EULER-POINCARÉ PAIRINGS AND ELLIPTIC REPRESENTATIONS OF WEYL GROUPS AND p-ADIC GROUPS

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Abstract. The space of elliptic virtual representations of a p-adic group is endowed with a natural inner product $EP(\ ,\ )$, defined analytically by Kazhdan and homologically by Schneider-Stuhler. Arthur has computed EP in terms of analytic R-groups. For Iwahori spherical representations, we show that EP can also be expressed in terms of a corresponding inner product on space of elliptic virtual representations of Weyl groups. This leads to an explicit description of both elliptic representation theories, in terms of the Kazhdan-Lusztig and Springer correspondences.

1. Introduction

Let G be a connected split adjoint group over a non-archimedean local field F of arbitrary characteristic. Schneider and Stuhler have defined a pairing $EP(V,V')$ between admissible representations of $G(F)$ by the formula

$$EP(V,V') = \sum_{n\geq 0}(-1)^n \dim \mathrm{Ext}^n(V,V'),$$

where Ext is taken in the category of smooth representations of $G(F)$, and they prove that $EP(V,V')$ is the trace on V' of a certain function f_V on $G(F)$. They also show, assuming the characteristic of F is zero, that $EP(V,V')$ equals the elliptic inner product of the characters of V,V'. (See [K] for background on elliptic representation theory.)

On the other hand, Arthur [A] has calculated the elliptic inner product in terms of elliptic characters of the analytic R-group R_{an} ("analytic" because it is defined by zeros of Plancherel measures). Arthur then suggests that his formula "... might also play a role in the general character theory of Weyl groups". One purpose of this paper is to confirm this prediction: We show that the Iwahori-spherical elliptic

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representation theory of $G(F)$ is equivalent to the elliptic representation theory of the Weyl groups of endoscopic groups of G.

In addition, this paper contains the following.

1. We describe explicitly the elliptic virtual representations of a Weyl group W, in terms of Springer representations, using a Weyl group analogue of the pairing EP.

2. We describe explicitly the elliptic Iwahori-spherical virtual representations of $G(F)$, in terms of Kazhdan-Lusztig parameters. This is in the same spirit as the classification of tempered and discrete series representations in [KL].

3. For Iwahori spherical representations, Arthur's formula now holds in any characteristic, provided we use the homological definition of EP, instead of the elliptic inner product of characters.

4. In order to apply Arthur's formula to the Kazhdan-Lusztig correspondence, we describe the analytic R-group R_{an}, and the cocycle η_{an} (arising in Harish-Chandra's theory of intertwining operators) in terms of a geometric R-group and cocycle attached to the Kazhdan-Lusztig parameter.

To give a more precise exposition, we begin with Weyl groups. Let $\hat{G}$ be a simply-connected Lie group with Weyl group W. Then W has an elliptic representation theory, in which proper Levi subgroups are stabilizers of nonzero vectors in the reflection representation E of W. Let $\mathcal{R}(W)$ be the span of the irreducible representations of W, and let $\bar{\mathcal{R}}(W)$ be the quotient of $\mathcal{R}(W)$ by the span of all induced representations from proper Levi subgroups. We define an analogue of the pairing EP on $\bar{\mathcal{R}}(W)$ as follows:

$$e_W(\chi, \chi') = \sum_{n \geq 0} (-1)^n \dim \operatorname{Hom}_W(\Lambda^n E \otimes \chi, \chi').$$

This pairing is initially defined on $\mathcal{R}(W)$, but its radical is exactly the kernel of the map $\mathcal{R}(W) \longrightarrow \bar{\mathcal{R}}(W)$, hence e_W is a non-degenerate pairing on $\bar{\mathcal{R}}(W)$.

For $x \in \hat{G}$, let A_x be the component group of the centralizer of x in the adjoint group of $\hat{G}$. The groups A_x also have "Levi" subgroups, hence their own elliptic representation theories, hence pairings e_{A_x} analogous to e_W.

Let $\mathcal{U}_G$ be the set of unipotent elements in $\hat{G}$, modulo conjugacy. Combining results of Borho-MacPherson, Lusztig and Springer, we have a decomposition

$$\mathcal{R}(W) = \bigoplus_{u \in \mathcal{U}_G} \mathcal{R}_u(W),$$

together with an isomorphism

$$H_u : \mathcal{R}_\circ(A_u) \longrightarrow \mathcal{R}_u(W)$$

where $H_u(\rho)$ is the Springer representation of W on $\operatorname{Hom}_{A_u}(\rho, H(\mathcal{B}^u))$, $\mathcal{B}^u$ denotes the fixed points of u in the flag manifold $\mathcal{B}$ of $\hat{G}$, $H(\mathcal{B}^u)$ is the cohomology of $\mathcal{B}^u$, with grading ignored, and $\mathcal{R}_\circ(A_u)$ is the span of the irreducible representations of A_u which appear in the natural action of A_u on $H(\mathcal{B}^u)$. The representations $H_u(\rho)$ have been calculated explicitly in [BS], [Sho1] for W of exceptional type, but are only given by a recursive algorithm in classical cases. See [Sho2] and references therein.

We have

$$\bar{\mathcal{R}}(W) = \sum_{u \in \mathcal{U}_G} \bar{\mathcal{R}}_u(W), \tag{1a}$$

where $\bar{\mathcal{R}}_u(W)$ is the image of $\mathcal{R}_u(W)$ in $\bar{\mathcal{R}}(W)$.

Say that a unipotent element u in a reductive group H is *quasi-distinguished* in H if there is a semisimple element $t \in H$ commuting with u such that tu centralizes no nontrivial torus in H. This forces H to be semisimple. Note that u is distinguished, in the usual sense, if it is quasi-distinguished with $t = 1$.

Proposition.

(1) *The sum in (1a) is orthogonal with respect to the pairing e_W, and is in particular a direct sum.*
(2) *The Springer map H_u induces a bijective isometry $\bar{\mathcal{R}}_\circ(A_u) \longrightarrow \bar{\mathcal{R}}_u(W)$.*
(3) *The space $\bar{\mathcal{R}}_u(W)$ is nonzero if and only if u is quasi-distinguished in $\hat{G}$.*
(4) *If u is in fact distinguished in $\hat{G}$, then H_u maps the irreducible representations in $\mathcal{R}_\circ(A_u)$ to an e_W-orthonormal basis of $\mathcal{R}_u(W)$.*

Corollary. *Assume that u is a non-regular unipotent element in $\hat{G}$. Then*

$$\sum_{n \geq 0} (-1)^n \dim \mathrm{Hom}_W(\Lambda^n E, H_u(\rho)) = 0$$

for every $\rho \in \mathcal{R}_\circ(A_u)$.

The above results are deduced from corresponding results for p-adic groups. Let $\mathcal{I}$ be an Iwahori subgroup of $G(F)$. Let $\mathcal{R}_{\mathrm{temp}}(G, \mathcal{I})$ be the $\mathbb{C}$-vector space spanned by the irreducible tempered $\mathcal{I}$-spherical representations of $G(F)$. Let $\mathcal{T}_G$ be the set of conjugacy classes of elements $x \in \hat{G}$ whose semisimple part lies in a compact subgroup of $\hat{G}$. The Kazhdan-Lusztig classification may be interpreted as a decomposition

$$\mathcal{R}_{\mathrm{temp}}(G, \mathcal{I}) = \bigoplus_{x \in \mathcal{T}_G} \mathcal{R}_x(G, \mathcal{I}),$$

together with an isomorphism

$$V_x : \mathcal{R}_\circ(A_x) \longrightarrow \mathcal{R}_x(G, \mathcal{I}),$$

where $\mathcal{R}_\circ(A_x)$ is the span of the representations of A_x appearing in the cohomology $H(\mathcal{B}^x)$.

Let $\bar{\mathcal{R}}(G, \mathcal{I})$ be the quotient of $\mathcal{R}_{\mathrm{temp}}(G, \mathcal{I})$ by the subspace spanned by induced representations from proper Levi subgroups, and let $\bar{\mathcal{R}}_x(G, \mathcal{I})$ be the image of $\mathcal{R}_x(G, \mathcal{I})$ in $\bar{\mathcal{R}}(G, \mathcal{I})$. It is easy to see that

$$\bar{\mathcal{R}}(G, \mathcal{I}) = \bigoplus_{x \in \mathcal{T}_G} \bar{\mathcal{R}}_x(G, \mathcal{I}),$$

and that the sum is orthogonal with respect to the pairing EP.

Let $x \in \mathcal{T}_G$ have Jordan decomposition $x = su$. Let $\hat{G}_s$ be the centralizer of s, and let W_s be the Weyl group of $\hat{G}_s$.

Main Theorem. *We have a diagram of vector space isomorphisms, commuting up to the sign* $(-1)^{\mathrm{rank} G}$

$$\begin{array}{ccc} \bar{\mathcal{R}}_\circ(A_x) & \xrightarrow{\bar{V}_x} & \bar{\mathcal{R}}_x(G,\mathcal{I}) \\ \Big\| & & \Big\downarrow r_x \\ \bar{\mathcal{R}}_\circ(A_x) & \xrightarrow[\bar{H}_x]{} & \bar{\mathcal{R}}_u(W_s), \end{array}$$

which preserve the elliptic pairings EP *on* $\bar{\mathcal{R}}_x(G,\mathcal{I})$, e_{W_s} *on* $R_u(W_s)$, *and* e_{A_x} *on* $\bar{\mathcal{R}}_\circ(A_x)$. *Moreover, these spaces are nonzero if and only if* u *is quasi-distinguished in* $\hat{G}_s$.

The map $\bar{V}_x$ is induced by the Kazhdan-Lusztig correspondence, the map $\bar{H}_x$ is induced by the Springer correspondence, and r_x is a kind of restriction map.

The fact that all these maps are well-defined on the elliptic spaces $\bar{\mathcal{R}}(\ )$, and commute up to sign, follows from results of Lusztig and Kazhdan-Lusztig, together with analysis of the connected components of certain fixed point varieties in $\mathcal{B}$, given in §10. The fact that V_x is an isometry follows from Arthur's formula and the comparison of analytic and geometric R-groups, along with their twisted group algebras, see §9. In §5, we prove that r_x is an isometry for $s = 1$ by a direct calculation of the trace of Schneider-Stuhler's function f_V. This involves a reduction to Hecke algebras, then to affine Weyl groups, in the spirit of [R1]. Then for arbitary s, where our calculation fails, we reduce to $s = 1$ by categorical equivalences, using [BaM], see §7.

The first part of the paper, sections 2 and 3, contains our results on Weyl groups, and can be read by those who are not familiar with p-adic groups. However, the proofs of several results here require the extra structure contained in the p-adic theory.

This paper can be viewed as evidence for the idea, based on Arthur's formula, that the conjectural Langlands parametrization of tempered L-packets should induce an isometry between elliptic representation spaces. Here we have tested the two simplest cases: Weyl groups and Iwahori-spherical representations of p-adic groups. The same pattern of argument should work whenever we can describe representations of $G(F)$ using Hecke algebra isomorphisms. For example, much of this paper could have been written in the more natural context of unipotent representations [L1]. At the moment, however, there are several technical obstacles that make our results in the unipotent case less complete than the Iwahori-spherical case considered here. Nevertheless, the comparison of analytic and geometric R-groups in §§8-10 is presented in a wider context, assuming when necessary a Langlands correspondence with certain naturality properties. These properties are either known, or are verified here, for Iwahori spherical representations.

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General Notation: When a group Γ acts on a set X, we let X^Γ be the points in X fixed by all of Γ, and let X^γ be the points fixed by a single element $\gamma \in \Gamma$. The set of irreducible representations of Γ, up to equivalence, is denoted $\mathrm{Irr}(\Gamma)$, and $\mathcal{R}(\Gamma)$ is the complex span of $\mathrm{Irr}(\Gamma)$. The identity component of a Lie group H is denoted

H°, and the centralizer of $x \in H$, resp. $S \subset H$ is denoted H_x resp. H_S. The Lie algebra of H is denoted by the corresponding gothic $\mathfrak{h}$.

2. Elliptic representations of finite groups

2.1 Let Γ be a finite group, and let E be a real representation of Γ. We allow $E = 0$. An element $\gamma \in \Gamma$ is *elliptic* if γ has no nonzero fixed vectors in E. Let Γ_{ell} be the set of elliptic elements in Γ. We will define an "elliptic representation theory" for the pair (Γ, E) which is dual to the set of conjugacy classes contained in Γ_{ell}. When $E = 0$, we get the usual relation between characters and conjugacy classes. All constructions will depend on E, but we suppress it from the notation.

Let $\mathcal{L}$ be the set of subgroups $\Delta \subseteq \Gamma$ for which $E^\Delta \neq 0$. For each $\Delta \in \mathcal{L}$ we have the induction map $\mathrm{Ind}_\Delta^\Gamma : \mathcal{R}(\Delta) \longrightarrow \mathcal{R}(\Gamma)$, hence a subspace

$$\mathcal{R}_{\mathrm{ind}}(\Gamma) = \sum_{\Delta \in \mathcal{L}} \mathrm{Ind}_\Delta^\Gamma[\mathcal{R}(\Delta)] \subseteq \mathcal{R}(\Gamma).$$

We set $\bar{\mathcal{R}}(\Gamma) := \mathcal{R}(\Gamma)/\mathcal{R}_{\mathrm{ind}}(\Gamma)$.

Let $\Lambda^n E$ be the n^{th} exterior power of E. The following lemma is elementary.

(2.1.1) Lemma. *Let $\Lambda E = \sum_{n \geq 0} (-1)^n \Lambda^n E \in \mathcal{R}(\Gamma)$. Then for $\gamma \in \Gamma$, we have*

$$\mathrm{tr}(\gamma, \Lambda E) = \det(1 - \gamma)_E \geq 0,$$

with equality if and only if $E^\gamma \neq 0$. In particular, Γ_{ell} is the support of the character of ΛE.

