FORMAL DEGREES AND L-PACKETS OF UNIPOTENT DISCRETE SERIES REPRESENTATIONS OF EXCEPTIONAL p-ADIC GROUPS.

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with an appendix by Frank Lübeck

Abstract. We calculate the formal degrees of square-integrable unipotent representations of exceptional Chevalley groups over p-adic fields. The representations can be uniquely partitioned into L-packets, so as to have the degrees proportional within a packet. The main ingredient is a theorem of Schneider and Stuhler, which gives the degree as a value of an Euler-Poincaré function. The latter is expressed in terms of restriction to parahoric subgroups, which we compute using Green polynomials, branching of Weyl group representations, and weight diagrams of modules over affine Hecke algebras. The computer program CHEVIE was involved in the more difficult cases.

0. Introduction and Statement of Results

Let F be a nonarchimedean local field of residue cardinality q. Let G be the F-rational points of a connected split simple group of adjoint type over F. An irreducible admissible representation V of G is *unipotent* if there is a parahoric subgroup $P < G$, with pro-unipotent radical U, such that the invariants of U in V contain a cuspidal unipotent representation σ of the finite reductive group P/U. Let ${\mathrm{Irr}_{\mathrm{upt}}}^2(G)$ denote the set of isomorphism classes of square-integrable unipotent representations of G.

Let $\hat{G}$ be the complex dual group of G. The conjugacy class $[x]$ of an element $x \in \hat{G}$ is *elliptic* if it meets no proper Levi subgroup of $\hat{G}$. Let $\hat{G}_x$ be the centralizer of x in $\hat{G}$, let $\hat{G}_x^\circ$ be its identity component, and let Z be the center of $\hat{G}$. Consider the finite group

$$A_x := \hat{G}_x / Z\hat{G}_x^\circ.$$

Then A_x is a subgroup of one of S_2^n, S_3, S_4, S_5, the latter three cases occuring only in exceptional groups. Let $\Psi(G)$ be the set of $\hat{G}$-conjugacy classes of pairs (x,ρ), where $[x]$ is elliptic in $\hat{G}$ and ρ is an irreducible character of A_x.

Assuming G is of exceptional type, we define, for each $(x,\rho) \in \Psi(G)$, a representation $V_{x,\rho} \in {\mathrm{Irr}_{\mathrm{upt}}}^2(G)$, in terms of an explicitly described module over an appropriate affine Hecke algebra, and we calculate the formal degree of $V_{x,\rho}$. These are tabulated in sections 10-13.

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The degree has a general form, but before writing it, we make four remarks about the correspondence

$$(x,\rho) \mapsto V_{x,\rho} \tag{0.1}$$

(i) For $G = G_2, F_4, E_6, E_7$, we prove that the map (0.1) is a bijection

$$\Psi(G) \leftrightarrow \mathrm{Irr_{upt}}^2(G),$$

and hence is a possible Langlands correspondence for square-integrable unipotent representations (cf. [B2,§10]). For E_8, I can at the moment only show that (0.1) is injective, though I believe it is bijective in this case as well. Without the adjective "square-integrable", and allowing non-elliptic conjugacy classes, such a bijection has been found for all split adjoint groups, by Lusztig [L1]. It is not known if Lusztig's bijection restricts to a square-integrable Langlands correspondence. If this were known for E_8, it would prove that (0.1) is bijective in that case as well, and then presumably (0.1) would be the restriction of Lusztig's bijection. For the other exceptional groups we can avoid this difficulty. (Added in proof: it has now been proved that (0.1) is bijective for E_8. Details will appear in joint work with Heckman and Opdam.)

(ii) The representation $V_{x,\rho}$ contains invariants under an Iwahori subgroup if and only if ρ appears in the action of A_x on the homology of the variety $\mathcal{B}^x$ of Borel subgroups of $\hat{G}$ containing x. The correspondence (0.1) for these representations was given by Kazhdan-Lusztig in [KL].

(iii) The formal degrees satisfy the relation

$$\deg(V_{x,\rho}) = \rho(1) \cdot \deg(V_{x,1}). \tag{0.2}$$

This was conjectured in [R2] and verified for small groups in [R2] and [R7]. In [Sh, 9.3], Shahidi shows how proportionality of formal degrees should follow from expected stability properties of L-packets.

(iv) The representation $V_{x,\rho}$ admits a Whittaker model if and only if $\rho = 1$ [R3]. In this case, $V_{x,1}$ contains a nice matrix coefficient coming from the Steinberg representation of the hyperspecial maximal compact subgroup. This coefficient was used by Heckman and Opdam [HO, Thm 5.1] to give a general formula for $\deg(V_{x,1})$ up to a constant independent of q (more on this below). One can deduce from [HO] that, at least for exceptional G, the degrees of $V_{x,1}$ and $V_{y,1}$ are not proportional if $[x] \neq [y]$. It follows then from (0.2) that if we define "L-packets" as

$$\Pi_x = \{V_{x,\rho} : \ \rho \in \hat{A}_x\},$$

then this is the unique way to partition the set of all of the $V_{x,\rho}$ so as to have formal degrees being proportional within a packet.

Additional properties of the correspondence (0.1) appear later in the introduction, and in section 14. These are sufficient to characterize the parametrization within a packet, i.e., the map $\rho \mapsto V_{x,\rho}$, for fixed x.

We turn now to the explicit degree formula, for which we need more notation. Let $x = su$ be the Jordan decomposition. By the Jacobson-Morozov theorem, there is a homomorphism

$$\phi_u : SL_2(\mathbb{C}) \longrightarrow \hat{G}_s$$

such that $u = \phi_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then ϕ_u maps the diagonal matrices into a maximal torus $\hat{T}$ containing s. Set

$$\tau = s\phi_u \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.$$

For a root α of $\hat{T}$ in $\hat{G}$ we let e_α denote the corresponding rational character of $\hat{T}$. Choose Haar measure to give unit mass to an Iwahori subgroup. The following is our main result.

Theorem. *Suppose G is of type $G_2, F_4, E_{6,7,8}$. Then*

$$\deg V_{x,\rho} = \frac{\rho(1)q^\nu}{|A_x||Z|} \frac{\prod'_\alpha e_\alpha(\tau) - 1}{\prod'_\alpha qe_\alpha(\tau) - 1}, \tag{0.3}$$

where ν is the number of positive roots of $\hat{G}$, and $\prod'_\alpha$ denotes the product of over those roots of $\hat{G}$ whose corresponding term is nonzero.

This is also known for classical groups of rank≤ 4 (and in those cases is easy to verify using the results herein). It has already been checked for G_2 in [R2].

Our formulation of (0.3) was inspired by the Heckman and Opdam formula [HO], valid for any split G, in the case $\rho = 1$. They assert that

$$\deg(V_{x,1}) = Cq^\nu \frac{\prod'_\alpha e_\alpha(\tau) - 1}{\prod'_\alpha qe_\alpha(\tau) - 1}, \tag{0.4}$$

where C is a nonzero rational number, independent of q, but otherwise unknown. Because we want to prove (0.2), we cannot use (0.4) directly, so we have computed $\deg(V_{x,1})$ in the same manner as the other $V_{x,\rho}$'s. Formula (0.3) shows, in the cases for which it is proved, that

$$C = \frac{1}{|A_x||Z|}. \tag{0.5}$$

In (7.2) we show that, as a power series in q, the right side of (0.3) is

$$\frac{\rho(1)|M_u^s|}{|A_x||Z|} q^{\dim \mathcal{B}^u} + \text{higher powers of } q.$$

Here M_u is the identity component of the centralizer of the image of ϕ_u. Since $x = su$ is elliptic in $\hat{G}$, it follows that M_u is a torus on which s acts by conjugation with finite fixed point set M_u^s (see (7.1)).

For exceptional groups, those u which are the unipotent parts of elliptic classes are determined by their $\dim \mathcal{B}^u$. Thus, the formal degree of V, as a rational function in q, determines the unipotent part of the Langlands parameter of V. A similar phenomenon was observed for finite Hecke algebras by Lusztig [L3], and predicted by him for affine Hecke algebras with equal parameters. Lusztig has mentioned to me that, assuming V is Iwahori-spherical, and also $x = u$, the formula (0.5) would follow, for all G, from his conjecture in [L6, (10.5)].

We verify (0.3) by computing each formal degree separately. The main tool is a special case of a theorem of Schneider and Stuhler [SS,§4], stating that if V has a nonzero invariant under the pro-unipotent radical of a special parahoric subgroup, then

$$\deg V = \sum_c (-1)^{\dim c} \frac{\dim V^{U_c}}{vol(P_c^+)}, \tag{0.6}$$

where the sum is over G-orbits of facets in the building, P_c^+ is the stabilizer of facet c, and U_c is the pro-unipotent radical of P_c^+. Actually, Schneider and Stuhler require char $F = 0$ in their general theorem, but at least for unipotent representations, formal degrees can be expressed in terms of Hecke algebra formal degrees, and for a given G, P, σ, the Hecke algebra depends only on the residue cardinality q. It follows that (0.6) holds for any F. Formula (0.6) applies to all unipotent representations of split exceptional groups, and to Iwahori-spherical representations of any reductive group.

There are about $2^{\ell+1}$ terms in the sum (0.6), where ℓ is the rank of G. However, using the duality theory for finite reductive groups developed by Alvis, Curtis and Kawanaka, (cf. [A2]) one can reduce to a sum of about $\ell+1$ terms (essentially a sum over vertices). Moreover, the "associativity of types" proved by Moy-Prasad [MP1] implies that if V contains some cuspidal unipotent (P, σ), then we only get a contribution from facets c containing the facet of P in their closure.

Thus, the main problem is to compute $\dim V^{U_c}$. We in fact compute V^{U_c} as a P_c-module. In effect, we are computing Schneider-Stuhler's formula for a pseudo-coefficient of V whose value at 1 is the right side of (0.6). The constituents of V^{U_c} are unipotent representations of P_c/U_c whose dimensions are tabulated in [C], for example.

There are two cases. First, let $\mathcal{I}$ be an Iwahori subgroup of G, and assume $V^{\mathcal{I}}$ is nonzero. Here all facets contribute to the formal degree, but there are additional results at our disposal, that lead to a recipe for computing V^{U_c}, valid for any split adjoint G. It is enough to let $q \to 1$ and determine $[V^{\mathcal{I}}]_{q=1}$ as a module over the affine Weyl group $\widetilde{W}$. Then V^{U_c} is determined by the restriction of $[V^{\mathcal{I}}]_{q=1}$ to the parahoric subgroup of $\widetilde{W}$ corresponding to P_c.

The next proposition follows from results of Lusztig and Kato. Let $\widetilde{W}_s$ be the full inverse image in $\widetilde{W}$ of the stabilizer W_s of s in W. We can view s as a character of $\widetilde{W}_s$. Let $\mathcal{B}_s^u$ be the fixed points of u in the flag manifold of $\hat{G}_s$. The group A_x acts naturally on $\mathcal{B}_s^u$. Let $H(\mathcal{B}_s^u)^\rho$ be the ρ-isotypic component in homology, with grading ignored, viewed as the Springer representation of W_s, pulled back to $\widetilde{W}_s$. The trivial representation of W_s corresponds to u regular in $\hat{G}_s$.

Proposition. *Assume $V_{x,\rho}$ has Iwahori fixed vectors. Then, up to semisimplification we have*

$$[V_{x,\rho}^{\mathcal{I}}]_{q=1} = \epsilon \otimes \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} \left[s \otimes H(\mathcal{B}_s^u)^\rho \right], \tag{0.7}$$

where ϵ is the sign character of W, pulled back to $\widetilde{W}$.

By induction, (0.7) reduces the Iwahori-spherical degree calculation to two problems.

(i) Calculate the W-modules $H(\mathcal{B}^u)^\rho$.

(ii) Determine the branching from affine Weyl groups to finite parahoric subgroups. Via Mackey theory, this boils down to branching from finite Weyl groups to subgroups arising as centralizers of elements in $\hat{T}$.

Given this information, it is in principal straightforward to compute $\dim V^{U_c}$, hence the formal degree of V, using (0.6). For $G \neq E_8$, I used the tables of [BS], [Sho], [A1], and naive computer algebra to compute the degrees. This was quite tedious, indeed unfeasible for some cases in E_8. However, these big calculations, and verification of the smaller ones, were easily performed with the computer program CHEVIE by Frank Lübeck, who describes his work in an appendix to this paper.

Assume now that V is not Iwahori spherical. This means V contains (P, σ), where P is not an Iwahori subgroup. Then $V^\sigma := \mathrm{Hom}_P(\sigma, V)$ is a module over a certain affine Hecke algebra $\mathcal{H}(G/P)$, usually having unequal parameters, which renders the previous proposition inapplicable. (The structure of this algebra depends only on P, not on σ, as one checks from the tables in [L1].) There is an intrinsic notion of square-integrable $\mathcal{H}(G/P)$ modules. We classify these (for G exceptional, P non-Iwahori) using the method of weight diagrams, developed in [R1],[R6], except that we have not proved that every square-integrable module of $\mathcal{H}(E_8/D_4)$ is one of the 18 modules on our list for that algebra. See (i) above.

The weight diagrams are given in tables, after the degrees. Given the diagram, it is a simple matter to calculate the branching, as in the Iwahori-spherical case, and find the degree using (0.6). Since the rank of $\mathcal{H}(G/P)$ is at most four (for exceptional G and non-Iwahori P), the computational difficulties of the Iwahori spherical case do not arise here.

Since there is no intrinsic definition for it, any alleged L-packet can only be convicted upon circumstantial evidence, of which the formal degrees are one piece. However, our tables also indicate a relation between Langlands parameters, and Lusztig's families of unipotent representations of finite groups, via the notion of "leading K-type". The rough idea is that a certain representation of K appearing in V should belong to a family attached to the Langlands parameter. We defer a more precise discussion until §14.

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1. Preliminaries

(1.1) Throughout the paper, F denotes a non-archimedean field, with residue field isomorphic to $\mathbb{F}_q$, and G is the group of F-rational points of a split connected simple group over F.

Let $\mathcal{I}$ be an Iwahori subgroup of G. The double cosets of $\mathcal{I}$ in G have the structure of a group $\mathcal{W}$, which is an extension of a Coxeter group $\mathcal{W}'$ by a finite abelian group Ω. Fix a set $\widetilde{\Sigma}$ of Coxeter generators of $\mathcal{W}'$. Then Ω may be viewed as the group of permutations of $\widetilde{\Sigma}$ induced by conjugation in $\mathcal{W}$, and also as the center of the Langlands dual group $\hat{G}$.

