \( c \) is \( W \)-equivariant. Since the sign character is absent from \( \mathcal{H}^{k-1} \), there is a possible root \( \alpha \) such that the reflection \( s_\alpha \) along the associated hyperplane does not act like \( -I \) on \( V \). We can then decompose \( V = V_+ \oplus V_- \) according to the eigenspaces of \( s_\alpha \). If \( V \neq 0 \), then \( V_+ \neq 0 \) so we may take some \( f \in V_+ \). Now \( c(\alpha f) = c(\alpha) c(f) = 0 \) and \( \alpha f \) is in degree \( k \), so \( \alpha f \in \mathcal{J} \) by assumption. Let \( h_1, \ldots, h_{|W|} \) be a basis for \( \mathcal{H} \) with \( h_1, \ldots, h_r \) \( s_\alpha \)-skew and the rest \( s_\alpha \)-invariant. By Chevalley’s Theorem, we can write \( \alpha f = \sum_i h_i \tau_i \), with \( \tau_i \) \( W \)-invariant of positive degree. Since \( \alpha f \) is \( s_\alpha \)-skew by construction, the sum only goes up to \( r \). For \( i \leq r \), the polynomial \( h_i \) must vanish on \( \ker \alpha \) and therefore \( h_i = \alpha h'_i \) for some \( h'_i \in \mathcal{D} \). Then it follows that \( f = \sum_{i=1}^r h'_i \tau_i \in \mathcal{J} \) and \( f \) is harmonic. Thus, we must have that \( f = 0 \) and \( c \) is injective on \( \mathcal{H}^{k-1} \). By induction, \( c \) is injective and since \( H(G/T) \) vanishes in odd degree, the proof is complete.

\[ \square \]