2.2 We define a pairing

$$\tilde{e}_\Gamma(\chi, \chi') = \sum_{n \geq 0} (-1)^n \dim \mathrm{Hom}_\Gamma(\Lambda^n E \otimes \chi, \chi'), \qquad \chi, \chi' \in \mathcal{R}(\Gamma).$$

It follows from (2.1.1) that $\mathcal{R}_{\mathrm{ind}}(\Gamma)$ is contained in the radical of $\tilde{e}_\Gamma$.

(2.2.1) Definition. *The pairing e_Γ on $\bar{\mathcal{R}}(\Gamma)$ induced by $\tilde{e}_\Gamma$ is called the* elliptic pairing *on* Γ.

Let C_{ell} be the set of conjugacy classes in Γ_{ell}, and let S_{ell} be the set of $\mathbb{C}$-valued functions on C_{ell}, identified with class functions on Γ which vanish off Γ_{ell}. We have a nondegenerate pairing on S_{ell} defined by

$$\langle f, g \rangle_{ell} = \sum_{c \in C_{ell}} f(c) \overline{g(c)} \frac{\mathrm{tr}(c, \Lambda E)}{z(c)}, \qquad f, g \in S_{ell},$$

where $z(c)$ is the order of the centralizer in Γ of an element in c.

The characters of representations in $\mathcal{R}_{\mathrm{ind}}(\Gamma)$ vanish on Γ_{ell}, so we have a map $rst : \bar{\mathcal{R}}(\Gamma) \longrightarrow S_{ell}$ given by restriction of characters.

(2.2.2)Proposition. *The map $rst : \bar{\mathcal{R}}(\Gamma) \longrightarrow S_{ell}$ is a bijective isometry. Hence the radical of $\tilde{e}_\Gamma$ is exactly $\mathcal{R}_{\text{ind}}(\Gamma)$, the elliptic pairing e_Γ is nondegenerate on $\bar{\mathcal{R}}(\Gamma)$, and the dimension of $\bar{\mathcal{R}}(\Gamma)$ equals the number of conjugacy classes in Γ_{ell}.*

Proof. It is clear that rst is a surjective isometry, and that its kernel is the radical of $\tilde{e}_\Gamma$. It suffices to prove that $\dim \bar{\mathcal{R}}(\Gamma) = \dim S_{ell}$. We identify elements of $\mathcal{R}(\Gamma)$ with their characters. If $f \in \mathcal{R}(\Gamma)$, then

$$\chi \mapsto \operatorname{tr}(f, \chi) := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma) \operatorname{tr}(\gamma, \chi)$$

is a linear functional on $\mathcal{R}(\Gamma)$. Thus we have an isomorphism $\operatorname{tr} : \mathcal{R}(\Gamma) \longrightarrow \mathcal{R}(\Gamma)^*$, under which $\bar{\mathcal{R}}(\Gamma)^*$ corresponds to the functions $f \in \mathcal{R}(\Gamma)$ such that $\operatorname{tr}(f, \operatorname{Ind}_\Delta^\Gamma(\sigma)) = 0$ for for each $\Delta \in \mathcal{L}$, and $\sigma \in \mathcal{R}(\Delta)$. By Frobenius reciprocity, this is equivalent to f vanishing on every $\Delta \in \mathcal{L}$. In turn, this is equivalent to f being in S_{ell}. Thus, $\bar{\mathcal{R}}(\Gamma)^* \simeq S_{ell}$. □

3. Elliptic representation theory of Weyl groups

Let $\hat{G}$ be a semisimple complex Lie group, and choose a maximal torus $\hat{T} \subset \hat{G}$. Let W denote the Weyl group of $\hat{T}$, and let E be the real span of the coroots of $\hat{T}$. Thus E affords the reflection representation of W. In this section we describe the elliptic representation theory of the pair (W, E). More precisely, we want to calculate the elliptic pairing e_W on the space $\bar{\mathcal{R}}(W)$ of elliptic virtual representations of W, as defined in §2. This will be done by reducing to the elliptic representation theory of the small groups A_u, via Springer representations.

3.1 If $\hat{L}$ is a Levi subgroup of $\hat{G}$, then some conjugate of $\hat{L}$ contains $\hat{T}$, and the Weyl group of this conjugate is a subgroup $W_L \subset W$, well-defined up to conjugacy in W. The non-elliptic elements of W are those belonging to some W_L, for $\hat{L}$ a proper Levi subgroup of $\hat{G}$.

The number of elliptic elements in W is the product of the exponents of W [S]. They are partitioned into conjugacy classes as follows [C].

If W is the symmetric group S_n, then the n-cycles form the unique elliptic class. If W has type $B_n = C_n$, then each elliptic class is represented by a product of Coxeter elements in $B_{\lambda_1} \times \cdots \times B_{\lambda_t} \subseteq B_n$, one class for each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0)$ of n. This class lies in D_n exactly when t is even, and then forms a single elliptic class in D_n. In the exceptional cases, the number of elliptic classes is as follows:

$$G_2 : 3, \qquad F_4 : 9, \qquad E_6 : 5, \qquad E_7 : 12, \qquad E_8 : 30.$$

3.2 For any $x \in \hat{G}$, we have a finite group

$$A_x := \hat{G}_x / Z_{\hat{G}} \hat{G}_x^\circ,$$

where $Z_{\hat{G}}$ is the center of $\hat{G}$. The essential case is when $x = u$ is unipotent. Let $\phi : SL_2(\mathbb{C}) \longrightarrow \hat{G}$ be a homomorphism such that $u = \phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let M denote the centralizer of the image of ϕ in $\hat{G}$, let $\hat{S} \subset B_M$ be a maximal torus contained in a Borel subgroup of M°, and let $N(\hat{S}, B_M)$ be the subgroup of M normalizing both $\hat{S}$ and B_M. The inclusions $N(\hat{S}, B_M) \subset M \subset \hat{G}_u$ induce isomorphisms

$$N(\hat{S}, B_M)/Z_{\hat{G}}\hat{S} \longrightarrow M/Z_{\hat{G}}M^\circ \longrightarrow A_u. \tag{3.2a}$$

Let $\mathfrak{s}_0$ denote the real span of the coroots of $\hat{S}$. Via (3.2a), we have an action of A_u on $\mathfrak{s}_0$, and we consider the elliptic representation theory of the pair $(A_u, \mathfrak{s}_0)$.

Let $\mathcal{L}_u$ be the set of proper Levi subgroups of $\hat{G}$ containing u. The centralizer of $\hat{S}$ in $\hat{G}$ is a minimal Levi subgroup $\hat{L}_1 \in \mathcal{L}_u$. All minimal Levi subgroups in $\mathcal{L}_u$ are conjugate under $\hat{G}_u^\circ$.

For any $\hat{L} \in \mathcal{L}_u$, let $A_u^L = \hat{L}_u/Z_{\hat{L}}\hat{L}_u^\circ$. Since $\hat{L}$ is the centralizer of a torus in $\hat{G}_u^\circ$, and connectedness is preserved by taking centralizers of tori, it follows that $\hat{L}_u^\circ = \hat{L} \cap \hat{G}_u^\circ$, so the inclusion $\hat{L} \hookrightarrow \hat{G}$ induces an injection $A_u^L \hookrightarrow A_u$, by which we view A_u^L as a subgroup of A_u. In particular, A_u has the canonical normal subgroup $A_u^{L_1}$. Note that $A_u^{L_1}$ is the kernel of the action of A_u on $\mathfrak{s}_0$, so the group $A_u/A_u^{L_1}$ acts faithfully on $\mathfrak{s}_0$.

(3.2.1) Lemma. *The elliptic elements of A_u are those lying outside $\bigcup_{L \in \mathcal{L}_u} A_u^L$.*

Proof. Suppose $a \in A_u^L$ for some $\hat{L} \in \mathcal{L}_u$. Conjugating by $\hat{G}_u^\circ$ if necessary, we may assume $\phi(SL_2(\mathbb{C})) \subset \hat{L}$. Let $\mathfrak{s}'$ be the Lie algebra of the center of $\hat{L}$. Conjugating by M°, we may assume $\mathfrak{s}' \subset \mathfrak{s}$. Then a fixes $\mathfrak{s}'_0$ pointwise, so a is not elliptic. Conversely, if $0 \neq h \in \mathfrak{s}_0^a$, the centralizer of h is a proper Levi subgroup $\hat{L} \in \hat{G}$, and $a \in A_u^L$. □

(3.2.2) Lemma. *The following are equivalent:*

(1) *A_u is not the union of the A_u^L for $\hat{L} \in \mathcal{L}_u$.*

(2) *There is a semisimple element $t \in \hat{G}_u$ such that tu is not contained in any proper Levi subgroup of $\hat{G}$.*

If these conditions hold, then $\mathfrak{m} = \mathfrak{s}$.

Proof. Suppose (2) fails. Let $a \in A_u$, and choose a semisimple representative $t \in M_u$ of a. Then $tu \in \hat{L}$ for some $\hat{L} \in \mathcal{L}$. Thus $t \in \hat{L}_u$, so $a \in A_u^L$, so (1) fails. The converse follows from (3.2.1). That $\mathfrak{m} = \mathfrak{s}$ is proved in [R1, (7.1)]. □

We say that u is *quasi-distinguished* in $\hat{G}$ if either of the conditions in (3.2.2) holds. Combining (3.2.1,2) and (2.2.2), we get

(3.2.3) Corollary. *$\bar{\mathcal{R}}(A_u) \neq 0$ if and only if u is quasi-distinguished in $\hat{G}$.*

3.3 Take a unipotent $u \in \hat{G}$, and $\hat{L} \in \mathcal{L}_u$, and let $\mathcal{B}$ and $\mathcal{B}_L$ denote the flag varieties of $\hat{G}$ and $\hat{L}$ respectively. Let $\mathcal{R}_\circ(A_u)$ be the span of the irreducible representations ρ of A_u which appear in the natural action of A_u on the cohomology $H(\mathcal{B}^u)$ (grading ignored) of the fixed point variety $\mathcal{B}^u$, and likewise define $\mathcal{R}_\circ(A_u^L)$ with respect to $\mathcal{B}_L^u$. Springer has constructed a natural action of W on $H(\mathcal{B}^u)$, commuting with A_u. We refer to the version of this construction given in [BM]. Thus, for each irreducible representation in $\rho \in \mathcal{R}_\circ(A_u)$ we have a W-module

$$H_u(\rho) := \mathrm{Hom}_{A_u}(\rho, H(\mathcal{B}^u)).$$

These representations are generally reducible, since W preserves the grading on $H(\mathcal{B}^u)$. However, we have the following paraphrased result of Borho and MacPherson [BM].

(3.3.1)Lemma. *The representations $H_u(\rho)$, for $u \in \mathcal{U}_G$, $\rho \in \mathrm{Irr}(A_u) \cap \mathcal{R}_\circ(A_u)$, form a basis of $\mathcal{R}(W)$.*

Proof. Choose representatives $u_1, u_2, \ldots, u_n$ for the conjugacy classes of unipotent elements in $\hat{G}$, such that if u_i is contained in the closure of the class of u_j then $i < j$. Let $d_i = \dim \mathcal{B}^{u_i}$, and for an irreducible representation $\rho \in \mathcal{R}_\circ(A_{u_i})$, set

$$\chi_{u_i,\rho} = H^{2d_i}_{u_i}(\rho).$$

By Springer's theory, the $\chi_{u_i,\rho}$'s are a complete list of the irreducible representations of W. By [BM, Cor.2], we have

$$\dim \mathrm{Hom}_W(\chi_{u_j,\rho}, H_{u_i}(\rho')) \neq 0 \Rightarrow i \leq j,$$

and it follows immediately from [BM,Cor. 1] that

$$\dim \mathrm{Hom}_W(\chi_{u_i,\rho}, H_{u_i}(\rho')) = 1$$

if $\rho = \rho'$ and is zero otherwise. This proves the lemma. □

An analogous fact holds for the W_L-representations

$$H^L_u(\sigma) := \mathrm{Hom}_{A^L_u}(\sigma, H(\mathcal{B}^u_L)).$$

Set

$$\mathcal{R}_u(W) := \{H_u(\rho) : \ \rho \in \mathcal{R}_\circ(A_u)\}, \qquad \mathcal{R}_u(W_L) := \{H^L_u(\sigma) : \ \sigma \in \mathcal{R}_\circ(A^L_u)\},$$

so that

$$\mathcal{R}(W) = \bigoplus_{u \in \mathcal{U}_G} \mathcal{R}_u(W), \qquad \mathcal{R}(W_L) = \bigoplus_{u \in \mathcal{U}_L} \mathcal{R}_u(W_L),$$

and we have vector space isomorphisms

$$H_u : \mathcal{R}_\circ(A_u) \longrightarrow \mathcal{R}_u(W), \qquad H^L_u : \mathcal{R}_\circ(A^L_u) \longrightarrow \mathcal{R}_u(W_L).$$

(3.3.2)Lemma. *The induction map $\mathrm{Ind}^{A_u}_{A^L_u}$ sends $\mathcal{R}_\circ(A^L_u)$ to $\mathcal{R}_\circ(A_u)$.*

Proof. See section 10.

The next result is due to Lusztig. It was stated in [AL], and follows from (5.11.1) and (6.2a) below.