(1.2) For each facet c in the building of G, let P_c^+ be the stabilizer of c in G. We have an exact sequence

$$1 \longrightarrow U_c \longrightarrow P_c^+ \longrightarrow M_c^+ \longrightarrow 1,$$

where U_c is the pro-unipotent radical of P_c^+, and M_c^+ is the group of k-rational points in a (possibly disconnected) reductive group $\mathbf{M}_c^+$ over k. The parahoric subgroup P_c is, by definition, the inverse image in P_c^+ of the group M_c of k-rational points of the identity component $\mathbf{M}_c$ of $\mathbf{M}_c^+$. We sometimes call "parahoric subgroup" the whole triple (P_c, U_c, M_c). If c is open in the building, then P_c is an Iwahori subgroup. All Iwahori subgroups are conjugate in G.

(1.3) The G-orbits of facets in the building are in one-to-one correspondence with G-conjugacy classes of parahoric subgroups, which are in turn bijective with the Ω-orbits of proper subsets of $\widetilde{\Sigma}$. If the facet c corresponds to $J \subsetneq \widetilde{\Sigma}$, then

$$P_c^+/P_c \simeq \Omega_J,$$

where Ω_J is the stabilizer of J in Ω, and (P_c, U_c, M_c) is conjugate to the parahoric subgroup (P_J, U_J, M_J), where P_J is generated by $\mathcal{I}$ and J.

2. Euler-Poincaré functions

Our calculations of formal degrees rely on an alternating sum formula, due Schneider and Stuhler [SS], which reduces the problem to counting invariants under certain compact open subgroups of G.

(2.1) Let V be an admissible representation of G. In particular, for any compact open subgroup U of G, the space of invariants V^U is finite dimensional. For any facet c, let χ_c be the character of P_c^+ afforded by the P_c^+ module V^{U_c} (this might be zero), and let $\epsilon_c : P_c^+ \longrightarrow \pm 1$ be the orientation character of P_c^+ on c. We extend all functions on P_c^+ to be zero on the rest of G. With then define a function f_V on G by

$$f_V = \sum_c (-1)^{\dim c} \frac{\epsilon_c \bar{\chi}_c}{vol(P_c^+)},$$

where the sum runs over all G-orbits of facets in the building, and volume is computed with respect to some fixed Haar measure dg on G. This paper is based on the following theorem of Schneider and Stuhler, especially part (1).

Theorem [SS]. *Assume that $V^{U_c} \neq 0$ for some special facet c. Then*

(1) *If V is square-integrable, then $f_V(1)$ is the formal degree of V with respect to dg.*
(2) *If V is tempered but not square-integrable, then $f_V(1) = 0$.*
(3) *If V is induced from discrete series on a proper parabolic subgroup, then $f_V(1) = 0$.*

Thus, when $V^{U_c} \neq 0$ for a special facet c and V is square-integrable, the formal degree is given by

$$\deg(V) = \sum_c (-1)^{\dim c} \frac{\dim V^{U_c}}{vol(P_c^+)}. \tag{2.1a}$$

(2.2) Normalize dg so that Iwahori subgroups have volume one. We prefer to sum over subsets of simple affine roots instead of facets. Let

$$W_J(q) = \prod_m \frac{q^{m+1} - 1}{q - 1},$$

where product is over the exponents m of the (finite) Coxeter subgroup of $\mathcal{W}$ generated by J. Then $W_J(q) = vol(P_J)$, and it follows from (2.1a) that

$$\deg(V) = \frac{1}{|\Omega|} \sum_J (-1)^{\ell - |J|} \frac{\dim V^{U_J}}{W_J(q)}, \tag{2.2a}$$

where the sum is over all subsets $J \subsetneq \widetilde{\Sigma}$.

3. Duality

At first glance, there are $2^{\ell+1} - 1$ terms in the sum (2.2a), where ℓ is the rank of G. We can reduce this to about $\ell + 1$ terms, using results of Alvis [A2], as follows.

(3.1) Let H be a finite reductive group. Given a finite dimensional complex representation χ of H, we consider the virtual representation

$$\chi^* = \sum_Q (-1)^{r(H) - r(Q)} \operatorname{Ind}_Q^H(\chi^{U_Q}),$$

where Q runs over the conjugacy classes of parabolic subgroups of H, $r(\,\cdot\,)$ denotes semisimple rank, U_Q is the unipotent radical of Q, and χ^{U_Q} is the space of U_Q-fixed vectors in χ, viewed as a representation of Q. When χ is irreducible, then χ^* is again an irreducible representation of H, up to sign, and $\chi^{**} = \chi$. Applying the definition to $H = M_J$, $\chi = V^{U_J}$, we get

Lemma 1. *Fix a subset $J \subsetneq \widetilde{\Sigma}$. Then*

$$\sum_{I \subseteq J} (-1)^{|I|} \frac{\dim V^{U_I}}{W_I(q)} = \frac{\dim\, [V^{U_J}]^*}{W_J(q)}.$$

Now suppose that $J = J_1 \cup J_2$, where each root of J_1 is orthogonal to each root in J_2. Then M_J is a commuting product $M_J = M_{J_1} M_{J_2}$, and for each irreducible representation χ of M_{J_1} appearing in V^{U_J}, there is a representation (possibly reducible) χ_2 of M_{J_2} such that

$$V^{U_J} = \bigoplus_{\chi} \chi \otimes \chi_2.$$

Abusing notation, we write

$$V_{J_1} \otimes V_{J_2}^* := \bigoplus_{\chi} \chi \otimes \chi_2^*,$$

where the duality involution is applied only on the J_2-factor. Applying Lemma 1 to J_2, it is straightforward to prove

Lemma 2. *With notation as above, we have*

$$\sum_{J_1 \subseteq I \subseteq J} (-1)^{|I|} \frac{\dim V^{U_I}}{W_I(q)} = (-1)^{|J_1|} \frac{\dim(V_{J_1} \otimes V_{J_2}^*)}{W_{J_1}(q) W_{J_2}(q)}.$$

Lemma 2 can be used to successively halve the number of terms in formula (2.2a), until about $\ell + 1$ terms remain.

(3.2)Example: Let $G = Sp_{2n}(F)$, and number the affine diagram of G as

$$0 \Rightarrow 1 — 2 — \cdots — (n-1) \Leftarrow n.$$

Since G is simply-connected, we have $\Omega = 1$, so G-orbits of facets in the building correspond bijectively to proper subsets of $[0, n]$. Each $i \in [0, n]$ corresponds to a a maximal parahoric subgroup (P_i, U_i, M_i), with reductive quotient

$$M_i = Sp_{2i}(\mathbb{F}_q) \times Sp_{2(n-i)}(\mathbb{F}_q).$$

Write $V^{U_i} = \oplus \chi^i \otimes \chi_2^i$ corresponding to the factorization of M_i, where χ^i runs over irreducible representations of $Sp_{2i}(\mathbb{F}_q)$, so $V^i \otimes V_2^{i*} = \oplus \chi^i \otimes \chi_2^{i*}$.

From Lemma 2, the sum over those subsets J with $[0, i-1] \subseteq J$, but $i \notin J$ is

$$\frac{\dim[V^i \otimes V_2^{i*}]}{vol(P_i)},$$

So we have

$$\deg(V) = \sum_{i=0}^{n} \frac{\dim[V^i \otimes V_2^{i*}]}{vol(P_i)}.$$

The same sort of formula holds for F_4 (section 10). For branched diagrams we cannot halve all the way to the end, and there will be a few more terms in the degree formula (see sections 11-13).

4. Reduction to Hecke algebras

The main task arising from the Schneider-Stuhler formula (2.2a) is calculating the dimension of each V^{U_J}. We in fact compute V^{U_J} as a P_J module, and thus determine the Euler-Poincaré function f_V. This can be reduced to a problem about affine Hecke algebras, as follows.

(4.1) Fix a standard parahoric subgroup (P, U, M) with $\mathcal{I} \subseteq P$, and a cuspidal representation σ of M, not required to be unipotent at this stage. Let $\check{\sigma}$ be the contragredient representation. Let $\mathcal{H}(G/P)$ be the algebra, under convolution, consisting of all locally constant compactly supported functions f on G taking values in $\mathrm{End}(\check{\sigma})$, transforming as $f(pgp') = \check{\sigma}(p)f(g)\check{\sigma}(p')$, for $g \in G,\ p, p' \in P$.

Let P_J be a parahoric subgroup containing P, and let $\mathcal{H}(P_J/P)$ be the subalgebra of $\mathcal{H}(G/P)$ consisting of functions supported on P_J. Note that

$$\mathcal{H}(P_J/P) \simeq \mathrm{End}_{P_J}(\mathrm{Ind}_P^{P_J} \sigma).$$

If χ is a representation of P_J, we have an associated $\mathcal{H}(P_J/P)$-module

$$\chi^\sigma := \mathrm{Hom}_P(\sigma, \chi) = \mathrm{Hom}_{P_J}(\mathrm{Ind}_P^{P_J} \sigma, \chi),$$

and a decomposition as $\mathcal{H}(P_J/P) \times P_J$-modules

$$\mathrm{Ind}_P^{P_J} \sigma \simeq \bigoplus_\chi \chi^\sigma \otimes \chi,$$

where χ runs over all irreducible P_J-modules χ with $\chi^\sigma \neq 0$. Note that

$$\mathrm{Hom}_{\mathcal{H}(P_J/P)}(\mathrm{Ind}_P^{P_J} \sigma, \chi^\sigma) \simeq \chi$$

as P_J-modules.

(4.2) Likewise, if V is any admissible representation of G, we have an associated $\mathcal{H}(G/P)$-module

$$V^\sigma := \mathrm{Hom}_P(\sigma, V).$$

Now since $U_J \subseteq U$, we have (letting $\langle\ ,\ \rangle$ denote multiplicities)

$$\begin{aligned} V^\sigma = \mathrm{Hom}_P(\sigma, V) = \mathrm{Hom}_P(\sigma, V^{U_J}) &\simeq \mathrm{Hom}_{P_J}(\mathrm{Ind}_P^{P_J} \sigma, V^{U_J}) \\ &\simeq \bigoplus_\chi \langle \chi, V^{U_J} \rangle_{P_J} \mathrm{Hom}_{P_J}(\mathrm{Ind}_P^{P_J} \sigma, \chi) \\ &\simeq \bigoplus_\chi \langle \chi, V^{U_J} \rangle_{P_J} \chi^\sigma. \end{aligned}$$

This proves

Lemma. *The multiplicity of the simple P_J-module χ in V^{U_J} is given by*

$$\langle \chi, V^{U_J} \rangle_{P_J} = \langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(P_J/P)}.$$

5. Unipotent representations

What has been said so far applies to any admissible representation with vectors fixed under the pro-unipotent radical of some parahoric subgroup. Among these are the unipotent representations, for which we have further simplifications.

(5.1) We say an irreducible representation V of G is *unipotent* if there is a parahoric subgroup (P, U, M) such that V^U contains a cuspidal unipotent representation of M.

We can reduce number of terms in the degree formula further using "associativity of types" proved in [MP]. A similar result holds for non-unipotent representations, but it is more complicated.

Lemma. *Suppose $I \subsetneq \widetilde{\Sigma}$, and that V^{U_I} contains the cuspidal unipotent representation σ of M_I. If $J \subseteq \widetilde{\Sigma}$, with $V^{U_J} \neq 0$, and J is minimal with respect to this property, then there is $\omega \in \Omega$ such that $I = \omega J$, and V^{U_J} consists of copies of σ^ω. Moreover, if G is exceptional, then $J = I$.*

Proof. The minimality of J forces V^{U_J} to consist of irreducible cuspidal representations of M_J. Let σ' be one of them. By the proof of [MP2,6.2], there is an isomorphism between the algebraic groups underlying M_I and M_J under which σ corresponds to σ'. In particular, σ' is unipotent. The classification of unipotent representations drastically restricts the possibilities for J, and considering each simple group, together with the action of Ω on the affine diagram [IM, 1.8], we find ω sending J to I. For exceptional groups, each I for which M_I admits a cuspidal unipotent representation is the unique subdiagram of its type, so $J = I$ in those cases. The fact that V^{U_J} is isotypic is a special case of [MP1, 5.2]. □

(5.2) Let (P, U, M) be a standard parahoric subgroup, say $P = P_I$, and let σ be a cuspidal unipotent representation of M. The Hecke algebra $\mathcal{H}(G/P)$ has the following structure [L1],[M]. There is a semisimple complex Lie group $\mathbf{G}$, with maximal torus $\mathbf{T}$, character group $X = \mathrm{Hom}(\mathbf{T}, \mathbb{C}^\times)$, Weyl group W, and affine Weyl group $\widetilde{W} = W \ltimes X$, such that $\mathcal{H}(G/P)$ has a linear basis $\{T_w : \ w \in \widetilde{W}\}$. The multiplication is as follows. The group $\widetilde{W}$ acts on X by affine motions, and the subgroup of $\widetilde{W}$ generated by affine reflections is a Coxeter group W' whose generators may be identified with $\tilde{S} := \tilde{\Sigma} - I$, and we have $\widetilde{W} \simeq \Omega_I \ltimes W'$, where Ω_I is the stabilizer of I in Ω (see (1.1)). Extend the length function on W' trivially across Ω_I. Then $T_x T_y = T_{xy}$ if the length of xy is the sum of those of x and y, and for $r \in \tilde{S}$, we have

$$(T_r + 1)(T_r - q^{c(r)}) = 0,$$

where $c(r)$ is a certain positive integer. The groups $\mathbf{G}$ and parameters $c(r)$ are tabulated in [L1]. They depend only on P and not on σ.

If $I \subseteq J \subsetneq \tilde{\Sigma}$, let $\widetilde{W}_J$ be the subgroup of $\widetilde{W}$ generated by $J - I$. Then the subalgebra $\mathcal{H}(P_J/P)$ is the span of $\{T_w : \ w \in \widetilde{W}_J\}$.

More details on the structure of $\mathcal{H}(G/P)$ will be given in (9.1) as they are needed.

6. Reduction to Weyl groups

In (4.2) we have reduced the calculation of V^{U_J} to a similar problem for Hecke algebras. In this section, we reduce this further to standard problems about Weyl group representations.

(6.1) Let v be an indeterminate, and let R be the ring of rational functions $f \in \mathbb{C}(v, v^{-1})$ which are holomorphic on the set of positive real numbers. Let $\mathcal{H}(G/P)_v$ be the Hecke algebra defined over R, with the same generators and relations as $\mathcal{H}(G/P)$, but q replaced by v^2. By evaluating v we have specializations

$$\mathcal{H}(G/P)_{\sqrt{q}} = \mathcal{H}(G/P), \qquad \mathcal{H}(G/P)_1 = \mathbb{C}\widetilde{W},$$

this last being the group algebra of $\widetilde{W}$.