(3.3.3)Proposition. *If $\hat{L} \in \mathcal{L}_u$ and $\sigma \in \mathcal{R}_\circ(A^L_u)$ then*

$$H_u(\mathrm{Ind}^{A_u}_{A^L_u} \sigma) = \mathrm{Ind}^W_{W_L} H^L_u(\sigma).$$

In particular, we have

$$\mathrm{Ind}^W_{W_L}(\mathcal{R}_u(W_L)) \subseteq \mathcal{R}_u(W).$$

3.4 We can now explain how the elliptic theories of W and A_u are compatible with the Springer maps H_u. Some of the proofs are deferred to later sections.

Let $\bar{\mathcal{R}}_\circ(A_u)$ be the image of $\mathcal{R}_\circ(A_u)$ in $\bar{\mathcal{R}}(A_u)$, and let $\bar{\mathcal{R}}_u(W)$ be the image of $\mathcal{R}_u(W)$ in $\bar{\mathcal{R}}(W)$.

(3.4.1)Proposition. *The Springer isomorphism $H_u : \mathcal{R}_\circ(A_u) \longrightarrow R_u(W)$ induces a vector space isomorphism*

$$\bar{H}_u : \bar{\mathcal{R}}_\circ(A_u) \longrightarrow \bar{\mathcal{R}}_u(W).$$

These spaces are nonzero if and only if u is quasi-distinguished in $\hat{G}$.

Proof. By (3.3.2) and (3.3.3), we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_\circ(A_u) & \xrightarrow[\simeq]{H_u} & \mathcal{R}_u(W) \\ {\scriptstyle I_A}\uparrow & & {\scriptstyle I_W}\uparrow \\ \bigoplus_{\hat{L}\in\mathcal{L}_u} \mathcal{R}_\circ(A_u^L) & \xrightarrow[\simeq]{\oplus H_u^L} & \bigoplus_{\hat{L}\in\mathcal{L}_u} \mathcal{R}_u(W_L), \end{array}$$

where I_A, I_W are the respective induction maps. It suffices to show that

$$\operatorname{im} I_A = \mathcal{R}_\circ(A_u) \cap \mathcal{R}_{\text{ind}}(A_u), \qquad \operatorname{im} I_W = \mathcal{R}_u(W) \cap \mathcal{R}_{\text{ind}}(W).$$

In both equations, the left side is contained in the right side. The diagram shows that H_u maps $\operatorname{im} I_A$ isomorphically onto $\operatorname{im} I_W$, so we need only verify the second equation. We have

$$\begin{aligned} \mathcal{R}_{\text{ind}}(W) &= \sum_{\hat{L}\subsetneq\hat{G}} \operatorname{Ind}_{W_L}^W \mathcal{R}(W_L) \\ &= \sum_{\hat{L}\subsetneq\hat{G}} \sum_{v\in\mathcal{U}_L} \operatorname{Ind}_{W_L}^W \mathcal{R}_v(W_L) \\ &= \sum_{u\in\mathcal{U}_G} \sum_{\hat{L}\in\mathcal{L}_u} \operatorname{Ind}_{W_L}^W \mathcal{R}_u(W_L). \end{aligned}$$

The term for a given $u \in \mathcal{U}_G$ belongs to the direct summand $\mathcal{R}_u(W)$, so

$$\mathcal{R}_{\text{ind}}(W) \cap \mathcal{R}_u(W) = \sum_{\hat{L}\in\mathcal{L}_u} \operatorname{Ind}_{W_L}^W \mathcal{R}_u(W_L) = \operatorname{im} I_W,$$

as we wished to show. The last assertion of (3.4.1) now follows from (3.2.3). □

The next two results will be proved in §§5,9, using p-adic groups.

(3.4.2)Proposition. *If u and v are non-conjugate unipotent elements in $\hat{G}$, then $\bar{\mathcal{R}}_u(W)$ and $\bar{\mathcal{R}}_v(W)$ are orthogonal with respect to the elliptic pairing e_W on $\bar{\mathcal{R}}(W)$.*

Proof. See 5.11.

(3.4.3)Proposition. *The isomorphism $\bar{H}_u$ of (3.4.1) is an isometry with respect to the elliptic pairings on $\bar{\mathcal{R}}_\circ(A_u)$ and $\bar{\mathcal{R}}_u(W)$.*

Proof. See (5.10.1) and (9.2.3).

Now suppose u is distinguished in $\hat{G}$, that is, u centralizes no nontrivial torus in $\hat{G}$. Then $\bar{\mathcal{R}}_u(W) = \mathcal{R}_u(W)$. Moreover $\mathfrak{s}_0 = 0$, so the elliptic theory of $(A_u, \mathfrak{s}_0)$ is the ordinary representation theory of A_u, and $\bar{\mathcal{R}}_\circ(A_u) = \mathcal{R}_\circ(A_u)$. Again using p-adic groups we will prove

(3.4.4) Proposition. *If u is distinguished in $\hat{G}$, then the set*

$$\{H_u(\rho) : \ \rho \in \mathcal{R}_\circ(A_u)\}$$

is an orthonormal basis of $\mathcal{R}_u(W)$, with respect to the elliptic pairing e_W.

Proof. See 5.11.

If u_r is regular in $\hat{G}$, then $\mathcal{R}_{u_r}(W)$ is spanned by the trivial representation of W. In this case (3.4.2) implies

(3.4.5) Corollary. *If u is a nonregular unipotent element in $\hat{G}$ and $\rho \in \mathcal{R}_\circ(A_u)$, then*

$$\sum_{n\geq 0}(-1)^n \dim \mathrm{Hom}_W\big(\Lambda^n E, H_u(\rho)\big) = 0.$$

This was already known in certain cases. For example if $\hat{G}$ is simply-laced and u is subregular, then $\mathcal{B}^u$ is the "Dynkin curve" (c.f. [St]) and $H_u(1) = \mathbb{C} \oplus E$. I understand that for $\hat{G}$ classical and u regular in a Levi subgroup, Lehrer (unpublished) has calculated the multiplicity of $\Lambda^n E$ in $H_u^i(\rho)$ for each n, i, using the theory of hyperplane arrangements.

4. p-adic groups

In this section we review some of the structure of p-adic groups. See [IM], [T] for additional details.

Let F be a non-archimedean local field with integers $\mathcal{O}$, units $\mathcal{O}^\times$ and residue field of cardinality q. Let G be a connected, split, semisimple, algebraic group over F. In G we choose a maximal torus T and Borel subgroup B containing T. Let $\Sigma \subset \Delta \subset Y$ denote the corresponding simple roots, roots, and rational characters of T. We assume the group $\hat{G}$ considered in section 3 is the Langlands dual of G.

Let

$$T(\mathcal{O}) = \{t \in T(F) : \ \chi(t) \in \mathcal{O}^\times, \text{ for all } \chi \in Y\}.$$

Let N be the normalizer of T in G. Then we have the Weyl group and affine Weyl group

$$W = N(F)/T(F), \qquad \widetilde{W} = N(F)/T(\mathcal{O}),$$

and a natural split surjection

$$\widetilde{W} \longrightarrow W$$

whose kernel is the free abelian group $X = T(F)/T(\mathcal{O})$, and $\widetilde{W} = WX$ (semidirect product) acts by affine motions on the vector space $E = \mathbb{R} \otimes X$. Let $\widetilde{W}^\circ$ be the subgroup of $\widetilde{W}$ generated by affine reflections. Then E has a simplicial structure given by the hyperplanes corresponding to the reflections in $\widetilde{W}^\circ$, and the open facets are permuted simply transitively by $\widetilde{W}^\circ$. For any facet $b \subset E$, the stabilizer $\widetilde{W}_b$ of b in $\widetilde{W}$ is a finite group.

The groups X, Y are in duality via the pairing

$$X \times Y \longrightarrow \mathbb{Z}, \quad (\lambda, \chi) \mapsto -val_F(\chi(\lambda)).$$

Thus we view roots in Δ as linear functionals on E. There is a unique open facet $c_0 \subset E$ which contains 0 in its closure, on which roots in Σ take positive values.

Let S be the set of reflections in $\widetilde{W}$ about the hyperplanes bounding c_0, and let $\widetilde{W}^\circ$ be the subgroup of $\widetilde{W}$ generated by S. Then $\widetilde{W}^\circ$ is a Coxeter group, and $\widetilde{W}^\circ$ is normal in $\widetilde{W}$. Let Ω be the stabilizer of c_0 in $\widetilde{W}$. It permutes the elements of S, and we have

$$\widetilde{W} = \Omega \ltimes \widetilde{W}^\circ.$$

Let $\mathbf{B}$ denote the building of $G(F)$. It is a simplicial complex containing E as a subcomplex, on which $G(F)$ acts via simplicial maps, and every facet in $\mathbf{B}$ may be translated by $G(F)$ into E. Let G_b be the stabilizer in $G(F)$ of a facet $b \subset \mathbf{B}$. It is a compact open subgroup of $G(F)$ described by the exact sequence

$$1 \longrightarrow U_b \longrightarrow G_b \longrightarrow M_b(\mathbb{F}_q) \longrightarrow 1,$$

where U_b is pro-unipotent, and M_b is a reductive group (possibly disconnected) over the residue field $\mathbb{F}_q$. Let M_b° be the identity component of M_b, and let G_b° be the full pre-image of $M_b^\circ(\mathbb{F}_q)$ in G_b.

Assume that b is contained in the closure of the facet c_0 described above. Let Ω_b be the stabilizer of b in Ω. Then

$$G_b/G_b^\circ \simeq \Omega_b.$$

If b' is another facet in the closure of c_0, then b is $G(F)$-conjugate to b' if and only if b is Ω-conjugate to b'. Thus, the G-orbits of facets in $\mathbf{B}$ are in bijection with Ω-orbits on S.

If $b = c_0$, then we write

$$\mathcal{I} := G_{c_0}^\circ.$$

For any facet b contained in the closure of c_0 we have

$$G_b = \mathcal{I}\widetilde{W}_b\mathcal{I}.$$

This makes sense because $T(\mathcal{O}) \subset \mathcal{I}$.

5. Calculation of EP for real central character

5.1 Let $\mathcal{A}(G,\mathcal{I})$ be the category of smooth $G(F)$ representations which are generated by their $\mathcal{I}$-invariants, and let $\mathcal{R}(G,\mathcal{I})$ be the Grothendieck group of representations of finite length in $\mathcal{A}(G,\mathcal{I})$. For each Levi subgroup L in G we have a natural induction map

$$\mathrm{I}_L^G : \mathcal{R}(L,\mathcal{I}_L) \longrightarrow \mathcal{R}(G,\mathcal{I}).$$

Here $\mathcal{I}_L$ is an Iwahori subgroup of L. Let $\mathcal{R}_{\mathrm{ind}}(G,\mathcal{I})$ be the span of the images of I_L^G as L ranges over Levi subgroups $\neq G$, and let

$$\bar{\mathcal{R}}(G,\mathcal{I}) = \mathcal{R}(G,\mathcal{I})/\mathcal{R}_{\mathrm{ind}}(G,\mathcal{I}).$$

Our aim is to calculate the pairing EP defined on $\mathcal{R}(G,\mathcal{I})$ by

$$EP(V,V') = \sum_{n\geq 0}(-1)^n \dim \mathrm{Ext}(V,V'),$$

where Ext is taken in $\mathcal{A}(G,\mathcal{I})$. Note this category is a direct summand of the category of all smooth $G(F)$ representations, so the definition above coincides with that of [SS]. The space $\mathcal{R}_{\text{ind}}(G,\mathcal{I})$ belongs to the radical of EP [SS, Lemma 18], and we again write EP for the induced pairing on $\bar{\mathcal{R}}(G,\mathcal{I})$.

5.2 Let dg be the Haar measure on $G(F)$ assigning volume one to $\mathcal{I}$. Let $\mathcal{H}$ be the convolution algebra of compactly-supported functions on $G(F)$ which are left and right invariant under $\mathcal{I}$. It is known [B] that $\mathcal{A}(G,\mathcal{I})$ is equivalent to the category of unital $\mathcal{H}$-modules, via the functor $V \mapsto V^{\mathcal{I}}$ of $\mathcal{I}$-invariants.

The algebra $\mathcal{H}$ has a linear basis $\{T_w : w \in \widetilde{W}\}$, where T_w is the characteristic function of $\mathcal{I}w\mathcal{I}$. For $\rho \in \Omega$, $w \in \widetilde{W}^\circ$, we have

$$T_{\rho w} = T_\rho T_w = T_{\rho w \rho^{-1}} T_\rho. \qquad (5.2a)$$

Let $\mathcal{H}_0$ be the span of $\{T_w : w \in W\}$, and let $\mathbb{C}[\hat{T}]$ be the affine coordinate ring of the complex torus $\hat{T} = \text{Hom}(X, \mathbb{C}^\times)$. There is an embedding of algebras

$$j : \mathbb{C}[\hat{T}] \hookrightarrow \mathcal{H}$$

(see 5.7 below) by means of which we view $\mathbb{C}[\hat{T}]$ as a subalgebra of $\mathcal{H}$. As vector spaces we have

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}[\hat{T}].$$

The center of $\mathcal{H}$ is $\mathcal{Z} = \mathbb{C}[\hat{T}]^W$. Thus $\text{Spec}(\mathcal{Z}) = \hat{T}/W$, and points in $\hat{T}/W$ are characters of $\mathcal{Z}$, i.e., central characters of $\mathcal{H}$-modules.

Except in the proof of (5.3.1) below, we will confound points in $\hat{T}$ with their images in $\hat{T}/W$. Let $\mathcal{R}(\mathcal{H},\tau)$ be the span of the finite-dimensional simple $\mathcal{H}$-modules with central character $\tau \in \hat{T}$.

5.3 The real central characters are those belonging to

$$\hat{T}(\mathbb{R}) := \text{Hom}(X, \mathbb{R}^\times_{>0}).$$

We want to show that $\mathcal{R}(\mathcal{H},\tau)$ is invariant under isogeny when τ is real.

Suppose we have an isogeny $G \longrightarrow G'$. This induces an isogeny $\hat{T}' \longrightarrow \hat{T}$ and an embedding of algebras $\mathcal{H} \hookrightarrow \mathcal{H}'$ such that $\mathcal{H}_0 = \mathcal{H}'_0$ and $\mathbb{C}[\hat{T}] \hookrightarrow \mathbb{C}[\hat{T}']$ corresponds to $\hat{T}' \longrightarrow \hat{T}$.

The induced map

$$\hat{T}'/W \longrightarrow \hat{T}/W \qquad (5.3a)$$

corresponds to restriction of central characters from $\mathcal{H}'$ to $\mathcal{H}$. On real points, (5.3a) is bijective.

Lemma(5.3.1). *Let τ' be a real central character of $\mathcal{H}'$, and let τ be its restriction to $\mathcal{Z}$. Then the restriction map*

$$\mathcal{R}(\mathcal{H}',\tau') \longrightarrow \mathcal{R}(\mathcal{H},\tau)$$

is bijective, and irreducible $\mathcal{H}'$-modules restrict to irreducible $\mathcal{H}$-modules.

Proof. Let C be the kernel of our isogeny $\hat{T}' \longrightarrow \hat{T}$. We have corresponding maximal ideals $\mathfrak{m}_{\tau'} \subset \mathbb{C}[\hat{T}']$, $\mathfrak{m}_\tau \subset \mathbb{C}[\hat{T}]$. Our isogeny is an isomorphism on tangent spaces, so

$$\mathfrak{m}_{\tau'} = \mathfrak{m}_\tau + \mathfrak{m}_{\tau'}^2. \qquad (5.3a)$$

Now C acts by multiplication on $\hat{T}'$ commuting with the W-action, so C acts on the branched covering $\hat{T}'/W \longrightarrow \hat{T}/W$. A real central character of $\mathcal{H}'$ is an example of a C-unramified point in $\hat{T}'/W$. We must be a little more precise about our notation for W-orbits. Let $\mathfrak{o}' \in \hat{T}'/W$ be any C-unramified point, and let $\mathfrak{o} \in \hat{T}/W$ be its image. Then any $\tau \in \mathfrak{o}$ has a unique lift $\tau' \in \mathfrak{o}'$.

Let Ψ be a simple $\mathcal{H}'$-module with central character $\mathfrak{o}'$. Restricting to $\mathbb{C}[\hat{T}']$ we have

$$\Psi = \bigoplus_{\tau' \in \mathfrak{o}'} \Psi_{\tau'},$$

where $\Psi_{\tau'}$ is the subspace of Ψ killed by some power of $\mathfrak{m}_{\tau'}$. Restricting further to $\mathbb{C}[\hat{T}]$, we have

$$\Psi_{\tau'} = \Psi_{\tau},$$

for each $\tau' \in \mathfrak{o}'$, since $\mathfrak{o}'$ is C-unramified. Let $\Psi^1_{\tau'}$ (resp. Ψ^1_{τ}) be the subspace of Ψ killed by $\mathfrak{m}_{\tau'}$ (resp. $\mathfrak{m}_{\tau}$). From (5.3a) it follows that

$$\Psi^1_{\tau'} = \Psi^1_{\tau}, \tag{5.3b}$$

for all $\tau' \in \mathfrak{o}'$. Let $\Phi \subset \Psi$ be a nonzero simple $\mathcal{H}$-submodule, and choose $0 \neq u \in \Phi^1_{\tau}$. Then $u \in \Psi^1_{\tau} = \Psi^1_{\tau'}$, so

$$\Phi = \mathcal{H}u = \mathcal{H}_0 u = \mathcal{H}' u = \Psi,$$

so Ψ restricts irreducibly to $\mathcal{H}$.

Now let Ψ be a simple $\mathcal{H}$-module with central character $\mathfrak{o}$. Then there is $\tau_1 \in \mathfrak{o}$ such that Ψ is a submodule of the principal series $M = \mathcal{H} \otimes_{\mathbb{C}[\hat{T}]} \tau_1$. Then M extends to $\mathcal{H}'$ by lifting τ_1. As in (5.3b), we have $M^1_{\tau'} = M^1_{\tau}$, for all $\tau' \in \mathfrak{o}'$, which implies

$$\mathfrak{m}_{\tau'} \Psi^1_{\tau} \subseteq \mathfrak{m}_{\tau'} M^1_{\tau} = 0.$$

Hence $\mathbb{C}[\hat{T}']\Psi^1_{\tau} = \Psi^1_{\tau}$, so

$$\mathcal{H}' \Psi^1_{\tau} = \mathcal{H}_0 \Psi^1_{\tau} = \mathcal{H} \Psi^1_{\tau} = \Psi,$$

showing that Ψ is stable under $\mathcal{H}'$. This completes the proof of the lemma. $\square$

Remark. *The restriction from $\mathcal{H}'$ to $\mathcal{H}$ for arbitrary central character is described in the recent preprints* [RR], [R4].

5.4 The functor $V \mapsto V^{\mathcal{I}}$ induces an isomorphism

$$\mathrm{Ext}^{\bullet}_{G}(V, V') = \mathrm{Ext}^{\bullet}_{\mathcal{H}}(V^{\mathcal{I}}, V'^{\mathcal{I}}).$$

Since there are no extensions between $\mathcal{H}$-modules with different central characters, it follows that we have an EP-orthogonal direct sum

$$\mathcal{R}(G, \mathcal{I}) = \bigoplus_{\tau \in \hat{T}/W} \mathcal{R}(G, \mathcal{I}, \tau),$$

where $\mathcal{R}(G,\mathcal{I},\tau)$ is the span of the irreducible representations $V \in \mathcal{R}(G,\mathcal{I})$ such that $\mathcal{Z}$ acts via the character τ on $V^{\mathcal{I}}$. Letting $\bar{\mathcal{R}}(G,\mathcal{I},\tau)$ denote the image of $\mathcal{R}(G,\mathcal{I},\tau)$ in $\bar{\mathcal{R}}(G,\mathcal{I})$, we have

$$\bar{\mathcal{R}}(G,\mathcal{I}) = \bigoplus_{\tau\in\hat{T}/W} \bar{\mathcal{R}}(G,\mathcal{I},\tau),$$

again EP-orthogonal.

5.5 We recall here the formula for $EP(V,V')$ given by Schneider and Stuhler [SS,§4]. For $V \in \mathcal{A}(G,\mathcal{I})$ of finite length, define a function f_V on $G(F)$ by

$$f_V = \sum_{b\in\mathcal{F}} (-1)^{\dim b} \frac{\epsilon_b \bar{\chi}_b}{vol(G_b)},$$

where $\mathcal{F}$ is a set of representatives of $G(F)$ orbits of facets in the building $\mathbf{B}$, $vol(G_b)$ is the volume of G_b with respect to dg,

$$\epsilon_b : G_b \longrightarrow \pm 1$$

is the orientation character of G_b acting on b (thus, $\epsilon_b(g) = 1$ iff g preserves orientation on b), and χ_b is the character of G_b on the invariants V^{U_b}, extended by zero to all of $G(F)$.

(5.5.1) Theorem [SS, §4, Prop.1]**.** *We have*

$$EP(V,V') = \mathrm{tr}(f_V, V'),$$

where the trace is taken with respect to the Haar measure used to define f_V.

Over the next few sections, we will calculate $\mathrm{tr}(f_V, V')$ in terms of Weyl group representations. To begin with, it is clear that

$$\mathrm{tr}(f_V, V') = \sum_{b\in\mathcal{F}} (-1)^{\dim b} \dim \mathrm{Hom}_{G_b}(\epsilon_b \otimes V_b, V'_b), \tag{5.5a}$$

where we have abbreviated $V_b = V^{U_b}$.

We choose $\mathcal{F}$ to consist of Ω-orbits of facets b which lie in the closure of c_0. Then $U_b \subset \mathcal{I} \subseteq G_b$ for each $b \in \mathcal{F}$. Let $\mathcal{H}_b$ be subalgebra of functions in $\mathcal{H}$ supported on G_b, and let $\mathcal{H}_b^\circ$ be the functions in $\mathcal{H}_b$ supported on G_b°. Then

$$\mathcal{H}_b = \mathbb{C}\Omega_b \tilde{\otimes} \mathcal{H}_b^\circ,$$

with cross multiplication as in (5.2a). The character $\epsilon_b : G_b \longrightarrow \pm 1$ is trivial on G_b° hence may be viewed as a character of Ω_b, by which we can twist $\mathcal{H}_b$-modules via tensor product. We have then

$$[\epsilon_b \otimes \phi]^{\mathcal{I}} \simeq \epsilon_b \otimes \phi^{\mathcal{I}} \tag{5.5b}$$

for any G_b-representation ϕ.

We may also view $\mathcal{H}_b$ as an intertwining algebra:

$$\mathcal{H}_b = \mathrm{End}_{G_b}(\mathrm{Ind}_{\mathcal{I}}^{G_b} \mathbb{C}),$$

and

$$\mathrm{Ind}_{\mathcal{I}}^{G_b} \mathbb{C} = \bigoplus_{\psi} \psi \otimes \psi^{\mathcal{I}},$$

where ψ runs over the irreducible G_b representations that contain nonzero $\mathcal{I}$-invariants. If ϕ_1, ϕ_2 are G_b representations, then

$$\mathrm{Hom}_{G_b}(\phi_1, \phi_2) \simeq \mathrm{Hom}_{\mathcal{H}_b}(\phi_1^{\mathcal{I}}, \phi_2^{\mathcal{I}}). \tag{5.5c}$$

Combining (5.5a,b,c), and recalling that $U_b \subset \mathcal{I}$, we find that

$$\mathrm{tr}(f_V, V') = \sum_{b \in \mathcal{F}} (-1)^{\dim b} \dim \mathrm{Hom}_{\mathcal{H}_b}(\epsilon_b \otimes V^{\mathcal{I}}, V'^{\mathcal{I}}). \tag{5.5d}$$

5.6 Let v be an indeterminate, and let $\mathcal{H}(v)$ be the generic affine Hecke algebra. It is the algebra over $\mathbb{C}[v, v^{-1}]$ with the same generators as $\mathcal{H}$, but now q is replaced by v^2 in the relations. We have analogous subalgebras $\mathcal{H}_b(v) \subset \mathcal{H}(v)$.

Let $V \in \mathcal{R}(G, \mathcal{I})$ be irreducible and tempered. When G has connected center, the construction of $V^{\mathcal{I}}$ in [KL] shows that there is an $\mathcal{H}(v)$-module $\Psi(v)$ such that

$$V^{\mathcal{I}} = \Psi(q) := \mathbb{C}_q \otimes_{\mathbb{C}[v,v^{-1}]} \Psi(v),$$

where $\mathbb{C}_q = \mathbb{C}$ and $\mathbb{C}[v, v^{-1}]$ acts by evaluating $v \mapsto \sqrt{q}$. If G is arbitrary (split) and V has real central character, then by (5.3.1) we have an analogous $\mathcal{H}'(v)$- module $\Psi(v)$ obtained by restriction from an isogenous group G' with connected center.

If we instead specialize to $v = 1$, then $\Psi(1)$ is a module over the group algebra

$$\mathcal{H}(1) = \mathbb{C}\widetilde{W}.$$

Since $\mathcal{H}_b(1) = \mathbb{C}\widetilde{W}_b$ and $\widetilde{W}_b$ is a finite group, we have

$$\dim \mathrm{Hom}_{\mathcal{H}_b}(\epsilon_b \otimes V^{\mathcal{I}}, V'^{\mathcal{I}}) = \dim \mathrm{Hom}_{\widetilde{W}_b}(\epsilon_b \otimes \Psi(1), \Psi'(1)),$$

so that

$$\mathrm{tr}(f_V, V') = \sum_{b \in \mathcal{F}} (-1)^{\dim b} \dim \mathrm{Hom}_{\widetilde{W}_b}(\epsilon_b \otimes \Psi(1), \Psi'(1)). \tag{5.6a}$$

5.7 We may identify $\mathbb{C}[\hat{T}]$ with the group algebra $\mathbb{C}X$ of X, and the embedding $\mathbb{C}[\hat{T}] \hookrightarrow \mathcal{H}$ is a specialization of a generic embedding of algebras

$$j_v : \mathbb{C}X \hookrightarrow \mathcal{H}(v)$$

of the form

$$j_v(\lambda) = v^{n(\lambda)} T_{\lambda_1} T_{\lambda_2}^{-1}, \qquad \lambda \in X,$$

where $n(\lambda)$ is an integer, and λ_i are dominant with respect to Σ, such that $\lambda = \lambda_1 - \lambda_2$. Note that, under the identification $\mathcal{H}(1) = \mathbb{C}\widetilde{W}$, we have $j_1(\lambda) = \lambda$. We have, as vector spaces,

$$\mathcal{H}(v) = \mathcal{H}_0(v) \otimes j_v(\mathbb{C}X).$$

Assume that Ψ is a simple tempered $\mathcal{H}$-module with real central character, so it is obtained by specialization from the $\mathcal{H}(v)$-module $\Psi(v)$ as described in (5.6). The eigenvalues of $j_v(\lambda)$ on $\Psi(v)$ are of the form $v^{m(\lambda)}$ with $m(\lambda) \in \mathbb{Z}$, so $\lambda = j_1(\lambda)$ acts on $\Psi(1)$ by unipotent transformations.