For any simple $\mathcal{H}(G/P)$-module E considered in this paper, there exists an $\mathcal{H}(G/P)_v$-module E_v such that $E \simeq E_v \otimes_R \mathbb{C}$, where $f \in R$ acts on $\mathbb{C}$ by $f(\sqrt{q})$. This holds for P an Iwahori subgroup, by the construction of simple tempered $\mathcal{H}(G/P)$- modules given in [KL]. It also holds for the square-integrable $\mathcal{H}(G/P)$-modules considered in this paper, by the matrix formulas in [R6], applied to the diagrams given in §9-13. A similar assertion holds for all simple modules over $\mathcal{H}(P_J/P)$, for any $I \subseteq J \subsetneq \tilde{\Sigma}$ (cf. [C,10.11.4]).

It follows that we have an operation "$q = 1$", taking simple modules over Hecke algebras to modules over the corresponding Weyl groups, obtained by setting $v = 1$ in all matrix coefficients of the generic module. It is clear that this process commutes with restriction. That is,

$$[E_{\mathcal{H}(P_J/P)}]_{q=1} \simeq [E_{q=1}]_{\widetilde{W}_J}.$$

It also follows from [C,10.11.4] that the map

$$\mathcal{H}(P_J/P) - \text{Mods} \xrightarrow{q=1} \widetilde{W}_J - \text{Mods}$$

is an isometry, i.e., for any irreducible P_J module χ, we have

$$\langle \chi^\sigma, V^\sigma \rangle_{\mathcal{H}(P_J/P)} = \langle \chi^\sigma_{q=1}, V^\sigma_{q=1} \rangle_{\widetilde{W}_J}. \tag{6.1a}$$

By Lemma (4.2), the decomposition of the M_J-module V^{U_J} is therefore determined by decomposing the restriction of the $\widetilde{W}$-module $V^\sigma_{q=1}$ to $\widetilde{W}_J$. It is enough to replace $V^\sigma_{q=1}$ by its semisimplification, since $\widetilde{W}_J$ is a finite group.

(6.2) The irreducible representations of $\widetilde{W}$ have the form

$$\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} s \otimes \chi,$$

where $s \in \mathbf{T}$, $\widetilde{W}_s$ is the stabilizer of s in $\widetilde{W}$, s is extended to a character of $\widetilde{W}_s$ by $s(xw) = s(x)$ for $x \in X, w \in W_s$, and χ is an irreducible representation of W_s, extended trivially to X. Moreover, by Mackey theory, The restriction of such an irreducible representation to $\widetilde{W}_J$ is given by

$$[\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} s \otimes \chi]\big|_{\widetilde{W}_J} \simeq \bigoplus_{w \in W_s \backslash W / W_J} \mathrm{Ind}_{\widetilde{W}_s^w \cap \widetilde{W}_J}^{\widetilde{W}_J} s^w \otimes \chi^w,$$

where W_J is the image of $\widetilde{W}_J$ in W under the natural projection $\widetilde{W} \longrightarrow W$.

7. Elliptic Conjugacy Classes in the Dual Group

The calculation of the preceeding Weyl group representations (at least in the Iwahori-spherical case), along with subsequent connections between formal degrees and Langlands parameters, are based upon the Springer correspondence and the properties of elliptic conjugacy classes in the dual group. We summarize the latter, and relate them to the Heckman-Opdam formula for generic formal degrees.

(7.1) Let $\hat{G}$ be a simply-connected complex semisimple Lie group, with center Z, and Lie algebra $\mathfrak{g}$. Fix a maximal torus $\hat{T}$ in $\hat{G}$. Let Δ denote the roots of $\hat{T}$ in $\hat{G}$, and let e_α be the rational character of $\hat{T}$ corresponding to $\alpha \in \Delta$. Denote the centralizer of any element $g \in \hat{G}$ by $\hat{G}_g$.

Following [A, §7], we say that an element $x \in \hat{G}$ or its conjugacy class is *elliptic* if x is contained in no proper Levi subgroup of $\hat{G}$. Equivalently, if $x = su$ is the Jordan decomposition, then $\hat{G}_s$ is semisimple, and the unipotent u belongs to no proper Levi subgroup of $\hat{G}_s$. There are only finitely many elliptic conjugacy classes in $\hat{G}$.

Let $\phi : SL_2(\mathbb{C}) \longrightarrow \hat{G}_s$ be a homomorphism with $\phi\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = u$. We can arrange that ϕ maps the diagonal matrices into $\hat{T}$. Let M be the centralizer of the image of ϕ, with identity component M_0, and let $\mathfrak{m}$ be the Lie algebra of M. Let A_u denote the component group of the centralizer of u in $\hat{G}/Z$. The inclusion $M \hookrightarrow \hat{G}_u$ induces an isomorphism $M/ZM_0 \simeq A_u$, and we identify these two groups.

Lemma. *Assume that $x = su$ is elliptic in $\hat{G}$. Then*

(1) *M_0 is a torus, hence the adjoint action of M on $\mathfrak{m}$ factors through A_u.*
(2) *There are no nonzero invariants of $Ad(s)$ in $\mathfrak{m}$. Hence, s has nontrivial image in A_u unless $\mathfrak{m} = 0$ and $s \in Z$.*
(3) *Taking fixed points under conjugation by s, we have $M^s/(M_0)^s \simeq (M/M_0)^s$.*
(4) $\det(1 - Ad(s)_{\mathfrak{m}}) = |M_0^s| =$ *the number of fixed points of s in M_0.*

Proof. The Lie algebra $\mathfrak{m}$ is reductive, and $Ad(s)\big|_{\mathfrak{m}}$ is a semisimple automorphism, so the fixed point algebra $\mathfrak{m}^s$ is also reductive. Suppose $z \in \mathfrak{m}^s$ is semisimple. Then the centralizer in $\mathfrak{g}$ of z is a proper Levi subgroup containing s and u, a contradiction. It follows that $\mathfrak{m}^s = 0$. Results of Steinberg [St, 10.13] imply that a nonzero semisimple Lie algebra must have nonzero invariants under a semisimple automorphism. Hence $[\mathfrak{m}, \mathfrak{m}] = 0$, proving (1). Assertion (2) follows from (1) and its proof. By (2), there are only finitely many fixed points of s in M_0. This shows that the map $t \mapsto tst^{-1}s^{-1}$ is an isogeny from the torus M_0 to itself, which then implies that $(M/M_0)^s = M^s/(M_0)^s$. Finally, since $(M_0)^s$ is a finite group, all fixed points of s must lie in the maximal compact subgroup of M_0, to which we apply the Lefschetz fixed point theorem to get (4). $\square$

(7.2) In the next result, we investigate the main term in the formal degree formula. According to Heckman and Opdam, the left side in Proposition (7.2) below is a nonzero rational multiple (independent of q) of the formal degree of the generic member of the L-packet. According to our conjecture, it should be similarly proportional to all formal degrees in the packet. The point of Proposition (7.2) is to show how the Langlands parameters appear in these formal degrees.

Set

$$\tau = s\phi \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \in \hat{T}.$$

Let Δ be the roots of $\hat{T}$ in $\hat{G}$, let ν be the number of positive roots of $\hat{G}$, and for each $\alpha \in \Delta$ let e_α be the corresponding character of $\hat{T}$. Let ℓ be the rank of $\hat{G}$.

Proposition. *Assume that $x = su$ is elliptic in $\hat{G}$. Then*

$$q^\nu \frac{\prod'_\alpha e_\alpha(\tau) - 1}{\prod'_\alpha q e_\alpha(\tau) - 1} = q^{\dim \mathcal{B}_u} \ (q-1)^\ell \ |M_0^s| \ R(q),$$

where $R(q)$ is a rational function of q with $R(0) = 1$, $R(1) \neq 0$, and $\prod'_\alpha$ denotes product over the roots in $\hat{G}$ whose term is nonzero.

Proof. Let $\mathfrak{g}_i$ be the $q^{-i/2}$ -eigenspace of $Ad\phi(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix})$. The image of ϕ is isomorphic to $PSL_2(\mathbb{C})$ by [C,5.7.6], hence $\mathfrak{g}_i = 0$ for odd i. Let $\mathfrak{p}$, respectively $\mathfrak{n}$, be the sum of the $\mathfrak{g}_i$ for $i \geq 0$, respectively $i \geq 2$. Then $\mathfrak{p}$ is the Lie algebra of a parabolic subgroup P, and $\mathfrak{n}$ is the Lie algebra of the unipotent radical N of P. Choose a Borel subgroup $B \supset \hat{T}$ such that $N \subset B \subset P$. Let $\mathcal{O}$ be the conjugacy class of u in $\hat{G}$, and let $\overline{\mathcal{O}}$ be its closure. We have maps

$$\pi_B : \hat{G} \times_B N \longrightarrow \overline{\mathcal{O}}, \quad \pi_P : \hat{G} \times_P N \longrightarrow \overline{\mathcal{O}},$$

induced by conjugation. The generic fibers of π_B (ie., over $\mathcal{O}$) are naturally isomorphic to $\mathcal{B}^u$, and the generic fibers of π_P are single points [H]. These two facts give the dimension relations

$$\dim(\mathcal{B}^u) + \dim \mathcal{O} = \dim \hat{G} - \dim B + \dim N,$$

$$\dim \mathcal{O} = \dim \hat{G} - \dim P + \dim N = 2 \dim N,$$

which imply $\nu - \dim \mathcal{B}_u = \dim N$.

For any root of unity ϵ and integer i, set

$$\Delta(\epsilon, i) = \{\alpha \in \Delta : \ e_\alpha(\tau) = \epsilon q^{-i/2}\}.$$

Since u is elliptic in $\hat{G}_s$, it follows (cf [C,5.7.4)]) that

$$\ell + |\Delta(1,0)| = |\Delta(1,2)|. \tag{7.2a}$$

Thus

$$\begin{aligned} q^{\nu - \dim \mathcal{B}_u} & \frac{\prod'_\alpha e_\alpha(\tau) - 1}{\prod'_\alpha q e_\alpha(\tau) - 1} \\ &= q^{\dim N} \prod_{(\epsilon,i) \neq (1,0)} (\epsilon q^{-i/2} - 1)^{|\Delta(\epsilon,i)|} \cdot \prod_{(\epsilon,i)\neq(1,2)} (\epsilon q^{1-i/2} - 1)^{-|\Delta(\epsilon,i)|} \\ &= q^{\dim N} \frac{(q^{-1}-1)^{|\Delta(1,2)|}}{(q-1)^{|\Delta(1,0)|}} \prod_{(\epsilon,i)\neq(1,0),(1,2)} \left(\frac{\epsilon q^{-i/2} - 1}{\epsilon q^{1-i/2} - 1}\right)^{|\Delta(\epsilon,i)|} \\ &= (1-q)^\ell q^{\dim N - |\Delta(1,2)|} \prod_{(\epsilon,i)\neq(1,0),(1,2)} \left(\frac{\epsilon q^{-i/2} - 1}{\epsilon q^{1-i/2} - 1}\right)^{|\Delta(\epsilon,i)|}. \end{aligned} \tag{7.2b}$$

In the last line, we used (7.2a) and the fact that $|\Delta(1,0)|$ is even.

Now take the limit as $q \to 0$. If $i < 0$, all (ϵ, i) terms in the product go to 1. For $i = 0, \epsilon \neq 1$, the (ϵ, i)-term goes to $1 - \epsilon$. As for the remaining terms, we note that $\dim N$ is the sum of $|\Delta(\epsilon, i)|$ over all $i \geq 2$ and all ϵ. Thus

$$q^{\dim N - |\Delta(1,2)|} \prod_{\substack{i \geq 2 \\ (\epsilon,i) \neq (1,2)}} \left(\frac{\epsilon q^{-i/2} - 1}{\epsilon q^{1-i/2} - 1}\right)^{|\Delta(\epsilon,i)|} = \prod_{\substack{i \geq 2 \\ (\epsilon,i) \neq (1,2)}} \left(\frac{\epsilon - q^{i/2}}{\epsilon - q^{i/2-1}}\right)^{|\Delta(\epsilon,i)|}$$
$$\longrightarrow \prod_{\epsilon \neq 1} \left(\frac{\epsilon}{\epsilon - 1}\right)^{|\Delta(\epsilon,2)|}.$$

We have an M-equivariant exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{g}_0 \xrightarrow{ad(\log u)} \mathfrak{g}_2 \longrightarrow 0,$$

which remains exact modulo $Ad(s)$-invariants. Since s belongs to the closed subgroup of $\hat{G}$ with Lie algebra $\mathfrak{g}_0$, it follows that

$$\prod_{\epsilon \neq 1} \epsilon^{|\Delta(\epsilon,2)|} = \det Ad(s)_{\mathfrak{g}_2} = \det Ad(s)_{\mathfrak{g}_0} / \det Ad(s)_{\mathfrak{m}} = \det Ad(s^{-1})_{\mathfrak{m}},$$

because $Ad(s)_{\mathfrak{g}_0} = 1$ since $\mathfrak{g}_0$ is reductive, and s belongs to the corresponding Levi subgroup of G. Likewise,

$$\prod_{\epsilon \neq 1} (\epsilon - 1)^{|\Delta(\epsilon,0)| - |\Delta(\epsilon,2)|} = \det[Ad(s)_{\mathfrak{m}} - 1].$$

Thus,

$$\prod_{(\epsilon,i) \neq (1,0),(1,2)} \left(\frac{\epsilon q^{-i/2} - 1}{\epsilon q^{1-i/2} - 1}\right)^{|\Delta(\epsilon,i)|} \longrightarrow (-1)^{\dim(\mathfrak{g}_0/\mathfrak{g}_s)} \det[1 - Ad(s^{-1})]_{\mathfrak{m}}.$$

Note that $\dim(\mathfrak{g}_0/\mathfrak{g}_0^s)$ is even, since $\mathfrak{g}_0^s$ contains a Cartan subalgebra of $\mathfrak{g}_0$, and we can replace s^{-1} by s, since the groups A_u are products of symmetric groups, in which every element is conjugate to its inverse. By Lemma (7.1) the determinant expression is $|M_0^s|$. Finally, (7.2b) also implies that $R(1) \neq 0$. □

8. The Iwahori-spherical case

In this section we assume that $P = \mathcal{I}$ is an Iwahori subgroup, and σ is the trivial representation of $\mathcal{I}$. Let $\mathcal{H} = \mathcal{H}(G/\mathcal{I})$ be the affine Hecke algebra. Since G is split, we have constant parameters $c(r) \equiv 1$ (see (5.2)). The results in this section apply to any affine Hecke algebra with constant parameters and simply-connected root datum. We describe a general method to explicitly calculate the restrictions of tempered $\mathcal{H}$-modules to parahoric subalgebras, as in §6. The basic principles are difficult geometric results due to Lusztig, but these can be put into a form suitable for effective computations, so that, armed with the Schneider-Stuhler formula (2.2a), one can, in principle, calculate the formal degree of any Iwahori spherical representation of a split adjoint p-adic group.