5.8 We therefore consider representations of $\widetilde{W}$ on which X acts by unipotent transformations. Let $\pi : \widetilde{W} \longrightarrow W$ be the natural projection. If b is a facet in E, then $\widetilde{W}_b$ is finite, hence projects isomorphically onto a subgroup $W_b = \pi\widetilde{W}_b \subset W$. On representations, we then have an isometry

$$\pi_*^b : \mathcal{R}(\widetilde{W}_b) \longrightarrow \mathcal{R}(W_b).$$

Lemma(5.8.1). *Let ψ be a finite dimensional representation of $\widetilde{W}$ on which all elements of X act by unipotent transformations. Let ψ_b, ψ_0 be the restrictions of ψ to $\widetilde{W}_b$ and W, respectively. Then as W_b-modules, we have*

$$\pi_*^b\psi_b = \psi_0|_{W_b},$$

this last being the restriction of ψ_0 to $W_b \subseteq W$.

Proof. Since X acts on ψ via a finitely generated abelian unipotent group, normalized by the action of W, there is a $\widetilde{W}$-stable filtration on the space of ψ such that X acts trivially on each quotient. Let $\mathrm{gr}\psi$ be the associated graded space. It is a $\widetilde{W}$-module on which X acts trivially, and upon restriction to any finite subgroup W_1 of $\widetilde{W}$, we have $\mathrm{gr}\psi \simeq \psi$ as W_1-modules.

Let $w \in W_b$, and let $w\lambda \in \widetilde{W}_b$, with $\lambda \in X$, be the unique lift of w in $\widetilde{W}_b$. Taking traces, we have

$$\mathrm{tr}(w, \pi_*^b\psi_b) = \mathrm{tr}(w\lambda, \psi) = \mathrm{tr}(w\lambda, \mathrm{gr}\psi) = \mathrm{tr}(w, \mathrm{gr}\psi) = \mathrm{tr}(w, \psi_0).$$

The lemma is proved. $\square$

Now suppose we have two $\widetilde{W}$-modules ψ, ψ', on which X acts by unipotent transformations. Let $(\ ,\ )_\Gamma$ denote multiplicities between virtual representations of a group Γ. By Frobenius reciprocity and (5.8.1) we have

$$\begin{aligned}(\psi_b \otimes \epsilon_b,\ \psi_b')_{\widetilde{W}_b} &= (\pi_*^b(\psi_b \otimes \epsilon_b),\ \pi_*^b\psi_b')_{W_b} \\ &= ((\psi_0|_{W_b}) \otimes \pi_*^b\epsilon_b,\ \psi_0'|_{W_b})_{W_b} \\ &= (\psi_0 \otimes \mathrm{Ind}_{W_b}^W \pi_*^b\epsilon_b,\ \psi_0')_W. \end{aligned} \tag{5.8a}$$

5.9 The representations $\mathrm{Ind}_{W_b}^W \pi_*^b\epsilon_b$ also appear in the following expression for $\Lambda E = \sum_{n\geq 0}(-1)^n\Lambda^n E$.

(5.9.1)Lemma. *As virtual W-modules, we have*

$$\Lambda E = \sum_{b\in\mathcal{F}} (-1)^{\dim b}\, \mathrm{Ind}_{W_b}^{W}\, \pi_*^b \epsilon_b.$$

Proof. The character of ΛE is the Lefschetz character of W on the cohomology of the compact torus E/X. Now E/X has a W-simplicial structure induced by the $\widetilde{W}$-simplicial decomposition of E. As W-modules, the oriented co-chain groups are

$$C^n(E/X) \simeq \bigoplus_{\dim b=n} [\mathrm{Ind}_{\widetilde{W}_b}^{\widetilde{W}}\, \epsilon_b]^X,$$

sum over facets in $\mathcal{F}$ of dimension n. Restriction of functions induces an isomorphism

$$[\mathrm{Ind}_{\widetilde{W}_b}^{\widetilde{W}}\, \epsilon_b]^X \simeq \mathrm{Ind}_{W_b}^{W}\, \pi_*^b \epsilon_b$$

as W-modules. The lemma follows. □

Let e_W be the elliptic pairing on $\mathcal{R}(W)$, described in §3. Combining (5.8a) and (5.9.1) we have

(5.9.2)Proposition. *Let ψ, ψ' be representations of $\widetilde{W}$ on which all elements of X act by unipotent transformations. Then*

$$\sum_{b\in\mathcal{F}} (-1)^{\dim b} (\epsilon_b \otimes \psi_b, \psi'_b)_{\widetilde{W}_b} = e_W(\psi_0, \psi'_0).$$

5.10 Let $\mathcal{R}_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R})$ be the span of the irreducible tempered representations in $\mathcal{R}(G,\mathcal{I})$ with real central character. We have a map

$$r : \mathcal{R}_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R}) \longrightarrow \mathcal{R}(W), \quad r(V) = \Psi(1)|_W,$$

where $\Psi(v)$ is the $\mathcal{H}(v)$-module such that $\Psi(\sqrt{q}) = V^{\mathcal{I}}$.

Combining (5.6a) and (5.9.2) we have proved

(5.10.1)Theorem. *The map $r : R_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R}) \longrightarrow \mathcal{R}(W)$ is an isometry between the pairing EP on $\mathcal{R}_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R})$ and the pairing e_W on $\mathcal{R}(W)$.*

(5.10.2)Corollary. *Let $G \longrightarrow G'$ be an isogeny. Then the restriction map*

$$\mathcal{R}_{\mathrm{temp}}(G',\mathcal{I}',\mathbb{R}) \longrightarrow \mathcal{R}_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R}),$$

in (5.3.1) is an isometry for the respective EP pairings.

Proof. Since $\mathcal{H}_0 = \mathcal{H}'_0$, the restriction commutes with the respective maps r defined above, and the pairing e_W on $\mathcal{R}(W)$ is independent of isogeny class. □

5.11 According to [KL,8.2], the irreducible representations in $\mathcal{R}_{\mathrm{temp}}(G,\mathcal{I},\mathbb{R})$ are parametrized by $\hat{G}$ conjugacy classes of pairs (u,ρ) where $u \in \hat{G}$ is unipotent and ρ is an irreducible representation $\mathcal{R}_\circ(A_u)$. This is proved in [KL] for G with connected center. By (5.10.2) it holds as well for any semisimple G, since we have real central character. The central character of the corresponding representation $V_u(\rho)$ is given as follows. We may choose u in its conjugacy class so that there is a homomorphism $\varphi : SL_2(\mathbb{C}) \longrightarrow \hat{G}$ mapping the diagonal matrices into $\hat{T}$, such that $u = \varphi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the central character of $V_u(\rho)$ is the W-orbit of

$$\tau_u = \varphi\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

It is known that τ_u is W-conjugate to $\tau_{u'}$ if and only if u is $\hat{G}$-conjugate to u'. Therefore if we define

$$\mathcal{R}_u(G,\mathcal{I}) = \{V_u(\rho) : \ \rho \in \mathcal{R}_\circ(A_u)\},$$

then

$$\mathcal{R}_u(G,\mathcal{I}) = \mathcal{R}_{\text{temp}}(G,\mathcal{I},\mathbb{R}) \cap \mathcal{R}(G,\mathcal{I},\tau_u)$$

and we have an EP-orthogonal direct sum

$$\mathcal{R}_{\text{temp}}(G,\mathcal{I},\mathbb{R}) = \bigoplus_{u\in\mathcal{U}_G} \mathcal{R}_u(G,\mathcal{I}). \tag{5.11a}$$

On the Weyl group side, recall from §3 that

$$\mathcal{R}(W) = \bigoplus_u \mathcal{R}_u(W),$$

and that $R_u(W)$ has the basis $\{H_u(\rho) : \ \rho \in \mathcal{R}_\circ(A_u)\}$.

Now Lusztig has calculated the map $r : \mathcal{R}_{\text{temp}}(G,\mathcal{I},\mathbb{R}) \longrightarrow \mathcal{R}(W)$ as follows.

(5.11.1) Theorem [L2],[L3]. *We have*

$$r(V_u(\rho)) = \epsilon \otimes H_u(\rho),$$

where ϵ is the sign character of W.

This, combined with (5.9.1) proves the orthogonality of the sum $\bigoplus_u \mathcal{R}_u(W)$, as claimed in (3.4.2).

The representation $V_u(\rho)$ is square-integrable if and only if u is distinguished in $\hat{G}$ [KL, 8.3]. Then the elliptic pairing on $\mathcal{R}_\circ(A_u)$ coincides with the ordinary pairing. By [SS,Thm 6] we know that $\{V_u(\rho) : \ \rho \in \mathcal{R}_\circ(A_u)\}$ is then an EP-orthonormal basis of $\mathcal{R}_u(G,\mathcal{I})$. This proves (3.4.4).

6. Elliptic theory for real central characters.

We now apply the results of §5 to the elliptic theory of $G(F)$. Let $\bar{\mathcal{R}}(G,\mathcal{I},\mathbb{R})$ be the image of $\mathcal{R}(G,\mathcal{I},\mathbb{R})$ in $\bar{\mathcal{R}}(G)$.

6.1 Let u be a unipotent element in $\hat{G}$, let $\hat{L}$ be a Levi subgroup containing u, and let L be the corresponding Levi subgroup of G. Let $\mathcal{I}_L$ be an Iwahori subgroup of $L(F)$. Let

$$V_u^L : \mathcal{R}_\circ(A_u^L) \longrightarrow \mathcal{R}(L,\mathcal{I}_L)$$

be the Kazhdan-Lusztig isomorphism for L. It follows from [KL,6.2] (a result whose hypotheses are verified for our situation in the last paragraph of [KL,p.213]) that

$$\mathrm{I}_L^G \circ V_u^L = V_u \circ \mathrm{Ind}_{A_u^L}^{A_u} . \tag{6.1a}$$

As in (3.4.1), V_u induces an isomorphism

$$\bar{V}_u : \bar{\mathcal{R}}_\circ(A_u) \longrightarrow \bar{\mathcal{R}}_u(G,\mathcal{I}),$$

where the right side is the image of $\mathcal{R}_u(G,\mathcal{I})$ in $\bar{\mathcal{R}}(G)$.

6.2 From §5, we have bijective isometries

$$r : \mathcal{R}_u(G,\mathcal{I}) \longrightarrow \mathcal{R}_u(W), \quad r_L : \mathcal{R}_u(L,\mathcal{I}_L) \longrightarrow \mathcal{R}_u(W_L).$$

From [J, 2.1.2] it follows that

$$r \circ \mathrm{I}_L^G = \mathrm{Ind}_{W_L}^W \circ r_L. \tag{6.2a}$$

Hence r induces a bijective isometry

$$\bar{r} : \bar{\mathcal{R}}_u(G,\mathcal{I}) \longrightarrow \bar{\mathcal{R}}_u(W).$$

From (3.4.1) we conclude that the space $\bar{\mathcal{R}}_u(G,\mathcal{I})$ is nonzero if and only if u is quasi-distinguished in $\hat{G}$.

Finally, Lusztig's theorem (5.11.1) implies that

$$\bar{r} \circ \bar{V}_u = \epsilon \otimes \bar{H}_u = (-1)^\ell \bar{H}_u,$$

where $\bar{H}_u$ is the elliptic Springer isomorphism from (3.4.1).

We summarize all of this in the following theorem.

(6.2.1)Theorem. *The following diagram of vector space isomorphisms commutes up to the sign* $(-1)^\ell$.

$$\begin{array}{ccc}
\bigoplus_u \bar{\mathcal{R}}_\circ(A_u) & \xrightarrow{\oplus \bar{V}_u} & \bar{\mathcal{R}}(G,\mathcal{I},\mathbb{R}) = \bigoplus_u \bar{\mathcal{R}}_u(G,\mathcal{I}) \\
\| & & \downarrow \bar{r} \\
\bigoplus_u \bar{\mathcal{R}}_\circ(A_u) & \xrightarrow[\oplus \bar{H}_u]{} & \bar{\mathcal{R}}(W) = \bigoplus_u \bar{\mathcal{R}}_u(W).
\end{array}$$

Moreover, the sums on the right side are orthogonal for the pairings EP and e_W, the map $\bar{r}$ is an isometry, and the nonzero summands are precisely those for which u is quasi-distinguished in $\hat{G}$.

7. Arbitrary central character.

We now assume that G has adjoint type. We want to generalize (6.2.1) to arbitrary central characters. The calculations of sections (5.7-11) don't work because X no longer acts by unipotent transformations. Instead, we will reduce to real central character on a smaller group, using results of [BaM].

Let $\mathcal{R}_{\text{temp}}(G,\mathcal{I})$ be the span of the irreducible tempered $\mathcal{I}$-spherical representations of $G(F)$, without restriction on central character. By [KL, 8.2] we have a direct sum

$$\mathcal{R}_{\text{temp}}(G,\mathcal{I}) = \bigoplus_x \mathcal{R}_x(G,\mathcal{I}),$$

together with an isomorphism

$$V_x : \mathcal{R}_\circ(A_x) \longrightarrow \mathcal{R}_x(G,\mathcal{I}),$$

where x runs over conjugacy classes of elements in $\hat{G}$ with compact semisimple part, and $\mathcal{R}_x(G,\mathcal{I})$ is the span of irreducible representations $V_x(\rho)$, $\rho \in \mathcal{R}_\circ(A_x)$. Here $\mathcal{R}_\circ(A_x)$ is the span of the irreducible representations of A_x appearing in $H(\mathcal{B}^x)$.

Let $x = su$ be the Jordan decomposition of such an x, and let $\hat{G}_s$ be the centralizer of s in $\hat{G}$. We may choose $\phi : SL_2(\mathbb{C}) \longrightarrow \hat{G}_s$, as in (3.2). Let $\tau = s\tau_u$. By [KL, 8.2] we have

$$\mathcal{R}_x(G,\mathcal{I}) = \mathcal{R}_{\text{temp}}(G,\mathcal{I}) \cap \mathcal{R}(G,\mathcal{I},\tau).$$

Let $\bar{\mathcal{R}}_x(G,\mathcal{I})$ be the image of $\mathcal{R}_x(G,\mathcal{I})$ in $\bar{\mathcal{R}}(G)$.