(8.1) The groups $\mathbf{G}$, $\mathbf{T}$ of (5.2) are now the dual group $\hat{G}$ and its maximal torus $\hat{T}$, respectively. Fix a Borel subgroup $\hat{T} \subseteq \hat{B}$. Then X is the character group of $\hat{T}$, W is the Weyl group of $\hat{T}$, and $\widetilde{W} = WX$ is the (extended) affine Weyl group $\mathcal{W}$ of (1.1). For $\lambda \in X$, we write t_λ when thinking of it as an element of $\widetilde{W}$, and e_λ when viewing it as a character of $\hat{T}$. Let α_0 be the highest short root of $\hat{T}$ in $\hat{B}$, let s_0 be its corresponding reflection in W, and let $\tilde{s}_0 = t_{\alpha_0} s_0$. Let $\alpha_1, \dots, \alpha_\ell$ be the simple roots of $\hat{T}$ in $\hat{B}$. Each $J \subsetneq \{\tilde{s}_0, s_{\alpha_1}, \dots, s_{\alpha_\ell}\}$ generates a finite subgroup $\widetilde{W}_J \subset \widetilde{W}$, which represents the $\mathcal{I}$-double cosets in the parahoric subgroup P_J (see (1.3)).

Let $x = su \in \hat{G}$, where s is semisimple, u is unipotent, and $su = us$. We assume $s \in \hat{T}$, and $|e_\lambda(s)| = 1$ for all $\lambda \in X$. The centralizer $\hat{G}_s$ is connected since $\hat{G}$ is simply connected, and its Weyl group is W_s= stabilizer in W of s. We set $\widetilde{W}_s = W_s X \subseteq \widetilde{W}$. Let $\mathcal{B}$ and $\mathcal{B}_s$ be the flag varieties of $\hat{G}$ and $\hat{G}_s$, respectively. We have

$$\mathcal{B}^s = \coprod_{W_s \backslash W} \hat{G}_s w \hat{B}, \qquad \mathcal{B}^x = \coprod_{W_s \backslash W} (\hat{G}_s w B)^u,$$

and each $\hat{G}_s w B \simeq \mathcal{B}_s$, as $\hat{G}_s$-varieties. In particular, $(\hat{G}_s w B)^u \simeq \mathcal{B}_s^u$; these are the connected components of $\mathcal{B}^x$. According to Kato, the singular cohomology spaces $H(\mathcal{B}_s^u)$ and $H(\mathcal{B}^x)$ afford actions of $\widetilde{W}_s \times A_x$ and $\widetilde{W} \times A_x$, respectively, where $A_x = \hat{G}_x / Z\hat{G}_x^\circ$. (Note that $\hat{G}_x \subseteq \hat{G}_s$.) This action extends the Lusztig-Springer action of W_s and W. For each irreducible representation ρ of A_x, we have a $\widetilde{W}$-module (possibly zero)

$$H(\mathcal{B}^x)^\rho = \mathrm{Hom}_{A_x}(\rho, H(\mathcal{B}^x)),$$

and a similarly defined $\widetilde{W}_s$-module $H(\mathcal{B}_s^u)^\rho$. Kato proved

Proposition [K]. *The restriction map $H(\mathcal{B}^x)^\rho \longrightarrow H(\mathcal{B}_s^u)^\rho$ is $\widetilde{W}_s$-equivariant, and induces an isomorphism of $\widetilde{W}_s$-modules*

$$H(\mathcal{B}^x)^\rho \simeq \mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} H(\mathcal{B}_s^u)^\rho, \tag{8.1a}$$

where X acts on $H(\mathcal{B}_s^u)^\rho$ as $s \cdot I$, and W_s acts by the Springer representation.

As in §7, let $\phi : SL_2(\mathbb{C}) \longrightarrow \hat{G}_s$ be a homomorphism such that $u = \phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and set $\tau = s\phi\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \in \hat{T}$. Since this is the canonical polar decomposition of τ, we have $W_\tau \subseteq W_s$. We put

$$\mathcal{B}^{\tau,u} = \mathcal{B}^\tau \cap \mathcal{B}^u, \qquad \mathcal{B}^{\tau,u}_s = \mathcal{B}^\tau_s \cap \mathcal{B}^u$$

Since $A_x = M^s/Z$, we have natural actions of A_x on $\mathcal{B}^{\tau,u}$ and $\mathcal{B}^{\tau,u}_s$.

The simple tempered $\mathcal{H}$-module $V^{\mathcal{I}}_{x,\rho}$ was constructed by Kazhdan-Lusztig in [KL]. They defined an $\mathcal{H} \times A_x$-module action on $K(\mathcal{B}^{\tau,u})$, where K stands for a certain equivariant K-homology functor, whose precise definition is not important here, and

$$V^{\mathcal{I}}_{x,\rho} = \mathrm{Hom}_{A_x}(\rho, K(\mathcal{B}^{\tau,u})).$$

We abbreviate the right side as $K(\mathcal{B}^{\tau,u})^\rho$. The action of X on $K(\mathcal{B}^{\tau,u})$ has the following form: Each component of $\mathcal{B}^{\tau,u}$ meets $\hat{G}_\tau w \hat{B}$ for a unique $w \in W_\tau \backslash W$, and the action of $\lambda \in X$ on the corresponding summand of $K(\mathcal{B}^{\tau,u})$ is by the scalar $e_{w\lambda}(\tau)$ times a unipotent transformation.

Now $\mathcal{B}^{\tau,u}_s$ is a union of connected components of $\mathcal{B}^{\tau,u}$, corresponding to cosets in $W_\tau \backslash W_s$, so we have an injection of $X \times A_x$-modules

$$K(\mathcal{B}^{\tau,u}_s) \longrightarrow K(\mathcal{B}^{\tau,u}), \tag{8.1b}$$

by which we identify $K(\mathcal{B}^{\tau,u}_s)^\rho$ with an X-stable summand of $K(\mathcal{B}^{\tau,u})^\rho$.

Now let $q \to 1$, obtaining a $\widetilde{W} \times A_x$-module

$$K(\mathcal{B}^{\tau,u})_{q=1} = \bigoplus_\rho \rho \otimes K(\mathcal{B}^{\tau,u})^\rho_{q=1}.$$

The X-action on the w^{th} summand of $K(\mathcal{B}^{\tau,u})_{q=1}$ is now by the scalar $e_{w\lambda}(s) = s^w(\lambda)$, times a unipotent transformation.

The following description of the W-action was stated in [L6] and proved in [L5].

Theorem [Lusztig]. *We have a W-module isomorphism*

$$K(\mathcal{B}^{\tau,u})^\rho_{q=1} \simeq \epsilon \otimes H(\mathcal{B}^x)^\rho,$$

where the action of W on $H(\mathcal{B}^x)$ is the Lusztig-Springer action recalled above, and ϵ is the sign character of W.

Let $V^1_{x,\rho}$ be the semisimplification of the $\widetilde{W}$-module $K(\mathcal{B}^{\tau,u})^\rho_{q=1}$. Applying Lusztig's theorem to W_s and $K(\mathcal{B}^{\tau,u}_s)^\rho$, we see that $V^1_{x,\rho}$ has a $\widetilde{W}_s$-stable subspace $U_{s,\rho}$ isomorphic to $\epsilon \otimes H(\mathcal{B}^u_s)^\rho$ as a W_s module, on which X acts as $s \cdot Id$. Since $\mathrm{Ind}^W_{W_s} H(\mathcal{B}^u_s)^\rho = H(\mathcal{B}^x)^\rho$, it follows that the W-translates of $U_{s,\rho}$ span $V^1_{x,\rho}$. Thus we have

Corollary. *After letting $q \to 1$ and then taking semisimplification, the simple $\mathcal{H}$-module $V^{\mathcal{I}}_{x,\rho}$ becomes Kato's $\widetilde{W}$-module $\epsilon \otimes H(\mathcal{B}^x)^\rho$.*

(8.2) By (6.2), the restriction of $V_{x,\rho}$ to P_J can thus be computed as follows. Let $\widetilde{W}_J$ be the (finite) subgroup of $\widetilde{W}$ generated by the reflections from J. By the previous corollary, it suffices to calculate the restriction

$$H(\mathcal{B}^x)^\rho|_{\widetilde{W}_J}.$$

We find it convenient to work inside W. Let $W_J \subset W$ be the image of $\widetilde{W}_J$ under the natural projection of $\widetilde{W}$ to W. This map is injective on $\widetilde{W}_J$, and we let $\psi_J : W_J \longrightarrow \widetilde{W}_J$ be its inverse. Thus $\psi_J(s_\alpha) = s_\alpha$ if $\alpha \in J$, $\alpha \neq -\alpha_0$, and $\psi_J(s_0) = \tilde{s}_0 = t_{\alpha_0} s_0$ if $\tilde{s}_0 \in J$. Let ψ_J^* be the pullback of representations of $\widetilde{W}_J$ to those of W_J.

For ς in the W-orbit of s, we set $W_{J,\varsigma} = W_\varsigma \cap W_J$ and define the character

$$\chi_\varsigma^J := \chi_\varsigma \circ \psi_J : W_{J,\varsigma} \longrightarrow \mathbb{C}^\times .$$

Then by Proposition (8.1) and Mackey theory we have

$$\psi_J^*[H(\mathcal{B}^x)^\rho|_{\widetilde{W}_J}] \simeq \bigoplus_{w \in W_s \backslash W / W_J} \mathrm{Ind}_{W_{J,s^w}}^{W_J} \chi_{s^w}^J \otimes [H(\mathcal{B}_s^u)^\rho]^w. \tag{8.2a}$$

To avert possible confusion: The labellings of representations of W_J will sometimes depend on a choice of which roots in J are to be "long". The correct choice of root-lengths in (8.2a) is made by viewing J as a subdiagram of the extended Dynkin diagram of the p-adic group G. In particular, the reflection s_0 is regarded as "long".

(8.3) To make effective use of (8.2a), we must calculate the groups $W_{J,\varsigma}$, the characters χ_ς^J, and the $W_{J,\varsigma}$-action on the homology.

The Coxeter group W_J is the Weyl group of a connected reductive group $\hat{G}_J$, defined as follows. Identify J with the corresponding subset of the simple affine roots $\{-\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$. Let $\check{J}$ be the corresponding co-roots, let $\Phi_J = W_J J$, $\check{\Phi}_J = W_J \check{J}$. Let $\check{X}$ be the co-root lattice of $\hat{G}$. Then the root datum of $\hat{G}_J$ is $(X, \check{X}, \Phi_J, \check{\Phi}_J)$. If $\hat{G}$ is simply-laced, then $\hat{G}_J$ is just the subgroup of $\hat{G}$ generated by $\hat{T}$ and the roots in $\pm J$.

By the theory of centralizers in reductive groups, we have a semidirect product

$$W_{J,\varsigma} \simeq W_{J,\varsigma}^\circ \ltimes R_{J,\varsigma},$$

where $W_{J,\varsigma}^\circ$ is generated by reflections about roots of $\hat{G}_J$ which are trivial on ς, and $R_{J,\varsigma}$ is the stabilizer in $W_{J,\varsigma}$ of some system of positive roots for the centralizer of ς in $\hat{G}_J$. Let $\hat{G}_J^{sc}$ be the simply-connected cover of $\hat{G}_J$, and let Z_J be the kernel of the covering map

$$\hat{G}_J^{sc} \longrightarrow \hat{G}_J.$$

Choose a lift ς' of ς in $\hat{G}_J^{sc}$. If $w \in W_{J,\varsigma}$, then $(\varsigma')^w = \varsigma' \xi_\varsigma^J(w)$, for some $\xi_\varsigma^J(w) \in Z_J$. This defines a homomorphism

$$\xi_\varsigma^J : W_{J,\varsigma} \longrightarrow Z_J$$

whose kernel is exactly $W_{J,\varsigma}^\circ$, and whose image is therefore isomorphic to $R_{J,\varsigma}$.

This leads to an explicit formula for χ_ς^J. We assume $-\alpha_0 \in J$, since otherwise χ_ς^J is trivial. Let $J_0 = J - \{-\alpha_0\}$. Any $w \in W_J$ may be expressed as

$$w = x_1 s_0 x_2 s_0 \cdots x_n s_0 x_{n+1},$$

where $x_i \in W_{J_0}$. Then $\psi_J(w) = t_\mu w$, where $t_\mu \in \widetilde{W}$ is translation by

$$\mu = x_1\alpha_0 + x_1 s_0 x_2 \alpha_0 + \cdots + (x_1 s_0 \cdots s_0 x_n)\alpha_0.$$

Thus

$$\chi_\varsigma^J(w) = \prod_{i=1}^n e_{\alpha_0}(\varsigma^{w_k}), \tag{8.3a}$$

where $w_k = x_1 s_0 x_2 \cdots s_0 x_k$.

This may be expressed canonically as follows. Let λ_0 be the fundamental dominant weight of $\hat{G}_J^{sc}$, with respect to the basis J, of the simple root $-\alpha_0 \in J$. Then one checks that (8.3a) may be written as

$$\chi_\varsigma^J(w) = e_{\lambda_0} \circ \xi_\varsigma^J(w). \tag{8.3b}$$

In particular, we have the

Lemma. *The character χ_ς^J is trivial on $W_{J,\varsigma}^\circ$. If the orders of s and Z_J are relatively prime, then χ_ς^J is trivial on $W_{J,\varsigma}$ for all $\varsigma \in W_s\backslash W/W_J$.*

(8.4) We explicate (8.2a) in two extreme cases. First suppose $s = 1$. Then χ_ς^J is always trivial, and we simply restrict from W to W_J:

$$\psi_J^*[H(\mathcal{B}^x)^\rho|_{\widetilde{W}_J}] = H(\mathcal{B}^u)^\rho|_{W_J}. \tag{8.4a}$$

The other extreme is when x is regular in $\hat{G}$. This means u is regular in $\hat{G}_s$, so the variety $\mathcal{B}_s^u$ is a single point. Only $\rho = 1$ appears in the homology of this point, and $H(\mathcal{B}_s^u)$ is just the trivial representation of $\widetilde{W}_s$. Thus,

$$\psi_J^*[H(\mathcal{B}^x)|_{\widetilde{W}_J}] = \bigoplus_{\varsigma \in W_s\backslash W/W_J} \mathrm{Ind}_{W_{J,\varsigma}}^{W_J} \chi_\varsigma^J. \tag{8.4b}$$

(8.5) We digress to point out that (8.4b) may be interpreted as a restriction to hyperspecial maximal compacts of endoscopic subgroups (no longer adjoint) as follows. For $\varsigma \in W_s\backslash W/W_J$, let $\hat{G}_{J,\varsigma}$ be the centralizer in $\hat{G}_J$ of ς. Choose a homomorphism

$$\phi_\varsigma : SL_2(\mathbb{C}) \longrightarrow \hat{G}_{J,\varsigma}$$

whose image contains a regular unipotent element of $\hat{G}_{J,\varsigma}$. Let

$$\tau_\varsigma = \varsigma\phi_\varsigma \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.$$

Then τ_ς may be viewed as an unramified character of a Borel subgroup B_J of the p-adic group G_J whose dual is $\hat{G}_J$. We form the normalized induced representation

$$I(\tau_\varsigma) := \mathrm{Ind}_{B_J}^{G_J}(\tau_\varsigma).$$

Let N_J be the unipotent radical of B_J, and for $\alpha \in J$ let $x_\alpha : F \longrightarrow N_J$ be the corresponding root group. Choose a character

$$\kappa : N_J \longrightarrow \mathbb{C}^\times$$

such that $\kappa \circ x_\alpha$ has conductor one if $\alpha = -\alpha_0$, and conductor zero for the other roots in J. Then there is a unique irreducible constituent $V_{J,\varsigma}$ of $I(\tau_\sigma)$ having a κ-Whittaker model. Let K_J be a hyperspecial maximal compact subgroup of G_J and let K_J^1 be its pro-unipotent radical. Then W_J may be viewed as the Weyl group of K_J/K_J^1, so representations of W_J may be identified with constituents of the principal series of K_J/K_J^1.