(7.1.1)Lemma. *If $\hat{G}_s$ is not semisimple, then $\bar{\mathcal{R}}_x(G,\mathcal{I}) = 0$.*

Proof. We are assuming G has adjoint type, in particular $\hat{G}$ is semisimple. If $\hat{G}_s$ is not semisimple, it must be contained in a proper Levi subgroup $\hat{L}$ of $\hat{G}$. Then $A_x^L = A_x$, and $\mathcal{R}_\circ(A_x^L) = \mathcal{R}_\circ(A_x)$. By [KL, bottom of p.213] we have

$$\mathrm{I}_L^G(V_x^L(\rho)) = V_x(\rho), \qquad \rho \in \mathcal{R}_\circ(A_x^L).$$

This proves the lemma. □

Assume now that $\hat{G}_s$ is semisimple. Let G_s be the split group over F whose root datum is dual to that of $\hat{G}_s$. Let $\mathcal{I}_s$ be an Iwahori subgroup of G_s. According to [BaM] there is an isomorphism

$$\iota : \mathcal{R}(G,\mathcal{I},\tau) \xrightarrow{\simeq} \mathcal{R}(G_s,\mathcal{I}_s,\tau_u). \tag{7.1a}$$

This isomorphism is induced by an equivalence of categories, hence it preserves the EP-pairing. This isomorphism also preserves tempered representations [BaM, 6.5], so (7.1a) restricts to an isomorphism

$$\mathcal{R}_x(G,\mathcal{I}) \xrightarrow{\simeq} \mathcal{R}_u(G_s,\mathcal{I}_s). \tag{7.1b}$$

Let G_s' be the adjoint group of G_s, and let $\mathcal{I}_s'$ be the unique Iwahori subgroup of G_s' containing the image of $\mathcal{I}_s$ under the isogeny $G_s \longrightarrow G_s'$. Combining (7.1b) with (5.10.2), we have a bijective EP-isometry

$$\mathcal{R}_x(G,\mathcal{I}) \longrightarrow \mathcal{R}_u(G_s',\mathcal{I}_s'), \tag{7.1c}$$

where τ_u' corresponds to τ_u as in (5.3.1). Lusztig's theorem (5.11.1) applies to $R_x(G,\mathcal{I})$ as well, and asserts that

$$r \circ V_x = \epsilon \otimes H_x.$$

The Weyl group of G'_s is the centralizer W_s of s in W. Let

$$r_s : \mathcal{R}_u(G'_s, \mathcal{I}'_s) \longrightarrow \mathcal{R}_u(W_s) \tag{7.1d}$$

be the analogue of the map r defined in 5.10. It is an isometry, by (5.10.1). By [BaM, Cor. 3.4], the isomorphism (7.1a) satisfies

$$r = \mathrm{Ind}_{W_s}^{W} \circ r_s \circ \iota. \tag{7.1e}$$

Finally, by [Ka] we have

$$H_x = \mathrm{Ind}_{W_s}^{W} \circ H_u^s, \tag{7.1f}$$

where $H_u^s : \mathcal{R}_\circ(A_x) \longrightarrow \mathcal{R}_u(W_s)$ is the Springer isomorphism for W_s. A diagram chase involving (7.1c-f) proves the following generalization of (6.2.1).

Theorem(7.1.2). *The spaces $\bar{\mathcal{R}}_x(G,\mathcal{I})$, $\bar{\mathcal{R}}_{x'}(G,\mathcal{I})$ are orthogonal with respect to EP unless x, x' are conjugate in $\hat{G}$. If $\bar{\mathcal{R}}_{su}(G,\mathcal{I}) \neq 0$ then $\hat{G}_s$ is semisimple, u is quasi-distinguished in $\hat{G}_s$, and we have a diagram of isomorphisms commuting up to the sign $(-1)^\ell$*

$$\begin{array}{ccc}
\bar{\mathcal{R}}_\circ(A_{su}) & \xrightarrow{\bar{V}_{su}} & \bar{\mathcal{R}}_{su}(G,\mathcal{I}) \\
\Big\| & & \Big\downarrow \bar{r}_s \\
\bar{\mathcal{R}}_\circ(A_{su}) & \xrightarrow[\bar{H}_u^s]{} & \bar{\mathcal{R}}_u(W_s).
\end{array}$$

The map $\bar{r}_s$ is an isometry with respect to the pairings EP and e_{W_s}.

In the next few sections, we will see that the horizontal maps in (7.1.2) are also isometries, and this will complete the proof of the Main Theorem as stated in the introduction. We need Arthur's formula, and a comparison of analytic and geometric R-groups. We must also prove (3.3.2). These issues make sense for arbitarary Langlands parameters. When necessary, we make assumptions (see 9.2) about the conjectural Langlands correspondence. These assumptions are known to hold in the Iwahori-spherical case (see 9.3). The results on the geometric side are, of course, independent of these assumptions.

8. Analytic R-groups and Arthur's formula

Let S be an F-split torus in G, with centralizer L. Let $P = LU$ be a parabolic subgroup, and let Δ^+ be the roots of S in the Lie algebra of U. Let β be a reduced root in Δ^+, and let L_β be the centralizer of the kernel of β. Then L is a maximal Levi subgroup of L_β. Let $\hat{S}_\beta$ be the complex torus of unramified characters of $[S/\ker\beta](F)$. Given a discrete series representation V_L of $L(F)$, the corresponding Plancherel measure $\mu_\beta(t)$ is a rational function of $t \in \hat{S}_\beta$. We set $\Delta(V_L) = \{\beta \in \Delta^+ :\ \mu_\beta(1) = 0\}$.

Let $W(S) = N(S)/Z(S) = N(L)/L$, and let $W(S, V_L) = \{w \in W(S) :\ (V_L)^w \simeq V_L\}$. The analytic R-group is defined as

$$R_{an}(V_L) = \{r \in W(S, V_L) :\ r\Delta(V_L) = \Delta(V_L)\}.$$

Harish-Chandra proved that

$$\mathrm{End}_{G(F)}(\mathrm{Ind}_{P(F)}^{G(F)} V_L) \simeq \mathbb{C}[R_{an}(V_L), \eta_{an}],$$

where η_{an} is a certain 2-cocycle on $R_{an}(V_L)$ defined by the composition of standard intertwining operators. We therefore have a decomposition

$$\mathrm{Ind}_{P(F)}^{G(F)} V_L \simeq \bigoplus_{\psi} \psi \otimes V(\psi),$$

as $\mathbb{C}[R_{an}(V_L), \eta_{an}] \times G(F)$ modules, where ψ runs over the irreducible representations of $\mathbb{C}[R_{an}(V_L), \eta_{an}]$, and

$$V(\psi) = \mathrm{Hom}_{\mathbb{C}[\mathcal{R}_{an}(V_L), \eta_{an}]}(\psi, \mathrm{Ind}_{P(F)}^{G(F)} V_L).$$

Assume that F has characteristic zero. Let C_{ell} be the set of elliptic regular conjugacy classes in $G(F)$. The Weyl integration formula may be viewed as a measure dc on C_{ell} [K,§3]. Let V, V' be admissible representations of $G(F)$, with respective characters $\Theta_V, \Theta_{V'}$. Schneider and Stuhler [SS,Thm 21, Cor 17] proved that

$$EP(V, V') = \int_{C_{ell}} \Theta_V \overline{\Theta}_{V'} \; dc.$$

Let $V = V(\psi), V' = V(\psi')$ be two constituents of $\mathrm{Ind}_{P(F)}^{G(F)} V_L$, as above. Arthur [A, Cor 6.3] has shown that

$$\int_{C_{ell}} \Theta_{V(\psi)} \overline{\Theta}_{V(\psi')} dc = e_{an}(\psi, \psi'),$$

where e_{an} is the pairing on elliptic representations of $\mathcal{R}_{an}(V_L)$, defined with respect to the real Lie algebra of S. Note that ψ, ψ' are only projective representations of $\mathcal{R}_{an}(V_L)$, but their common multiplier is a root of unity, so that if $r \in \mathcal{R}_{an}(V_L)$, the quantity $\mathrm{tr}(r, \psi)\overline{\mathrm{tr}(r, \psi')}$, is well defined, hence so is the pairing e_{an}.

Thus, in characteristic zero, we have a calculation of EP as follows:

$$EP(V(\psi), V(\psi')) = e_{an}(\psi, \psi'). \tag{8.1a}$$

In some sense (8.1a) gives a calculation of EP that is far more general than the Iwahori spherical case considered previously. However, the calculation of $R_{an}(V_L)$ depends on the zeros of Plancherel measures, which are quite subtle. See [S], for V_L generic. A few non-generic examples are calculated in [R2].

In contrast, the geometric R-group is very simple, but its connection to the analytic R-group is based on expected properties of the conjectural Langlands correspondence, as we shall describe.

9. Geometric R-groups

9.1 Let $\mathcal{W}_F$ be the Weil group of our non-archimedean local field F, and suppose we have a homomorphism

$$\Phi : \mathcal{W}_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

mapping $\mathcal{W}_F$ to a bounded subgroup $\Gamma \subset \hat{G}$, generated by semisimple elements.

Let M be the centralizer of the image of Φ. Then M is reductive, though possibly disconnected. Let $\hat{S}$ be a maximal torus in the identity component M° of M, and let $\hat{L}$ be the centralizer of $\hat{S}$. Then $(\hat{L}_\Gamma)^\circ$ is a Levi subgroup of $(\hat{G}_\Gamma)^\circ$ minimally containing $\Phi(SL_2(\mathbb{C}))$. Set $M_L := M \cap \hat{L}$. Then $M_L^\circ = \hat{S}$.

Choose a triangular decomposition of Lie algebras

$$\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}.$$

That is, $\bar{\mathfrak{u}}$ and $\mathfrak{u}$ are the nilradicals of opposite parabolic subalgebras of $\mathfrak{g}$ with $\mathfrak{l}$ as Levi subalgebra. Then

$$\mathfrak{m} = (\bar{\mathfrak{u}} \cap \mathfrak{m}) \oplus \mathfrak{s} \oplus (\mathfrak{u} \cap \mathfrak{m}) \tag{9.1a}$$

and the subalgebra

$$\mathfrak{b}_\mathfrak{m} := \mathfrak{s} \oplus (\mathfrak{u} \cap \mathfrak{m})$$

is a Borel subalgebra of $\mathfrak{m}$.

Let $N(\mathfrak{s}, \mathfrak{b}_\mathfrak{m})$ be the normalizer in M of the pair $(\mathfrak{s}, \mathfrak{b}_\mathfrak{m})$. The inclusion of $N(\mathfrak{s}, \mathfrak{b}_\mathfrak{m})$ in M induces an isomorphism

$$N(\mathfrak{s}, \mathfrak{b}_\mathfrak{m})/\hat{S} \simeq M/M^\circ.$$

Generalizing the group A_x, we define

$$A = A_\Phi := M/Z_{\hat{G}}M^\circ.$$

Replacing $\hat{G}$ by $\hat{L}$, we have the analogous group

$$A_L := M_L/(Z_{\hat{L}}M_L^\circ) = M_L/Z_{\hat{L}},$$

which is naturally identified with a normal subgroup of A. We set

$$R := A/A_L.$$

Thus

$$R \simeq M/M^\circ M_L \simeq N(\mathfrak{s}, \mathfrak{b}_\mathfrak{m})/M_L.$$

Let R_σ be the stabilizer in R of $\sigma \in \mathrm{Irr}(A_L)$. We call R_σ the *geometric R-group* of the pair (Φ, σ).

We have

$$\mathrm{End}_A(\mathrm{Ind}_{A_L}^A \sigma) = \bigoplus_{r \in R_\sigma} \mathrm{Hom}_{A_L}(\sigma, \sigma^r).$$

A choice of nonzero $T_r \in \mathrm{Hom}_{A_L}(\sigma, \sigma^r)$, for each $r \in R_\sigma$, determines a 2-cocycle η on R_σ such that

$$T_x T_y = \eta(x, y) T_{xy}.$$

If $\mathbb{C}[R_\sigma, \eta]$ denotes the group algebra of R_σ twisted by η, we have algebra isomorphisms

$$\mathbb{C}[R_\sigma, \eta] \simeq \mathrm{End}_A(\mathrm{Ind}_{A_L}^A(\sigma)) \simeq \bigoplus_{\rho \in \mathrm{Irr}(A)} \mathrm{End}_\mathbb{C}(\psi_\rho), \tag{9.1b}$$

where $\psi_\rho = \mathrm{Hom}_A(\rho, \mathrm{Ind}_{A_L}^A(\sigma))$.

9.2 This subsection contains some conjectural properties of the Langlands correspondence, and consequences thereof. All of the properties are known to hold in the Iwahori-spherical case, see (9.3). For the real case, where all the following conjectures are proved, see [KZ].

The irreducible representations of A are supposed to parametrize an L-packet

$$\Pi_\Phi = \{V(\rho) : \ \rho \in \mathrm{Irr}(A)\}$$

of irreducible representations $V(\rho)$ of $G(F)$. These representations should all be tempered, since Γ is bounded. Thus, we expect an isomorphism

$$V : \mathcal{R}(A) \longrightarrow \mathcal{R}_\Phi(G),$$

where $\mathcal{R}_\Phi(G)$ denotes the span of the representations in Π_Φ.

Likewise, the set $\mathrm{Irr}(A_L)$ of irreducible representations of A_L is supposed to parametrize an L-packet

$$\Pi_\Phi^L = \{V_L(\sigma) : \ \sigma \in \mathrm{Irr}(A_L)\}$$

of irreducible representations $V_L(\sigma)$ of $L(F)$. Moreover, since the image of Φ lies in no proper Levi subgroup of $\hat{L}$, the representations in Π_Φ^L are supposed to be square-integrable modulo the center of $L(F)$.

Let $\Delta(\mathfrak{u})$ denote the roots of $\hat{S}$ in $\mathfrak{u}$. Since the groups G and $\hat{G}$ are dual, we have a canonical bijection $\beta \mapsto \hat{\beta} : \Delta^+ \to \Delta(\mathfrak{u})$.

Our first assumption about the Langlands correspondence is that

$$\Delta(V_L(\sigma))\hat{} = \Delta(\mathfrak{u} \cap \mathfrak{m}), \tag{9.2a}$$

the latter being the roots of $\hat{S}$ in $\mathfrak{u} \cap \mathfrak{m}$. Assumption (9.2a) would follow from Langlands' conjecture describing Plancherel measures in terms of Artin L-functions. For generic Iwahori spherical representations, (9.2a) was proved by Shahidi [Sh,Thm. 3.5]. It will be verified for all Iwahori-spherical representations in section 9.3. For real groups, (9.2a) was proved by Langlands [KZ,Thm. 3.3].

Let $N(L)$, $N(\hat{L})$ be the respective normalizers of L, $\hat{L}$ in G, $\hat{G}$. Both $N(L)/L$ and $N(\hat{L})/\hat{L}$ are naturally isomorphic to a subgroup $W(L) \subset W$, and we identify $N(L)/L = N(\hat{L})/\hat{L} = W(L)$.

Our second assumption is that

$$[V_L(\sigma)]^n = V_L(n\sigma), \qquad n \in N(L)/L \tag{9.2b}$$

where $n\sigma$ is the conjugated representation of $n(A_\Phi^L)n^{-1} = A_{n\Phi n^{-1}}^L$.

Assumptions (9.2a,b) imply that the analytic R-group $R_{an}(V_L(\sigma))$ for the discrete series representation $V_L(\sigma)$ consists of those $n\hat{L} \in N(\hat{L})/\hat{L}$ such that

(1) $\Phi^n \simeq \Phi$
(2) $n\Delta(\mathfrak{u} \cap \mathfrak{m}) \subseteq \Delta(\mathfrak{u})$
(3) $n\sigma \simeq \sigma$.

Now (1) says that $n \in M/M_L$, and (2) says that $n \in N(\mathfrak{s}, \mathfrak{b}_m)$. Taking (3) into account, we get equality of analytic and geometric R-groups:

(9.2.1) Proposition. *If assumptions (9.2a,b) hold, then* $R_{an}(V_L(\sigma)) = R_\sigma$.

The next assumption is about the relation between Φ and Φ_L. Namely, we should have

$$\Pi_\Phi = \mathrm{I}_L^G\, \Pi_\Phi^L,$$

in the sense that Π_Φ consists of the constituents of the representations induced from those in Π_Φ^L.

In fact, based on (6.1a), we expect a refinement of this. Namely, for each $\sigma \in \mathcal{R}(A_L)$, we should have

$$V(\mathrm{Ind}_{A_L}^A\, \sigma) \simeq \mathrm{I}_L^G\, V_L(\sigma). \tag{9.2c}$$

Moreover, isomorphism (9.2c) should have the following property: Let $r \in R_\sigma$, and let T_r be an element of $\mathrm{End}_A(\mathrm{Ind}_{A_L}^A\, \sigma)$ supported on rA_L. Invoking (9.2c), we have a $G(F)$-endomorphism T'_r of $\mathrm{I}_L^G\, V_L(\sigma)$. On the other hand, by (9.2.1), we may view r as an element of $R_{an}(V_L(\sigma))$, which corresponds to an intertwining operator $\mathcal{A}_r$ of $\mathrm{I}_L^G\, V_L(\sigma)$, given by the standard integral. The final refinement is the assumption

$$T'_r = c_r \mathcal{A}_r \tag{9.2d}$$

for some nonzero scalar $c_r \in \mathbb{C}$.

By assumption (9.2c), the $G(F)$-endomorphism algebra of $\mathrm{I}_L^G\, V_L(\sigma)$ is

$$\mathrm{End}_{G(F)}(V(\mathrm{Ind}_{A_L}^A\, \sigma)) = \bigoplus_{\rho \in \mathrm{Irr}(A)} \mathrm{End}_{G(F)}(\psi_\rho \otimes V(\rho)) = \bigoplus_{\rho \in \mathrm{Irr}(A)} \mathrm{End}_{\mathbb{C}}(\psi_\rho).$$

In view of (9.1a), this implies an analogue of the Harish-Chandra commuting algebra theorem, but in terms of geometric R-groups instead of analytic ones.

(9.2.2)Proposition. *If (9.2a-c) hold, then* $\mathrm{End}_{G(F)}(\mathrm{Ind}_{P(F)}^{G(F)}\, V_L(\sigma))$ *is isomorphic to the twisted group algebra* $\mathbb{C}[R_\sigma, \eta]$. *If in addition (9.2d) holds, then this isomorphism sends* $r \in R_\sigma$ *to a scalar multiple of the standard intertwining operator* $\mathcal{A}_r$.

Since the analytic cocycle η_{an} arises from the multiplication of the $\mathcal{A}_r$'s, (9.2.2) implies that η_{an} is cohomologous to η. If e_A denotes the elliptic pairing on A, and e_{an} that on $R_{an}(V_L(\sigma))$, then for two representations ρ, ρ' of A which appear in $\mathrm{Ind}_{A_L}^A\, \sigma$, we have

$$e_A(\rho, \rho') = e_{an}(\psi_\rho, \psi_{\rho'}).$$

Then by Arthur's formula (8.1a), we have

$$e_A(\rho, \rho') = EP(V_\Phi(\rho), V_\Phi(\rho')).$$

In summary then, we have shown

(9.2.3)Proposition. *If char(F)=0, and we have a Langlands correspondence* $\rho \mapsto V(\rho)$ *satisfying (9.2a-d) as above, then it induces an isometry between the elliptic pairings* e_A *on* $\bar{\mathcal{R}}(A)$ *and* EP *on* $\bar{\mathcal{R}}_\Phi(G)$.

In the next section we will see that (9.2a-d) hold for Iwahori-spherical representations, if we restrict to $\mathcal{R}_\circ(A_x)$. In this setting, EP can be computed in the category of $\mathcal{H}$-modules (see (5.4)), hence is independent of $\mathrm{char}(F)$. This proves that the horizontal maps in (7.1.2) are also isometries, and this completes the proof of the Main Theorem, as stated in the introduction, modulo the proof of (3.3.2), which is given in section 10.

9.3 Suppose $\Phi : \mathcal{W}_F \times SL_2(\mathbb{C}) \longrightarrow \hat{L}$ is unramified on $\mathcal{W}_F$. We may choose dual maximal tori $T \subset G$, $\hat{T} \subset \hat{G}$, such that Φ maps $\mathcal{W}_F$ and the diagonal matrices of $SL_2(\mathbb{C})$ into $\hat{T}$.

Let $s \in \hat{T}$ be the image of a Frobenius element under Φ, and set

$$x = s\Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \hat{L}.$$

Suppose that σ is an irreducible representation of $A^L_\Phi = A^L_x$ appearing in the homology of $\mathcal{B}^x$.

To check (9.2a), we may assume that L is maximal in G, i.e., that $G = L_\beta$, as in section 8. Proving (9.2a) then amounts to showing that $\mu_\beta(1) = 0$ if and only if $\mathfrak{u} \cap \mathfrak{m} = 0$.

Since $\Phi(SL_2(\mathbb{C})) \subset \hat{L}^s$, the space $\mathfrak{u}^s$ is an $\mathfrak{sl}_2(\mathbb{C})$-module. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and let $\mathfrak{u}^s(i)$ be the $2i$-eigenspace of h in $\mathfrak{u}^s$. For $i \geq 0$, the map

$$e : \mathfrak{u}^s(i) \longrightarrow \mathfrak{u}^s(i+1)$$

is surjective. Since $\mathfrak{u} \cap \mathfrak{m}$ is the kernel of e on $\mathfrak{u}^s(0)$, we see that $\mathfrak{u} \cap \mathfrak{m} = 0$ if and only if

$$\dim \mathfrak{u}^s(0) = \dim \mathfrak{u}^s(1).$$

On the other hand, the Plancherel measure $\mu_\beta(t)$ is described in terms of intertwining operators as follows. To ease the notation, we write

$$\pi = V_L(\sigma).$$

Let w be the element of W sending $\mathfrak{u}$ to $\bar{\mathfrak{u}}$. Let $\gamma : \mathbb{C}^\times \longrightarrow \hat{S}$ be a one-parameter subgroup generating $\hat{S}$. For each $t \in \mathbb{C}^\times$, we view γ_t as an unramified character of $\hat{L}$. The standard intertwining operator

$$\mathcal{A}_w(\pi, t) : \mathrm{Ind}_{P(F)}^{G(F)}(\pi \otimes \gamma_t) \longrightarrow \mathrm{Ind}_{P(F)}^{G(F)}(w\pi \otimes w\gamma_t)$$

is given by analytic continuation of the integral

$$\mathcal{A}_w(\pi, t) f(g) = \int_{U(F)} f(\dot{w}^{-1} u g)\, du. \tag{9.3a}$$

Here $\dot{w}$ is a representative of w, and du is a Haar measure on $U(F)$. The composition $\mathcal{A}_w(w\pi, t^{-1}) \circ \mathcal{A}_w(\pi, t)$ is a scalar operator, given by a nonzero constant times $\mu_\beta(t)^{-1}$.

Since Φ is unramified, the representation π is contained in an unramified principal series for $L(F)$. More precisely, let

$$\tau = s\Phi(1 \times \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}),$$

viewed as an unramified character of $T(F)$. There is a Borel subgroup B_L of L containing T, such that

$$\pi \hookrightarrow \mathrm{Ind}_{B_L(F)}^{L(F)} \tau.$$

Thus,

$$\mathrm{Ind}_{P(F)}^{G(F)}(\pi \otimes \gamma_t) \hookrightarrow \mathrm{Ind}_{B(F)}^{G(F)}(\tau\gamma_t), \tag{9.3b}$$

where $B = B_L U$ is a Borel subgroup of G. Under the embedding (9.3b), the intertwining operator $A_w(\pi, t)$ is the restriction of the operator

$$A_w(\tau\gamma_t) : \mathrm{Ind}_{B(F)}^{G(F)}(\tau\gamma_t) \longrightarrow \mathrm{Ind}_{B(F)}^{G(F)}(w(\tau\gamma_t)),$$

given by the same formula (9.3a).

Now from [Ca] we know, for generic t, that

$$A_{w^{-1}}(w(\tau\gamma_t)) \circ A_w(\tau\gamma_t) = C_w(\tau\gamma_t) C_w(w(\tau\gamma_t)),$$

where C_w is the rational function on $\hat{T}$ defined by

$$C_w(\chi) = \prod_{\alpha \in \Delta(\mathfrak{u})} \frac{1 - q^{-1}\alpha(\chi)}{1 - \alpha(\chi)}, \qquad \chi \in \hat{T}.$$

It is straightforward to check that both $C_w(\tau\gamma_t)$ and $C_w(w(\tau\gamma_t))$ have order

$$\dim \mathfrak{u}^s(1) - \dim \mathfrak{u}^s(0) \leq 0$$

at $t = 1$. Thus, they have poles exactly when $\mathfrak{u} \cap \mathfrak{m} \neq 0$. This completes the verification of (9.2a) in the Iwahori spherical case.

We prove (9.2b) on the level of affine Hecke algebras. Let us replace L by G, and suppose we have a rational automorphism n of $\hat{G}$. The action of n on the root datum of $\hat{G}$ induces an automorphism of $\mathcal{H}$, so for any $\mathcal{H}$-module M we have the twisted $\mathcal{H}$ module M^n. Let $M_{\tau,u}$ be the Kazhdan-Lusztig standard module [KL], where $u = \Phi(1 \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$. The underlying space of $M_{\tau,u}$ is the K-homology group $K(\mathcal{B}^{\tau,u})$, and the action of $\mathcal{H}$ is given by certain natural K-theoretic operations. The automorphism n of $\hat{G}$ induces a linear isomorphism (pushforward map)

$$n_* : K(\mathcal{B}^{\tau,u}) \longrightarrow K(\mathcal{B}^{n\cdot\tau, n\cdot u}).$$

One checks, using the naturality properties [KL,1.3], that n_* intertwines the $\mathcal{H}$-action so as to be an $\mathcal{H}$-module isomorphism

$$M_{\tau,u}^n \simeq M_{n\cdot\tau, n\cdot u}. \tag{9.3c}$$

The group A_Φ acts naturally on the space $\mathcal{B}^{\tau,u}$, and the induced action on $K(\mathcal{B}^{\tau,u})$ commutes with the $\mathcal{H}$-action. Thus, for $\rho \in \mathrm{Irr}(A_\Phi)$, we have an $\mathcal{H}$- module

$$V_\Phi(\rho) = \mathrm{Hom}_{A_\Phi}(\rho, M_{\tau,u}),$$

which is the simple tempered $\mathcal{H}$-module attached to (Φ,ρ) by Kazhdan-Lusztig. Again using the properties [KL,1.3], one checks that (9.3c) induces an isomorphism

$$V_\Phi(\rho)^n \simeq V_{n\cdot\Phi}(n\cdot\rho),$$

and this proves (9.2b).