It follows from [R5, (7.4)] that the space of K_J^1 invariants in $V_{J,\varsigma}$ has corresponding W_J-module

$$\epsilon \otimes \operatorname{Ind}_{W_{J,\varsigma}}^{W_J} \chi_\varsigma^J,$$

which is the ς-summand of (8.4b), tensored with ϵ as in Corollary (8.1).

9. The unequal parameter case

The results in the preceding section do not apply to the Hecke algebras arising from non-Iwahori parahoric subgroups. However, in the limited examples arising in exceptional groups, we can construct square-integrable Hecke modules and determine their restrictions to parahoric subalgebras using the direct methods in [R1], [R6].

Let (P, U, M) be a parahoric subgroup in G, and let σ be a cuspidal unipotent representation of M. At various points we use the fact that G is exceptional, to simplify the discussion. In particular, P is contained in a hyperspecial maximal compact subgroup K.

(9.1) We will need the theory of weight diagrams for Hecke modules (cf. [R1]).

Let $\mathcal{H}(K/P)$ be the subalgebra of $\mathcal{H}(G/P)$ consisting of functions supported on K. Then $\mathcal{H}(K/P)$ is the span of $\{T_w : w \in W\}$ (see (5.2)) and is generated by T_r, for $r \in S := \tilde{S} \cap W$. Let $\mathbb{C}[\mathbf{T}]$ be the affine coordinate ring of the torus $\mathbf{T}$. Then we have an isomorphismm [L4]

$$\mathcal{H}(G/P) \simeq \mathcal{H}(K/P) \tilde{\otimes} \mathbb{C}[\mathbf{T}],$$

where $\tilde{\otimes}$ denotes the tensor product of $\mathbb{C}$-vector spaces with a twisted multiplication between the factors, given by

$$(T_r - q^{c(r)})\theta - \theta^r (T_r - q^{c(r)}) = (\theta^r - \theta)\zeta_r \in \mathbb{C}[\mathbf{T}],$$

where $\zeta_r \in (\theta^r - \theta)^{-1}\mathbb{C}[\mathbf{T}]$ is a rational function on $\mathbf{T}$, defined as follows. Let β be the root for r, and let e_β be the corresponding character of $\mathbf{T}$. In every case but one, we have

$$\zeta_r = \frac{q^{c(r)} - e_\beta}{1 - e_\beta}.$$

The exception for exceptional groups occurs only for $\mathcal{H}(F_4/B_2)$. Then $\mathbf{G} = SO_5(\mathbb{C})$, $\hat{T} = \mathbb{C}^\times \times \mathbb{C}^\times$, with simple roots

$$e_{\beta_1}(z_1, z_2) = z_1/z_2, \qquad e_{\beta_2}(z_1, z_2) = z_2,$$

$$\zeta_{r_1} = \frac{q^3 - e_{\beta_1}}{1 - e_{\beta_1}}, \qquad \zeta_{r_2} = \frac{(q^2 - e_{\beta_2})(q - e_{\beta_2})}{1 - e_{2\beta_2}}.$$

(9.2) Let E be a finite dimensional $\mathcal{H}(G/P)$-module. The *weight diagram* of E is a labelled graph $\Gamma(E)$, defined as follows. The restriction of E to $\mathbb{C}[\mathbf{T}]$ is a direct sum

$$E\Big|_{\mathbb{C}[\mathbf{T}]} = \bigoplus_{\tau \in \mathbf{T}} E(\tau),$$

where $E(\tau)$ consists of those vectors in E killed by some power of the maximal ideal of $\mathbb{C}[\mathbf{T}]$ at τ. The vertices of $\Gamma(E)$ are the $\tau \in \mathbf{T}$ for which $E(\tau) \neq 0$. The edges of $\Gamma(E)$ are formed by connecting τ to τ^r if $r \in S$ is such that $\zeta_r(\tau)\zeta_r(\tau^{-1}) \neq 0$, and we label such an edge by r. Let $[\tau]$ be the connected component of $\Gamma(E)$ containing τ. Note that $[\tau]$ is simply an equivalence class in the W-orbit of τ, defined independently of E.

Lemma[R1,(3.6)]. *The function $\tau \mapsto \dim E(\tau)$ is constant on the components of $\Gamma(E)$.*

Thus we define the *multiplicity* of a component $[\tau]$ of $\Gamma(E)$ to be the dimension of $E(\tau)$. All of the simple Hecke modules that we consider are determined by their labelled graphs $\Gamma(E)$, of which many examples occur in the following sections.

(9.3) There is a notion of "square integrability" of simple $\mathcal{H}(G/P)$-modules [Mat], which is compatible with the same notion for G. Namely, the G-module V is square-integrable if and only if the $\mathcal{H}(G/P)$-module $E = V^\sigma$ is so. The square-integrability of $\mathcal{H}(G/P)$-modules is checked using the following "Casselman criteria", proved for arbitary Hecke algebras in [Mat].

Lemma. *The $\mathcal{H}(G/P)$-module E is square-integrable iff for every fundamental dominant weight λ of $\mathbf{G}$, we have $|e_\lambda(\tau)| < 1$ for every vertex in $\Gamma(E)$.*

Here "dominant" refers to the set of simple roots $\tilde{\Sigma} - I$.

To give a very simple example, we list the graphs of all of the square integrable representations of the algebra $\mathcal{H}(F_4/B_2)$, described in (9.1). The labels are Langlands parameters, to be explained in (10.1).

$$\begin{aligned}
[C_3A_1, -]:&\quad (-q^{-4}, -q^{-1}) \\
[C_3(42)A_1, -+]:&\quad (-q^{-2}, -q) \\
[B_4, -]:&\quad (q^{-5}, q^{-2}) \\
[A_1A_3, -1]:&\quad (q^{-2}, -q^{-1}) \overset{1}{-} (-q^{-1}, q^{-2}) \\
[B_4(531), r]:&\quad (q^{-1}, q^{-2}) \overset{1}{-} (q^{-2}, q^{-1}) \overset{2}{-} (q^{-2}, q)
\end{aligned}$$

To prove exhaustion, note that if E is square integrable then one of $E(\tau)$ and $E(\tau^{r_1r_2r_1})$ is zero by the above Lemma, from which one easily works out the listed possibilities. The existence of $\mathcal{H}(G/P)$-modules with these weight diagrams follows immediately from Lemma 1 in the next section.

(9.4) We recall some results from [R1], showing how simple $\mathcal{H}(G/P)$-modules can be classified by their weight diagrams if τ has mild singularities. The first result is due to Rodier [Ro] when all $c(r) = 1$, and proved for arbitrary parameters in [R1,§3].

Lemma 1. *Suppose $\tau \in \mathbf{T}$ is* regular, *in the sense that $\tau^r \neq \tau$ for all reflections r in W. Then there is exactly one simple $\mathcal{H}(G/P)$-module E containing τ upon restriction to $\mathbb{C}[\mathbf{T}]$. Moreover, $[\tau]$ is the unique component of $\Gamma(E)$, and its multiplicity in E is one.*

Next suppose we have a W-orbit $\mathcal{O} \subset \mathbf{T}$ such that for each $\tau \in \mathcal{O}$ there is a unique positive root β_τ such that ζ_{β_τ} has a pole at τ. The corresponding reflection must then fix τ. This was called *rank-one singularity* in [R1]. We say two components c, c' in $\mathcal{O}$ are *adjacent* if there is $\tau \in c$, $r \in S$ such that $\tau^r \in c'$ and $r\beta_\tau \neq \beta_\tau$. Let $\nu(\tau)$ be the number of positive roots α such that $s_{\beta_\tau}\alpha < 0$ and $\zeta_\alpha(\tau) = 0$. The function $\tau \mapsto \nu(\tau)$ is constant on components in $\mathbf{T}$. We say that $[\tau]$ has *type-0* if $\nu(\tau) = 0$ and *type-2* if $\nu(\tau) \geq 2$. If $\nu(\tau) = 1$, then it turns out that $[\tau]$ is adjacent to some type-0 component. If $[\tau]$ is adjacent to exactly one type-0 component we say it has *type-1*, otherwise $[\tau]$ *no type*. The type-0 components are those containing a vertex τ for which β_τ is simple [R1,(10.8)].

Lemma 2. *Suppose that $[\tau]$ has type-0. Then there is a unique simple $\mathcal{H}(G/P)$-module E containing $[\tau]$, and $[\tau]$ is the unique component in $\Gamma(E)$ with multiplicity two. The other components in E are exactly those which are adjacent to $[\tau]$, and these have multiplicity one in E.*

We call the modules in Lemma 2 *type-0* as well.

Lemma 3. *Suppose $[\tau]$ has type-1 or type-2. In either case, there is a unique simple $\mathcal{H}(G/P)$-module E with $\Gamma(E) = [\tau]$. In type-2, E is the unique simple module containing τ. In type-1, the simple modules containing $[\tau]$ are E and the type-0 module E_0 whose graph $\Gamma(E_0)$ contains the unique type-0 component adjacent to $[\tau]$.*

All but two of the square-integrable $\mathcal{H}(G/P)$-modules arising in exceptional groups for non-minimal parahorics contain a τ which is either regular or has rank-one singularity, and are therefore classified by the above results.

The exceptions are the modules labelled $[A_1E_7(a_5), -21]$ and $[A_1E_7(a_5), -3]$ in section 13. The weights in the component for the former module are regular for every rank-two parabolic subalgebra. Applying Lemma 1 to each rank two subsystem shows there is a module having this as its unique component with multiplicity one. (I learned this argument from A. Ram, [Ra]). For $[A_1E_7(a_5), -3]$ we need extensions of [R1] found in [R6], to which we defer the detailed treatment of this example.

(9.5) The graphs $\Gamma(E)$ can be used to determine the $\mathcal{H}(K/P)$ action on E as follows. Let $J \subset S$, and let w_J be a Coxeter element of the parabolic subgroup $W_J \subseteq W$. Let $\mathcal{H}_J(K/P)$ be the subalgebra of $\mathcal{H}(K/P)$ supported on W_J. Erase those edges of $\Gamma(E)$ whose label is not in J. The resulting graph is that of a module over a parabolic subalgebra $\mathcal{H}_J(G/P)$ generated by $\mathbb{C}[\mathbf{T}]$ and $\mathcal{H}_J(K/P)$, whose $\mathcal{H}_J(K/P)$ action is known by induction, so we can calculate the trace of w_J on $E_{q=1}$. In practice, a small number of such traces determine the restriction of $E_{q=1}$ to W, hence the restriction of E to $\mathcal{H}(K/P)$.

For example, let $J = \{\alpha\}$ be a singleton, and let $r = s_\alpha$. Let

$$C = \{\tau \in \mathbf{T} : \ |e_\alpha(\tau)| < 1\}.$$

Then

$$\operatorname{tr}(r, E_{q=1}) = \sum_{\tau \in C} \dim E(\tau^r) - \dim E(\tau). \tag{9.5a}$$

This is seen as follows. The parabolic subalgebra $\mathcal{H}_\alpha(G/P)$ is generated by T_r and $\mathbb{C}[\mathbf{T}]$. Our erasing leaves either edges of the form τ—τ^r or isolated points. In the former case, the graph is that of an $\mathcal{H}_\alpha(G/P)$-module whose composition factors are irreducibly induced from $\mathbb{C}[\mathbf{T}]$. These restrict to the regular representation of $\mathcal{H}_\alpha(K/P)$, so these edges contribute zero to $\operatorname{tr}(r, E_{q=1})$. The isolated points may be paired (by adding one member of the pair if it is not part of the graph) as $\{\tau, \tau^r\}$, with no edge, such that $|\alpha(\tau)| < 1$. This pair contributes $\dim E(\tau^r) - \dim E(\tau)$ to $\operatorname{tr}(r, E_{q=1})$, because the module $M = \mathcal{H}_\alpha(G/P) \otimes_{\mathbb{C}[\mathbf{T}]} \tau$ has two T_r-eigenvectors, with eigenvalues -1 in $M(\tau)$ and $q^{c(r)}$ in $M(\tau^r)$.

(9.6) We find additional information about the $\widetilde{W}$-module $E_{q=1}$ by considering induction from intermediate parabolics, as follows.

Suppose we have a vertex $\tau \in \Gamma(E)$ with $\dim E(\tau) = 1$, having a small number of edges attached to it. Let J be the largest subset of Σ such that $\mathcal{H}_J(G/P)$ has a one-dimensional representation with unique eigenvalue τ. The scarcity of edges means there is a good chance of finding such an J which is rather large. Then τ extends to a character of $\mathcal{H}_J(G/P)$ and E is a quotient of the induced module $\mathcal{H}(G/P) \otimes_{\mathcal{H}_J(G/P)} \tau$. In particular, $E_{q=1}$ is contained in a representation induced from a one-dimensional character of XW_J.

(9.7) We illustrate all of this with some examples in the Hecke algebra

$$\mathcal{H}(G/P): \quad 1 - 1 - 1 \Rightarrow 4 - 4.$$

Refer to (13.2) for notation and graphs $\Gamma(E)$. We have, up to semisimplification,

$$E_{q=1} \simeq \bigoplus_{\chi} \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} [s \otimes \chi], \tag{9.7a}$$

where χ runs over irreducible W_s-representations and s is extended to $\widetilde{W}_s$ as before. Moreover, each summand is irreducible. To find the χ's, we play (9.7a) against the W-action determined as in (9.5).