We have mentioned that (9.2c) in the Iwahori-spherical case is just (6.1a). Finally, for (9.2d), consider $\gamma \otimes V_L(\sigma)$, where $\gamma \in \hat{S}$ is an unramified character of $\hat{L}$, chosen in general position. The analogous component group $A_L(\gamma)$ is unchanged, but now $A(\gamma) = A_L(\gamma)$, and both sides of (9.2d) are irreducible. By the construction in [KL], (9.2c) is the localization at γ of a projective $\mathbb{C}[\hat{S}]$ module, so that $T_r'(\gamma)$ is a polynomial function of γ, which, by irreducibility, must be a scalar $c_r(\gamma)$ times $\mathcal{A}_r(\gamma)$. The latter is known to be analytic near $\gamma = 1$, so (9.2d) follows by taking $\gamma \to 1$.

10. On certain fixed point varieties

This section is purely on the geometric side, and does not rely on conjectures. Retain the notation of 9.1, and suppose that Γ is contained in a maximal torus of $\hat{G}$. In (3.3.2) we have the special case where Γ is generated by a semisimple element $s \in \hat{G}$. Let u, $\tau = \tau_u$ be as in (5.11).

Let $\mathcal{B}, \mathcal{B}_L$ be the respective varieties of Borel subgroups of $\hat{G}$ and $\hat{L}$. We consider the varieties $\mathcal{B}^{\Gamma,\tau,u}$, $\mathcal{B}_L^{\Gamma,\tau,u}$ of Borel subgroups normalized by Γ, τ, u. These varieties are non-empty, by Borel's fixed point theorem. We want to show that, if $\sigma \in \mathrm{Irr}(A_L)$ occurs in the representation of A_L on the cohomology $H(\mathcal{B}_L^{\Gamma,\tau,u})$, and $\rho \in \mathrm{Irr}(A)$ occurs in $\mathrm{Ind}_{A_L}^{A}\sigma$, then ρ also occurs in the representation of A on the cohomology $H(\mathcal{B}^{\Gamma,\tau,u})$. We will prove a bit more:

(10.1.1)Proposition. *There is an open and closed M-stable subvariety $\mathcal{C} \subset \mathcal{B}^{\Gamma,\tau,u}$ such that*

$$H(\mathcal{C}) \simeq \mathrm{Ind}_{A_L}^{A} H(\mathcal{B}_L^{\Gamma,\tau,u}),$$

as A-modules.

Proof. Let $H = \hat{G}_{\Gamma,\tau}, H_L = \hat{L}_{\Gamma,\tau}$ denote the respective centralizers of both Γ and τ. Let X_j, j in some index set J, denote the H-orbits in $\mathcal{B}^{\Gamma,\tau}$. By [R3, (2.3)] each X_j is a disjoint union of copies of the flag variety of H°. Let $\hat{P} = \hat{L}\hat{U}$ be a parabolic subgroup of $\hat{G}$ with Levi $\hat{L}$, as in (9.1), and set $\mathcal{B}_P = \{B' \in \mathcal{B} :\ B' \subset \hat{P}\}$. Projecting to $\hat{L}$ gives an isomorphism $\mathcal{B}_P \simeq \mathcal{B}_L$. Let Y_i, $i \in I$, be the H_L-orbits in $\mathcal{B}_P^{\Gamma,\tau}$, let $J_L = \{j \in J :\ X_j \cap \mathcal{B}_P \neq \varnothing\}$, and put

$$X_L = \bigcup_{j\in J_L} X_j.$$

There is a surjective map $f : I \longrightarrow J_L$ with the properties that

$$HY_i = X_{f(i)}, \qquad X_j \cap \mathcal{B}_P = \bigcup_{i\in f^{-1}(j)} Y_i.$$

For each $j \in J$, the variety $X_j^u := X_j \cap \mathcal{B}^u$ is projective, smooth [DLP] and M-stable. Likewise, each $Y_i^u := Y_i \cap \mathcal{B}_P^u$ is smooth and M_L-stable.

The M-action defines, for each $i \in I$, a map

$$\psi_i'' : M \times Y_i^u \longrightarrow X_{f(i)}^u, \qquad \psi_i''(m, B') = mB'm^{-1}.$$

Recall from 9.1 that $B_M = \hat{S}\hat{U}^\Phi$ is a Borel subgroup of M°. Note that B_M acts trivially on $\mathcal{B}_P$. Hence ψ_i'' induces a map

$$\psi_i' : (M/B_M) \times Y_i^u \longrightarrow X_{f(i)}^u.$$

Since M_L normalizes B_M, there is a right action of M_L on M/B_M commuting with the left action of M, and the kernel of this action is exactly $Z_{\hat{L}}$. It follows that A_L acts freely on M/B_M. Meanwhile, the left action of M_L on Y_i^u also restricts trivially to $Z_{\hat{L}}$. It follows that A_L acts freely on $(M/B_M) \times Y_i^u$, commuting with the left M-action. We write

$$(M/B_M) \times_{A_L} Y_i^u$$

for the quotient variety. It follows that ψ_i' induces a map

$$\psi_i : M/B_M \times_{A_L} Y_i^u \longrightarrow X_{f(i)}^u. \tag{10.1a}$$

(10.1.2)Lemma. *For every $i \in I$, the map ψ_i in (10.1a) is an isomorphism of M-varieties.*

Before starting the proof, we note that (10.1.1) follows from (10.1.2). Indeed, the left action of M on M/B_M induces the regular representation of A on $H(M/B_M)$, so

$$H((M/B_M) \times_{A_L} Y_i^u) \simeq [H(M/B_M) \otimes H(Y_i^u)]^{A_L} \simeq \operatorname{Ind}_{A_L}^{A} H(Y_i^u),$$

and

$$H(\mathcal{B}_L^{\Gamma,\tau,u}) = \bigoplus_{i \in I} H(Y_i^u)$$

as A_L-modules. Thus

$$\mathcal{C} = X_L^u = \bigcup_{i \in I} X_{f(i)}^u$$

has the properties claimed in (10.1.1).

Fix $i \in I$. The proof of (10.1.2) will be completed in three steps. First, ψ_i is injective. Second, the connected components of $M/B_M \times_{A_L} Y_i^u$ are smooth, and all have the same dimension, equal to the (likewise common) dimension of the components of $X_{f(i)}^u$. Third, MY_i^u meets every connected component of $X_{f(i)}^u$. Then (10.1.2) follows from the theorem on invariance of domain, applied to each component.

For step one, suppose $B' \subset \hat{P}$, and that $\psi'(mB_M, B') = \psi'(m_1 B_M, l_1 B')$, for some $m, m_1 \in M$, $l_1 \in L$. Then $m_1^{-1}m \in M \cap \hat{P} = (M_L)(M \cap U)$. Write $m_1^{-1}m = l_m u_m$ accordingly. Now, $ml_m^{-1} = m_1(l_m u_m l_m^{-1}) \in m_1 B_M$. Moreover,

$$B' = l_1^{-1}m_1^{-1}mB' = l_1^{-1}l_m u_m B' = l_1^{-1}l_m B',$$

since $\hat{U}$ acts trivially on $\mathcal{B}_P$. Thus $(m_1 B_M, l_1 B') = (m l_m^{-1} B_M, l_m B')$, hence the fibers of ψ_i' are A_L-orbits, and ψ_i is injective.

For the second step, we use the methods of [DLP], viewing Y_i^u and X_i^u as generic fibers of a collapsing of a vector bundle into a pre-homogeneous vector space.

Let $\mathfrak{q}$ be the q-eigenspace of τ in the centralizer subalgebra $\mathfrak{g}_\Gamma$. It is known that H° has finitely many Ad-orbits in $\mathfrak{q}$, in particular, there is a unique dense orbit $\mathfrak{q}^\times$. Moreover, $\mathfrak{q}^\times$ contains $e := \log(u)$. Now H/H° permutes the H°-orbits, but $\mathfrak{q}^\times$ is stable under H, by its uniqueness. Analogous statements hold for $\mathfrak{q} \cap \mathfrak{l}$, and $e \in (\mathfrak{q} \cap \mathfrak{l})^\times$.

Choose $B_L \in Y_i^u$, and let $B \in \mathcal{B}_P^{\Gamma,\tau,u}$ be the corresponding Borel subgroup of $\hat{G}$. On the Lie algebra level we have $\mathfrak{b} = \mathfrak{b}_L \oplus \mathfrak{u}$. Let $\mathfrak{n}_L$ be the nilradical of $\mathfrak{b}_L$. Let $B_H = B \cap H^\circ$, $B_H^L = B_L \cap H^\circ$; these are Borel subgroups of H°, H_L°, respectively.

Now $B \in X_{f(i)}$. Let D be the component of $X_{f(i)}$ containing B. Then D is isomorphic to the flag variety of H° and the components of $X_{f(i)}$ are the H/H° translates of D. Likewise $D \cap \mathcal{B}_P$ is a union of flag varieties for H_L°. Let D_L be the component of $D \cap \mathcal{B}_P$ containing B.

We have two diagrams

$$\begin{array}{ccc} H^\circ \times_{B_H} (\mathfrak{b} \cap \mathfrak{q}) & \xrightarrow{\ \pi\ } & \mathfrak{q} \\ {\scriptstyle p}\big\downarrow & & \\ D & & \end{array} \tag{10.1b}$$

$$\begin{array}{ccc} H_L^\circ \times_{B_H^L} (\mathfrak{b}_L \cap \mathfrak{q}) & \xrightarrow{\ \pi_L\ } & \mathfrak{q} \cap \mathfrak{l} \\ {\scriptstyle p_L}\big\downarrow & & \\ D_L & & \end{array} \tag{10.1c}$$

given by $\pi(h,x) = Ad(h)x$, $p(h,x) = hBh^{-1}$, $\pi_L(h,x) = Ad(h)x$, $p_L(h,x) = hB_Lh^{-1}$. For every $x \in \mathfrak{q}$, resp. $x \in \mathfrak{q} \cap \mathfrak{l}$, we have

$$p\pi^{-1}(x) = D^x, \quad \text{resp.} \quad p_1\pi_L^{-1}(x) = D_L^x$$

where D^x, D_L^x are the fixed points of $\exp(x)$ in D, D_L. For $x \in \mathfrak{q}^\times$, the variety D^x is smooth, and all connected components have the same dimension [DLP,2.2]. If $x \in (\mathfrak{q} \cap \mathfrak{l})^\times$, an analogous statement holds for D_L^x. Now $e \in \mathfrak{q}^\times \cap (\mathfrak{q} \cap \mathfrak{l})^\times$, since $\tau = \tau_u$. Hence we can relate the dimension of D^e to the dimension of D_L^e, as follows.

We will use three exact sequences, in each of which, the second map is $ad(e)$.

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{q} \longrightarrow 0, \tag{10.1d}$$

$$0 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{h} \cap \mathfrak{l} \longrightarrow \mathfrak{q} \cap \mathfrak{l} \longrightarrow 0, \tag{10.1e}$$

$$0 \longrightarrow \mathfrak{u} \cap \mathfrak{m} \longrightarrow \mathfrak{u} \cap \mathfrak{h} \longrightarrow \mathfrak{u} \cap \mathfrak{q} \longrightarrow 0. \tag{10.1f}$$

In the following calculation, we write (V) instead of $\dim V$, for a vector space V.

$$\begin{aligned}\dim D^e &= (\mathfrak{h}) - (\mathfrak{q}) - (\mathfrak{b}\cap\mathfrak{h}) + (\mathfrak{b}\cap\mathfrak{q}) \quad \text{by } (10.1b)\\ &= (\mathfrak{m}) - (\mathfrak{b}_L\cap\mathfrak{h}) - (\mathfrak{u}\cap\mathfrak{h}) + (\mathfrak{n}_L\cap\mathfrak{q}) + (\mathfrak{u}\cap\mathfrak{q}) \quad \text{by } (10.1d)\\ &= (\bar{\mathfrak{u}}\cap\mathfrak{m}) + (\mathfrak{s}) + (\mathfrak{u}\cap\mathfrak{m}) - (\mathfrak{b}_L\cap\mathfrak{h}) - (\mathfrak{u}\cap\mathfrak{h}) + (\mathfrak{n}_L\cap\mathfrak{q}) + (\mathfrak{u}\cap\mathfrak{q})\\ &= (\bar{\mathfrak{u}}\cap\mathfrak{m}) + (\mathfrak{s}) - (\mathfrak{b}_L\cap\mathfrak{h}) + (\mathfrak{n}_L\cap\mathfrak{q}) \quad \text{by } (10.1f)\\ &= (\bar{\mathfrak{u}}\cap\mathfrak{m}) + (\mathfrak{h}\cap\mathfrak{l}) - (\mathfrak{q}\cap\mathfrak{l}) - (\mathfrak{b}_L\cap\mathfrak{h}) + (\mathfrak{b}_L\cap\mathfrak{q}) \quad \text{by } (10.1e)\\ &= \dim(M/B_M) + \dim D_L^e, \quad \text{by } (10.1c)\end{aligned}$$

This completes the proof of the second step.

For the third and final step of the proof of (10.1.2), we let $D_1, \dots, D_n$ be the components of $X_{f(i)}$. These are permuted transitively by H. For each k let B_k be the unique Borel subgroup in D_k stabilized by B_H. By [DLP, 2.2iii)], M permutes transitively the components of each D_k^u. Let C_j, C_k be components in D_j^u, D_k^u. Choose $h \in H$ so that $hD_j = D_k$. Since H preserves the dense H°-orbit $\mathfrak{q}^\times$, it follows that $H = MH^\circ$, so we may assume $h \in M$. Now hC_j is a component of D_k^u, so there is $m \in M$ such that $mhC_j = C_k$. Thus M permutes transitively the components of $X_{f(i)}^u$, and the proofs of (10.1.2), and (3.3.2), are finished.

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