First consider the module with Langlands parameter $[D_8(5,11), -1]$. There is only one component, with multiplicity one, and $\dim E = 15$. Let $\tau = [-\bar{6}, 1, 4, 4]$. Then $E(\tau)$ is one dimensional, and is preserved by $\mathcal{H}_{C_3}(G/P)$, which acts on $E(\tau)$ via a twist of its Steinberg representation. Moreover, $W_{C_3} \subset W_s$, so up to semisimplification,

$$E_{q=1} \subset \operatorname{Ind}_{XW_{C_3}}^{\widetilde{W}} s \otimes \epsilon = \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} [s \otimes \operatorname{Ind}_{\widetilde{W}_{C_3}}^{\widetilde{W}_s} \epsilon].$$

The inner induced module is $\operatorname{Ind}_{W_{C_3}}^{W_s} \epsilon$ with trivial X action. Now W_s has type B_4, and

$$\operatorname{Ind}_{W_{C_3}}^{W_s} \epsilon = [-, 1^4] + [1, 1^3] + [-, 211],$$

so we have the irreducible decomposition

$$\operatorname{Ind}_{XW_{C_3}}^{\widetilde{W}} s \otimes \epsilon = \operatorname{Ind}_{\widetilde{W}_s}^{\widetilde{W}} s \otimes ([-, 1^4] + [1, 1^3] + [-, 211]).$$

These constituents have dimensions $3, 12, 9$ respectively, so (9.7a) becomes

$$E_{q=1} \underset{ss}{=} \mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}}\ s \otimes ([-, 1^4] + [1, 1^3]).$$

Note that this example only requires the trace of the identity element on $E_{q=1}$.

Next consider the module with Langlands parameter $[D_8(1357), r]$. The graph has four components: One with multiplicity two, the others have multiplicity one, and $\dim E = 66$. The method described in (9.5) gives the following traces on $E_{q=1}$:

$$\mathrm{tr}(r_1) = -10, \quad \mathrm{tr}(r_4) = -18, \quad \mathrm{tr}(r_1 r_2) = 0, \quad \mathrm{tr}(r_3 r_4) = -1.$$

Let $\tau = [1, -\bar{4}, 4, 4]$. Then $E(\tau)$ is a twist of Steinberg for $\mathcal{H}_{A_1 A_2}(G/P)$, and W_s is again of type B_4 (conjugate to the B_4 in the previous example). We calculate

$$\mathrm{Ind}_{W_{A_1 A_2}}^{W_s} \epsilon = [-, 1^4] + [-, 211] + [1^3, 1] + [1, 21] + [1, 1^3] + [11, 2] + [11, 11],$$

so the constituents of $E_{q=1}$ have multiplicity one, and are among the representations

$$\mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} (s \otimes [-, 1^4]), \quad \cdots, \quad \mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} (s \otimes [11, 11]).$$

Calculating traces of $r_1, r_4, r_1 r_2, r_3 r_4$ on the latter representations, we find

$$E_{q=1} \underset{ss}{=} \mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}}\ s \otimes ([-, 1^4] + [-, 211] + [1, 21] + [1, 1^3] + [11, 11]).$$

(9.8) The algebras $\mathcal{H}(G/P)$ are part of a one parameter family $\mathcal{H}^c$ of affine Hecke algebras, where $\mathcal{H}^1$ is the equal parameter algebra, discussed in §8. Some $\mathcal{H}(G/P)$ modules E are also part of a one parameter family of $\mathcal{H}^c$-modules E^c, such that $\mathrm{tr}(T_w, E^c)$ is continuous for some real c-interval containing 1, and the parameter for $\mathcal{H}(G/P)$. In this case we say E is "obtained by deformation". The most obvious example is the one dimensional $\mathcal{H}^c$-module on which $T_r = -1$ for all $r \in \tilde{S}$, and Ω_I acts by an arbitrary character. New deformations may be obtained by inducing this one, as in (8.4b).

If E is obtained by deformation, the restriction of $E_{q=1}$ to $\widetilde{W}_J$ does not change as the parameter c varies, so we can apply the equal-parameter results from §8 to calculate the restriction. Some interesting problems arise in this method, however. Not all square-integrable modules are obtained by deformation, and those that do may not have a deformation that remains square-integrable as c varies. (The equal-parameter F_4 module $[C_3(42)A_1, ++]$ is an example of the latter case.) For example, we sometimes find that the $\mathcal{H}^1$-module is contained in an induced from a maximal parabolic, and the other constituent of this induced becomes square-integrable after the deformation to the unequal parameter algebra $\mathcal{H}(G/P)$. This phenomenon is detected by the Euler-Poincaré value $f_V(1)$, which is zero on any properly induced representation (see (2.1)), hence is negative on the non-square-integrable constituent of an induced representation of length two.

(9.9) Using (9.3,4), one may check that each of the graphs in (9.3), §12,13 is indeed that of a unique square-integrable $\mathcal{H}(G/P)$-module E. Though it is not logically necessary, we wish to give a rough idea of how these graphs were found, emphasizing in particular some general methods distilled from many ad-hoc calculations.

Start with the modules over the equal parameter algebra $\mathcal{H}^1$, as in (9.8), and calculate their restrictions to W_J's, as in (8.2a). Use these to calculate the sum in the Euler-Poincaré formula (2.1a), but now using the generic degrees for parahoric subalgebras of the original unequal-parameter algebra $\mathcal{H}(G/P)$. If the answer looks like a formal degree, then search for a deformation of the graph of the equal parameter module as in (9.8). These searches were successful, in our examples. In most cases one finds a τ as in (9.6), deforms it so as to preserve J, and checks if the component $[\tau]$ satisfies the conditions of Lemmas 1,2 or 3 in (9.4). A few deformations were found only after considerable trial and error.

If the sum (2.1a) cannot be a formal degree, because, for example, it is negative, then there can be no square-integrable deformation. In this case, we find candidates for $\Gamma(E)$, as follows. Take an elliptic $x = su \in \hat{G}$ whose L-packet Π_x is suspected to have a representation containing (P, σ), and let $\tau = s\phi\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$, as in (7.2). Let L be the Levi subgroup of G of the same type as P, and let K_L be a hyperspecial parahoric in L. Let τ_L be the analogue for τ in the Langlands parameter of the supercuspidal obtained by compact induction of σ from K_L to the derived group L'. We may choose $\tau_L \in \hat{L}' \cap \hat{T}$. On the other hand the torus $\mathbf{T}$ in the tensor product decomposition of $\mathcal{H}(G/P)$ may be identified with the set of unramified characters of L, and hence $\mathbf{T}$ is a subtorus of $\hat{T}$, via restriction of characters from L to T. We then have the following heuristic principle:

For any vertex $\tau_1 \in \Gamma(E)$ we should have $\tau_1\tau_L$ in the W-orbit of τ.

Thus, we expect a relation between τ, which *a priori* is only relevant to the Iwahori-spherical part of the L-packet, to the central characters of modules over other Hecke algebras appearing in the packet.

For τ_1 satisfying the above heuristic, the component of τ_1 belongs to some $\mathcal{H}(G/P)$-module E (possibly several E's). For each fundamental dominant weight $\lambda \in X$, we evaluate e_λ on each vertex in the component of τ_1. If one such value is ≥ 1 in absolute value, then no E containing $[\tau_1]$ can be square-integrable, so we choose a new τ_1 satisfying the heuristic, and repeat the process. Lemma (9.3) provides a guide to making reasonable choices for τ_1.

10. F_4

In this section we give the degrees of all square integrable unipotent representations of the p-adic group $G = F_4$, and to each Langlands parameter $(x, \rho) \in \Psi(G)$ we attach a representation $V_{x,\rho} \in \mathrm{Irr_{upt}}^2(G)$ as in the main theorem in the introduction. We also give the action of the hyperspecial parahoric K on the invariants of its pro-unipotent radical in V. This indicates which parahoric has a cuspidal unipotent in V, and also shows the "leading K-type" which will be discussed in §14.

The affine diagram of G is labelled

$$0 - 1 - 2 \Longrightarrow 3 - 4.$$

For each of the reflection subgroups of type $J = F_4,\ A_1C_3,\ A_2\tilde{A}_2,\ A_3\tilde{A}_1,\ B_4$, (where $\tilde{}$ denotes short roots), let $J(q)$ be the corresponding Poincaré polynomial. It is also the volume of the maximal parahoric corresponding to J. As in (3.2), the simplified formal degree formula (with $vol(\mathcal{I}) = 1$) is

$$\deg(V) = \frac{\dim[V^*_{F_4}]}{F_4(q)} + \frac{\dim[V_{A_1} \otimes V^*_{C_3}]}{A_1C_3(q)} + \frac{\dim[V_{A_2} \otimes V^*_{\tilde{A}_2}]}{A_2\tilde{A}_2(q)} + \frac{\dim[V_{A_3} \otimes V^*_{\tilde{A}_1}]}{A_3\tilde{A}_1(q)} + \frac{\dim[V_{B_4}]}{B_4(q)},$$

where the subscripts on V indicate invariants under the pro-unipotent radical of the corresponding parahoric subgroup, and $*$ is the duality operation described in §3.

We label Weyl group representations and K-types as in [C]. For F_4 they are $\phi_{d,b}$, of dimension d and harmonic birthday b. Primes $'$ indicate different representations with the same d, b. For the symmetric group S_n, $n \geq 3$, the representations are identified with partitions λ of n. The trivial representation is n. For S_2 we write the characters as $\pm$. For $B_n = C_n$, they are $[\alpha, \beta]$, where α and β are partitions whose union is a partition of n. The trivial representation is $[n, -]$.

There are three relevant affine Hecke algebras: The Iwahori-spherical algebra $\mathcal{H}(F_4/\mathcal{I})$ of F_4 (with 18 L^2-representations), the algebra $\mathcal{H}(F_4/B_2)$ described in §9 (with five L^2 representations), and the algebra $\mathcal{H}(F_4/F_4) = \mathbb{C}$, corresponding to the 7 cuspidal unipotent representations of F_4. The formal degrees for $\mathcal{H}$ were computed using §8 and Shoji's calculation [Sho] of Green polynomials for F_4, which gives the restrictions to the hyperspecial parahoric. We give the restrictions to the other maximal parahorics in another table after the degrees. For example, the entries $3, \bar{2}$ in the first column of the A_1C_3 table are the multiplicities of $\mathrm{trivial}_{A_1} \otimes \mathrm{Steinberg}_{C_3}$ and $\mathrm{Steinberg}_{A_1} \otimes \mathrm{Steinberg}_{C_3}$, respectively. The degrees for $\mathcal{H}(F_4/B_2)$ are found using §9, and those of last algebra are trivial to compute.

As in (7.2), the lowest power of q in the formal degree is the dimension of $\mathcal{B}^u$, for a unique u which is the unipotent part of an elliptic conjugacy class in $\hat{G} = F_4(\mathbb{C})$. Therefore, we group the L-packets according to u (indicated by its label in [C]), one packet for each possible semisimple part s, the pair being described by the label of u, viewed in $\hat{G}_s$. For example, the label $C_3(42)A_1$ denotes the unipotent class in subgroup of $\hat{G} = F_4(\mathbb{C})$ isogenous to $Sp_6(\mathbb{C}) \times SL_2(\mathbb{C})$ which has partition 42 in the first factor, and is regular in the second. Thus, each representation is given a name (Langlands parameter) of the form $[u \in \hat{G}_s, \rho]$, where ρ is an irreducible character of A_{su}.

Finally, Φ_d is the d^{th} cyclotomic polynomial evaluated at q.

$$\mathbf{u=F_4},\quad \dim\mathcal{B}_{\mathbf{u}}=\mathbf{0},\quad \mathbf{A_u=1}$$

$[u\in\hat{G}_s,\rho]$	$K-$types	Degree
$[F_4,1]$	$\phi_{(1,24)}$	$\frac{\Phi_1^4\Phi_5\Phi_7\Phi_{11}}{\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}}$

$$\mathbf{u=F_4(a_1)},\quad \dim\mathcal{B}_{\mathbf{u}}=\mathbf{1},\quad \mathbf{A_u=S_2}$$

$[F_4(a_1),+]$	$\phi_{(4,13)}+\phi_{(1,24)}$	$\frac{q\Phi_1^4\Phi_5^2\Phi_7}{2\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8}$
$[F_4(a_1),-]$	$\phi'_{(2,16)}$	$\frac{q\Phi_1^4\Phi_5^2\Phi_7}{2\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8}$
$[B_4,+]$	$\phi''_{(2,16)}+\phi_{(1,24)}$	$\frac{q\Phi_1^4\Phi_5\Phi_7\Phi_{10}}{2\Phi_2^4\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}}$
$[B_4,-]$	$B_2[\epsilon]$	$\frac{q\Phi_1^4\Phi_5\Phi_7\Phi_{10}}{2\Phi_2^4\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}}$

$$\mathbf{u=F_4(a_2)},\quad \dim\mathcal{B}_{\mathbf{u}}=\mathbf{2},\quad \mathbf{A_u=S_2}$$

$[F_4(a_2),+]$	$\phi_{(9,10)}+\phi_{(4,13)}+\phi_{(1,24)}$	$\frac{q^2\Phi_1^4\Phi_5}{2\Phi_2^4\Phi_3^2\Phi_6^2}$
$[F_4(a_2),-]$	$\phi''_{(2,16)}$	$\frac{q^2\Phi_1^4\Phi_5}{2\Phi_2^4\Phi_3^2\Phi_6^2}$
$[C_3A_1,+]$	$\phi_{(9,10)}+\phi'_{(2,16)}+\phi_{(1,24)}$	$\frac{q^2\Phi_1^4\Phi_5\Phi_8}{2\Phi_2^4\Phi_4^2\Phi_6^2\Phi_{12}}$
$[C_3A_1,-]$	$B_2[\epsilon]$	$\frac{q^2\Phi_1^4\Phi_5\Phi_8}{2\Phi_2^4\Phi_4^2\Phi_6^2\Phi_{12}}$

$$\mathbf{u = F_4(a_3)}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{4}, \quad \mathbf{A_u = S_4}$$

$[F_4(a_3), 4]$	$\phi_{(12,4)} + \phi'_{(8,9)} + \phi''_{(8,9)} + \phi_{(9,10)} + \phi_{(4,13)} + \phi_{(1,24)}$	$\frac{q^4\Phi_1^4}{24\Phi_2^4\Phi_3^2\Phi_4^2}$
$[F_4(a_3), 31]$	$\phi'_{(9,6)} + \phi'_{(8,9)} + \phi'_{(2,16)}$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_3^2\Phi_4^2}$
$[F_4(a_3), 22]$	$\phi''_{(6,6)} + \phi_{(4,13)}$	$\frac{q^4\Phi_1^4}{12\Phi_2^4\Phi_3^2\Phi_4^2}$
$[F_4(a_3), 211]$	$\phi'_{(1,12)}$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_3^2\Phi_4^2}$
$[F_4(a_3), 1^4]$	$F_4^{II}[1]$	$\frac{q^4\Phi_1^4}{24\Phi_2^4\Phi_3^2\Phi_4^2}$
$[C_3(42)A_1, ++]$	$\phi_{(16,5)}+\phi'_{(8,9)}+\phi''_{(8,9)}+\phi_{(9,10)}+\phi_{(4,13)}+\phi'_{2,16}+\phi_{(1,24)}$	$\frac{q^4\Phi_1^4}{4\Phi_2^4\Phi_6^2\Phi_8}$
$[C_3(42)A_1, +-]$	$\phi'_{(4,7)} + \phi'_{(8,9)}$	$\frac{q^4\Phi_1^4}{4\Phi_2^4\Phi_6^2\Phi_8}$
$[C_3(42)A_1, -+]$	$B_2[\epsilon'']$	$\frac{q^4\Phi_1^4}{4\Phi_2^4\Phi_6^2\Phi_8}$
$[C_3(42)A_1, --]$	$F_4[-1]$	$\frac{q^4\Phi_1^4}{4\Phi_2^4\Phi_6^2\Phi_8}$
$[B_4(531), 1]$	$\phi''_{(9,6)} + \phi''_{(8,9)} + \phi_{(9,10)} + \phi_{(4,13)} + \phi''_{(2,16)} + \phi_{(1,24)}$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_4^2\Phi_6^2}$
$[B_4(531), \epsilon'']$	$\phi_{(4,8)} + \phi''_{(2,16)}$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_4^2\Phi_6^2}$
$[B_4(531), \epsilon']$	$\phi''_{(1,12)} + \phi''_{(2,16)}$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_4^2\Phi_6^2}$
$[B_4(531), r]$	$B_2[r] + B_2[\epsilon]$	$\frac{q^4\Phi_1^4}{4\Phi_2^4\Phi_4^2\Phi_6^2}$
$[B_4(531), \epsilon]$	$F_4^I[1]$	$\frac{q^4\Phi_1^4}{8\Phi_2^4\Phi_4^2\Phi_6^2}$
$[2A_2, 1]$	$\phi'_{(6,6)} + \phi'_{(8,9)} + \phi''_{(8,9)} + \phi_{(9,10)} + \phi_{(1,24)}$	$\frac{q^4\Phi_1^4}{3\Phi_3^2\Phi_6^2\Phi_{12}}$
$[2A_2, \theta]$	$F_4[\theta]$	$\frac{q^4\Phi_1^4}{3\Phi_3^2\Phi_6^2\Phi_{12}}$
$[2A_2, \theta^2]$	$F_4[\theta^2]$	$\frac{q^4\Phi_1^4}{3\Phi_3^2\Phi_6^2\Phi_{12}}$
$[A_1A_3, 1]$	$\phi''_{(4,7)} + \phi''_{(8,9)} + \phi_{(9,10)} + \phi''_{(2,16)} + \phi_{(1,24)}$	$\frac{q^4\Phi_1^4}{4\Phi_4^2\Phi_8\Phi_{12}}$
$[A_1A_3, -1]$	$B_2[\epsilon'] + B_2[\epsilon]$	$\frac{q^4\Phi_1^4}{4\Phi_4^2\Phi_8\Phi_{12}}$
$[A_1A_3, i]$	$F_4[i]$	$\frac{q^4\Phi_1^4}{4\Phi_4^2\Phi_8\Phi_{12}}$
$[A_1A_3, -i]$	$F_4[-i]$	$\frac{q^4\Phi_1^4}{4\Phi_4^2\Phi_8\Phi_{12}}$

Iwahori-spherical A_1C_3-types

V	$[-,1^3]$	$[1^3,-]$	$[11,1]$	$[1,11]$	$[21,-]$	$[-,21]$	$[2,1]$	$[1,2]$	$[-,3]$
$[F_4,1]$	$\bar{1}$								
$[F_4(a_1),+]$	$1,\bar{1}$			$\bar{1}$					
$[F_4(a_1),-]$						$\bar{1}$			
$[B_4,+]$	$1,\bar{1}$	$\bar{1}$							
$[F_4(a_2),+]$	$1,\bar{2}$		$\bar{1}$	$1,\bar{1}$		$\bar{1}$			
$[F_4(a_2),-]$	$\bar{1}$	1							
$[C_3A_1,+]$	$1,\bar{1}$		$\bar{1}$	$\bar{1}$		$1,\bar{1}$			
$[F_4(a_3),4]$	$2,\bar{2}$	$\bar{1}$	$1,\bar{2}$	$2,\bar{3}$		$1,\bar{1}$	$\bar{1}$	$1,\bar{1}$	
$[F_4(a_3),31]$	$1,\bar{1}$	$2,\bar{1}$	$1,\bar{1}$	$1,\bar{1}$	1				
$[F_4(a_3),22]$	1			$1,\bar{1}$			$\bar{1}$		
$[F_4(a_3),211]$		1							
$[C_3(42)A_1,++]$	$2,\bar{2}$	$\bar{1}$	$1,\bar{2}$	$2,\bar{3}$	$\bar{1}$	$2,\bar{2}$	$\bar{1}$	$1,\bar{1}$	
$[C_3(42)A_1,+-]$				$\bar{1}$		$\bar{1}$		$1,\bar{1}$	$\bar{1}$
$[B_4(531),1]$	$3,\bar{2}$	$1,\bar{2}$	$1,\bar{2}$	$2,\bar{2}$	$\bar{1}$	1			
$[B_4(531),\epsilon'']$					$\bar{1}$	$1,\bar{1}$			
$[B_4(531),\epsilon']$	1	$1,\bar{1}$							
$[2A_2,1]$	$1,\bar{2}$	$\bar{1}$	$1,\bar{2}$	$1,\bar{2}$		$1,\bar{1}$		$1,\bar{1}$	
$[A_1A_3,1]$	$2,\bar{2}$	$1,\bar{2}$	$1,\bar{2}$	$1,\bar{1}$		1			

Iwahori-spherical A_2A_2-types

(u,s,ρ)	$[1^3,1^3]$	$[1^3,21]$	$[1^3,3]$	$[21,1^3]$	$[21,21]$	$[21,3]$	$[3,1^3]$	$[3,21]$
$[F_4,1]$	1							
$[F_4(a_1),+]$	1	1		1				
$[F_4(a_1),-]$		1						
$[B_4,+]$	1			1				
$[F_4(a_2),+]$	2	2		2	1			
$[F_4(a_2),-]$				1				
$[C_3A_1,+]$	2	2		1	1			
$[F_4(a_3),4]$	4	4	1	4	4	1	1	1
$[F_4(a_3),31]$			1	1	2	3	2	
$[F_4(a_3),22]$		1	1	1	1		1	
$[F_4(a_3),211]$							1	
$[C_3(42)A_1,++]$	4	5	1	4	5	1	1	1
$[C_3(42)A_1,+-]$	1	2	1		1	1		
$[B_4(531),1]$	3	2		5	3		2	1
$[B_4(531),\epsilon'']$		1			1			
$[B_4(531),\epsilon']$				1			1	
$[2A_2,1]$	4	3		3	3	1		1
$[A_1A_3,1]$	3	1		4	2		1	1

Iwahori-spherical A_3A_1-types

(u,s,ρ)	$[1^4]$	$[211]$	$[22]$	$[31]$	$[4]$
$[F_4,1]$	$\bar{1}$				
$[F_4(a_1),+]$	$1,\bar{1}$	$\bar{1}$			
$[F_4(a_1),-]$	$1,\bar{1}$				
$[B_4,+]$		$\bar{1}$			
$[F_4(a_2),+]$	$1,\bar{2}$	$1,\bar{2}$	$\bar{1}$		
$[F_4(a_2),-]$			$\bar{1}$		
$[C_3A_1,+]$	$1,\bar{2}$	$1,\bar{2}$			
$[F_4(a_3),4]$	$2,\bar{3}$	$3,\bar{5}$	$1,\bar{1}$	$1,\bar{2}$	
$[F_4(a_3),31]$		$\bar{1}$	$1,\bar{2}$	$1,\bar{2}$	$\bar{1}$
$[F_4(a_3),22]$	1	$1,\bar{1}$		$\bar{1}$	
$[F_4(a_3),211]$					$\bar{1}$
$[C_3(42)A_1,++]$	$2,\bar{3}$	$4,\bar{6}$	$1,\bar{1}$	$1,\bar{2}$	
$[C_3(42)A_1,+-]$	$2,\bar{2}$	$1,\bar{1}$	1		
$[B_4(531),1]$	$\bar{1}$	$2,\bar{4}$	$\bar{1}$	$1,\bar{3}$	
$[B_4(531),\epsilon'']$		$1,\bar{1}$			
$[B_4(531),\epsilon']$				$\bar{1}$	
$[2A_2,1]$	$1,\bar{3}$	$2,\bar{4}$	$1,\bar{1}$	$1,\bar{1}$	
$[A_1A_3,1]$	$\bar{1}$	$1,\bar{3}$	$\bar{1}$	$1,\bar{2}$	

Iwahori-spherical B_4-types

(u,s,ρ)	$[\cdot,1^4]$	$[\cdot,211]$	$[\cdot,22]$	$[\cdot,31]$	$[\cdot,4]$	$[1^4,\cdot]$	$[1,1^3]$	$[1^3,1]$	$[1,21]$	$[11,11]$	$[11,2]$	$[2,11]$	$[2,2]$	$[211,\cdot]$
$[F_4,1]$	1													
$(F_4(a_1),+]$	1						1							
$[F_4(a_1),-]$	1					1								
$[B_4,+]$		1												
$[F_4(a_2),+]$	1	1					1			1				
$[F_4(a_2),-]$			1											
$[C_3A_1,+]$	1	1					1	1						
$[F_4(a_3),4]$	1	1					2	1	1	1	1	1		
$[F_4(a_3),31]$			1	1					1				1	
$[F_4(a_3),22]$							1					1		
$[F_4(a_3),211]$					1									
$[C_3(42)A_1,++]$	1	2					2	1	1	1	1	1		1
$[C_3(42)A_1,+-]$	1					1		1		1				
$[B_4(531),1]$		2		1			1		1		1	1		
$[B_4(531),\epsilon'']$		1												1
$[B_4(531),\epsilon']$				1										
$[2A_2,1]$	1	1					1	1	1	1	1			
$[A_1A_3,1]$		1		1			1		1		1			

11. E_6

In this section we give the degrees of all square integrable unipotent representations of the split adjoint p-adic group $G = E_6$, the action of the hyperspecial parahoric K on the invariants of its pro-unipotent radical (representations labeled as in [C]), and to each representation we attach a Langlands parameter in the form $[u \in \hat{G}_s, \rho]$, as in §10.

There are three relevant affine Hecke algebras: The Iwahori-spherical algebra of E_6 (with 6 L^2-representations, up to twists by unramified characters), the Hecke algebra $\mathcal{H}(E_6/D_4)$ of type A_2 with constant parameter q^4 (one L^2-representation, up to twists), and the algebra $\mathbb{C}$, corresponding to the 2 cuspidal unipotent representations of E_6. The group of unramified characters of G has order three; its twists permute the L-packets, preserving formal degrees. Those packets where u is elliptic in $\hat{G}$ have three distinct twists, and we just list one of each. There is one more packet, with $u = D_4(a_1)$, whose members are invariant under unramified twists.

As in §3 but with a few extra terms due to branching in the affine diagram, the simplified degree formula (with $vol(\mathcal{I}) = 1$) is

$$\begin{aligned} 3\deg(V) = {} & 3\frac{\dim[V^*_{E_6}]}{E_6(q)} + 3\frac{\dim[V^*_{D_5}]}{D_5(q)} + \frac{\dim[V^*_{D_4}]}{D_4(q)} + 3\frac{\dim[V_{A_1}\otimes V^*_{A_5}]}{A_1A_5(q)} \\ & + 6\frac{\dim[V_{A_1}\otimes V^*_{A_4}]}{A_1A_4(q)} + 3\frac{\dim[V_{A_1}\otimes V^*_{A_3}]}{A_1A_3(q)} + 3\frac{\dim[V_{2A_1}\otimes V^*_{A_3}]}{A_1A_1A_3(q)} \\ & + 3\frac{\dim[V_{2A_1}\otimes V^*_{A_2}]}{A_1A_1A_2(q)} + \frac{\dim[V_{3A_1}\otimes V^*_{A_1}]}{A_1A_1A_1A_1(q)} + \frac{\dim[V_{3A_2}]}{A_2A_2A_2(q)}. \end{aligned}$$

The rest of the notation is as in §10.

$$\mathbf{u = E_6}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{0}, \quad \mathbf{A_u = 1}$$

$[u \in \hat{G}_s, \rho]$	$K-$types	Degree
$[E_6, 1]$	$\phi_{(1,36)}$	$\frac{\Phi_1^6\Phi_7\Phi_{11}}{3\Phi_2^2\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}}$

$$\mathbf{u = E_6(a_1)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{1}, \quad \mathbf{A_u = 1}$$

$[E_6(a_1), 1]$	$\phi_{(6,25)} + \phi_{(1,36)}$	$\frac{q\Phi_1^6\Phi_5\Phi_7}{3\Phi_2^2\Phi_3^3\Phi_6^2\Phi_9}$

$\mathbf{u = E_6(a_3)}$, $\dim \mathcal{B}_\mathbf{u} = \mathbf{3}$, $\mathbf{A_u = S_2}$

$[E_6(a_3),+]$	$\phi_{(30,15)} + \phi_{(20,20)} + \phi_{(6,25)} + \phi_{(1,36)}$	$\frac{q^3\Phi_1^6}{6\Phi_2^2\Phi_3^3\Phi_6^2}$
$[E_6(a_3),-]$	$\phi_{(15,17)} + \phi_{(6,25)}$	$\frac{q^3\Phi_1^6}{6\Phi_2^2\Phi_3^3\Phi_6^2}$

$[A_1A_5,+]$	$\phi_{(15,16)} + \phi_{(20,20)} + \phi_{(1,36)}$	$\frac{q^3\Phi_1^6}{6\Phi_2^2\Phi_3\Phi_6^2\Phi_{12}}$
$[A_1A_5,-]$	$D_4(111)$	$\frac{q^3\Phi_1^6}{6\Phi_2^2\Phi_3\Phi_6^2\Phi_{12}}$

$\mathbf{u = D_4(a_1)}$, $\dim \mathcal{B}_\mathbf{u} = \mathbf{7}$, $\mathbf{A_{su} = Z_3}$

$[3A_2,1]$	$\phi_{(10,9)}+2\phi_{(60,11)}+\phi_{(24,12)}+\phi_{(30,15)}+\phi_{(15,16)}+2\phi_{(20,20)}+\phi_{(1,36)}$	$\frac{q^7\Phi_1^6}{3\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}}$
$[3A_2,\theta]$	$E_6[\theta]$	$\frac{q^7\Phi_1^6}{3\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}}$
$[3A_2,\theta^2]$	$E_6[\theta^2]$	$\frac{q^7\Phi_1^6}{3\Phi_3^3\Phi_6^2\Phi_9\Phi_{12}}$

Iwahori-spherical A_1A_5-types

V	$[1^6]$	$[21^4]$	$[2^21^2]$	$[222]$	$[31^3]$	$[321]$	$[333]$
$[E_6,1]$	$\bar{1}$						
$[E_6(a_1),1]$	$\bar{1},1$	$\bar{1}$					
$[E_6(a_3),+]$	$\bar{2},2$	$\bar{3},1$	$1,\bar{1}$	$\bar{1}$	$\bar{1}$		
$[E_6(a_3),-]$	1	$\bar{1},1$			$\bar{1}$		
$[A_1A_5,+]$	$\bar{2},1$	$\bar{2}$	$1,\bar{1}$	$\bar{1}$			
$[3A_2,1]$	$1,\bar{4}$	$2,\bar{5}$	$3,\bar{6}$	$1,\bar{4}$	$1,\bar{1}$	$2,\bar{2}$	$1,\bar{1}$

Iwahori-spherical $A_2A_2A_2$-types
(multiplicities are invariant under permutation of factors)

V	$[\epsilon\epsilon\epsilon]$	$[\epsilon\epsilon r]$	$[\epsilon\epsilon 1]$	$[\epsilon rr]$	$[\epsilon r1]$	$[\epsilon 11]$	$[rrr]$	$[rr1]$	$[r11]$
$[E_6,1]$	1								
$[E_6(a_1),1]$	1	1							
$[E_6(a_3),+]$	4	3	1	2			1		
$[E_6(a_3),-]$		1	1	1					
$[A_1A_5,+]$	4	2		1			1		
$[3A_2,1]$	12	7		7	1		9	1	1

12. E_7

In this section we give the degrees of all square integrable unipotent representations of the split adjoint p-adic group $G = E_7$, and to each representation we attach a Langlands parameter in the form $[u \in \hat{G}_s, \rho]$, as in §10. We also indicate the action of the hyperspecial parahoric K on the invariants of its pro-unipotent radical (in some cases, just the leading K-type, see §14). We do not include the restrictions to non-hyperspecial parahorics of Iwahori spherical representations, as these are quite lengthy to tabulate.

(12.1) The restrictions to K are described by representations of the Weyl group of E_7, for which d_b denotes the unique d-dimensional representation with harmonic birthday b. The group of unramified characters of G has order two. If the representations are distinct from their twists, we write the name as $[\pm u \in \hat{G}_s, \rho]$. No $\pm$ means the representation is isomorphic to its twist.

There are four relevant affine Hecke algebras: The Iwahori-spherical algebra $\mathcal{H}(E_7/\mathcal{I})$ (with 15 L^2-representations, up to twists by characters), the Hecke algebra $\mathcal{H}(E_7/D_4)$ (with five L^2-representations, up to twists, classified in (12.2)), the algebra $\mathcal{H}(E_7/E_6)$ with $\mathbf{G} = SL_2(\mathbb{C})$ with equal parameters q^9 (one L^2-representation for each of the two cuspidal unipotent representations of E_6) and the algebra $\mathbb{C}$, corresponding to the 2 cuspidal unipotent representations of E_7. For the first algebra, we used the Green polynomial calculations of Beynon-Spaltenstein [BS], and the induce/restrict tables of Alvis [A1], as discussed in §8. The second algebra requires §9, and we give the weight diagrams of its square-integrable modules in (12.2) below.

The simplified degree formula (with $vol(\mathcal{I}) = 1$) is

$$\begin{aligned}2\deg(V) = {} & \frac{\dim[V_{E_7} \oplus V_{E_7}^*]}{E_7(q)} + \frac{\dim[V_{A_1} \otimes V_{D_6}^* \oplus V_{A_1}^* \otimes V_{D_6}]}{A_1 D_6(q)} \\ & + \frac{\dim[V_{A_2} \otimes V_{A_5}^* \oplus V_{A_2}^* \otimes V_{A_5}]}{A_2 A_5(q)} - \frac{\dim[V_{A_4} \otimes V_{A_2}^*]}{A_2 A_4(q)} - \frac{\dim[V_{A_5} \otimes V_{A_1}^*]}{A_5 A_1(q)} \\ & + \frac{\dim[V_{A_3} \otimes V_{A_1}^* \otimes V_{A_3}^*]}{A_1 2 A_3(q)} + \frac{\dim[V_{A_7}}{A_7(q)} - \frac{\dim[V_{A_6}]}{A_6(q)}.\end{aligned}$$

The rest of the notation is as in §10." Same" means equal to the degree directly above.

$$\mathbf{u = E_7}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{0}, \ \mathbf{A_u = 1}$$

$[u \in \hat{G}_s, \rho]$	$K-$types	Degree
$[\pm E_7, 1]$	1_{63}	$\frac{\Phi_1^7 \Phi_{11} \Phi_{13} \Phi_{17}}{2\Phi_2^7 \Phi_3^2 \Phi_4^2 \Phi_6^3 \Phi_8 \Phi_{10} \Phi_{12} \Phi_{14} \Phi_{18}}$

$$\mathbf{u = E_7(a_1)}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{1}, \ \mathbf{A_u = 1}$$

$[\pm E_7(a_1), 1]$	$7_{46} + 1_{63}$	$\frac{q\Phi_1^7 \Phi_5 \Phi_{11} \Phi_{13}}{2\Phi_2^7 \Phi_3^2 \Phi_4^2 \Phi_6^3 \Phi_{10} \Phi_{12} \Phi_{14}}$

$$\mathbf{u = E_7(a_2)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{2},\ \mathbf{A_u = 1}$$

$[\pm E_7(a_2), 1]$	$27_{37} + 7_{46} + 1_{63}$	$\frac{q^2\Phi_1^7\Phi_7^2\Phi_{11}}{2\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}}$

$$\mathbf{u = E_7(a_3)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{3},\ \mathbf{A_u = S_2}$$

$[\pm E_7(a_3), 1]$	$56_{30} + 27_{37} + 7_{46} + 1_{63}$	$\frac{q^3\Phi_1^7\Phi_5\Phi_7^2}{4\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}}$
$[\pm E_7(a_3), -1]$	$21_{33} + 7_{46}$	same
$[\pm A_1D_6, 1]$	$35_{31} + 27_{37} + 1_{63}$	$\frac{q^3\Phi_1^7\Phi_5\Phi_7\Phi_9\Phi_{14}}{4\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}}$
$[\pm A_1D_6, -1]$	$D_4(-, 111)$	same

$$\mathbf{u = E_6(a_1)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{4},\ \mathbf{A_{su} = S_2}$$

$[A_7, 1]$	$15_{28} + 35_{31} + 21_{36} + 1_{63}$	$\frac{q^4\Phi_1^7\Phi_4\Phi_8\Phi_{16}}{2\Phi_2^7\Phi_6^3\Phi_{10}\Phi_{14}\Phi_{18}}$
$[A_7, -1]$	$D_4(111, -) + D_4(-, 111)$	same

$$\mathbf{u = E_7(a_4)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{5},\ \mathbf{A_u = S_2}$$

$[\pm E_7(a_4), 1]$	$189_{22}+120_{25}+56_{30}+35_{31}+21_{36}+27_{37}+7_{46}+1_{63}$	$\frac{q^5\Phi_1^7\Phi_5^2}{4\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8}$
$[\pm E_7(a_4), -1]$	15_{28}	same
$[\pm A_1D_6(39), 1]$	$189_{22}+105_{26}+56_{30}+35_{31}+21_{36}+27_{37}+7_{46}+1_{63}$	$\frac{q^5\Phi_1^7\Phi_5\Phi_7\Phi_{10}}{4\Phi_2^7\Phi_4^2\Phi_6^3\Phi_8\Phi_{12}\Phi_{14}}$
$[\pm A_1D_6(39), -1]$	$D_4(111, -)$	same

$\mathbf{u} = \mathbf{E_7(a_5)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{7}, \ \mathbf{A_u} = \mathbf{S_3}$

$[\pm E_7(a_5),3]$	$315_{16}+105_{21}+168_{21}+210_{21}+105_{26}+56_{30}+27_{37}+[E_7(a_4),1]$	$\frac{q^7\Phi_1^7}{12\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3}$
$[\pm E_7(a_5),21]$	$280_{18}+210_{21}+120_{25}+105_{26}+56_{30}+21_{33}+27_{37}+7_{46}$	$\frac{q^7\Phi_1^7}{6\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3}$
$[\pm E_7(a_5),111]$	$35_{22}+21_{33}$	$\frac{q^7\Phi_1^7}{12\Phi_2^7\Phi_3^2\Phi_4^2\Phi_6^3}$

$[\pm A_1D_6(57),1]$	$\mathrm{Ind}_{A_1D_6}^{E_7}(-)\otimes([1^6,\cdot]+[1^5,1]+[1^4,11]) = 280_{17}+\cdots$	$\frac{q^7\Phi_1^7\Phi_5}{4\Phi_2^7\Phi_4^2\Phi_6^3\Phi_{10}\Phi_{12}}$
$[\pm A_1D_6(57),-1]$	$D_4(1,11)+D_4(-,111)$	same

$[\pm A_2A_5,1]$	$70_{18}+105_{21}+168_{21}+189_{22}+56_{30}+35_{31}+21_{36}+27_{37}+1_{63}$	$\frac{q^7\Phi_1^7}{6\Phi_2\Phi_3^2\Phi_6^3\Phi_{12}\Phi_{18}}$
$[\pm A_2A_5,\theta]$	$E_6[\theta](-)$	same
$[\pm A_2A_5,\theta^2]$	$E_6[\theta^2](-)$	same

$\mathbf{u} = \mathbf{A_4A_1}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{11}, \ \mathbf{A_{su}} = \mathbf{Z_4}$

$[A_1A_3^2,1]$	$210_{13}+\cdots$	$\frac{q^{11}\Phi_1^7}{2\Phi_2^7\Phi_6^3\Phi_{10}\Phi_{14}\Phi_{18}}$
$[A_1A_3^2,-1]$	$D_4(1,11)+D_4(-,21)+D_4(-,111)$	same
$[A_1A_3^2,\pm i]$	$E_7[\pm\xi]$	same

(12.2) Here we classify the square integrable representations of the Hecke algebra

$$\mathcal{H}(E_7/D_4): \quad 1 \Rightarrow 4 - 4 \Leftarrow 1$$

according to their weight diagrams, as discussed in §9, and we give the restrictions to maximal parahorics needed to compute the formal degrees as shown in 12.1. The relevant representations of E_7 are those with K-types of the form $D_4(\lambda, \mu)$, [C,13.9].

We have $\mathbf{G} = Spin_7(\mathbb{C})$. Weights are points in a maximal torus $\mathbf{T}$ of $\mathbf{G}$, with simple roots $\alpha_1, \alpha_2, \alpha_3$, α_3 being short, and r_i is the reflection corresponding to α_i. We explain our notation for elements of $\mathbf{T}$ by an example: $[-\bar{3}, 4, \bar{0}]_\pm$ is the unique $\tau \in \mathbf{T}$ satisfying $e_{\alpha_1}(\tau) = -q^3,\ e_{\alpha_2}(\tau) = q^{-4},\ e_{\alpha_3}(\tau) = -1,\ \pm e_{\lambda_3}(\tau) > 0$, where λ_3 is the fundamental weight dual to α_3. The weight diagrams of square integrable representations of $\mathcal{H}(E_7/D_4)$ and corresponding Langlands parameters of E_7 are as follows.

$$\begin{array}{rl}
V & \qquad\qquad \Gamma(V^\sigma) \\[2ex]
[\pm A_1 D_6, -1]: & [4,4,1]_\pm \\
[A_7, -1]: & [4,4,\bar{0}]_+ \overset{3}{—} [4,4,\bar{0}]_- \\
[\pm A_1 D_6(39), -1]: & [-2,4,1]_\pm \overset{1}{—} [2,2,1]_\pm \overset{2}{—} [4,-2,3]_\pm \overset{3}{—} [4,4,3]_\pm \\[2ex]
[\pm A_1 D_6(57), -1]: & [4,4,-1]_\pm
\end{array}$$

$$\begin{array}{rccccc}
[A_1 A_3 A_3, -1]: & [-\bar{3},4,\bar{0}]_+ & \overset{1}{—} & [\bar{3},\bar{1},\bar{0}]_+ & \overset{2}{—} & [4,-\bar{1},1]_+ \\
& \Big| 3 & & \Big| 3 & & \\
& [-\bar{3},4,\bar{0}]_- & \overset{1}{—} & [\bar{3},\bar{1},\bar{0}]_- & \overset{2}{—} & [4,-\bar{1},1]_-
\end{array}$$

We now verify, using the results recalled in (9.3) and (9.4), that these exhaust all square-integrable representations of the algebra $\mathcal{H}(E_7/D_4)$. It causes no harm, and is convenient to change to slightly abusive notation for the calculation. We now write $\tau = (x, y, z)$ in a maximal torus of $SO_7(\mathbb{C})$, with roots $x/y, y/z, z$, and dominant weights x, xy, xyz. The L^2 condition (9.3) for an $\mathcal{H}(E_7/D_4)$-module E is that $\tau \in \Gamma(E)$ must have $|x|, |xy|, |xyz| < 1$.

Suppose (1): $|z| < 1$ for all $\tau \in \Gamma(E)$. Then in fact all $z = q^{-1}$. Suppose (1.1): $yq \neq q^{\pm 4}$ for some $(x, y, q^{-1}) \in \Gamma(E)$. Then $(x, q^{-1}, y) \in \Gamma(E)$, so $y = q^{-1}$. Then $(x, y, z) = (x, q^{-1}, q^{-1})$ has type-0, and we can apply r_3, to get $(x, q^{-1}, q) \in \Gamma(E)$, a contradiction. Suppose (1.2): $(x, q^3, q^{-1}) \in \Gamma(E)$. Then $|x| < q^{-3}$, so $|xq^{-3}| \neq q^{\pm 4}$ so $(q^3, x, q^{-1}) \in \Gamma(E)$, contradicting L^2. Thus, in case (1) all weights must be of the form (x, q^{-5}, q^{-1}). We cannot have $x = q^{-5}$, lest (q^{-5}, q^{-1}, q^{-5}) appear in $\Gamma(E)$ by (9.4), Lemma 3. Hence $xq^5 = q^{\pm 4}$. The same Lemma 3 says the weight (q^{-1}, q^{-5}, q^{-1}) appears in exactly two modules E_1, E_2. These have rank-one singularities and contain the type-0 weights $(q^{-5}, q^{-1}, q^{-1}) \in \Gamma(E_1)$ and

$(q^{-1},q^{-1},q^{-5}) \in \Gamma(E_2)$. Using Lemma (9.2), one finds weights $(q,q^{-5},q^{-1}) \in E_1$ and $(q^5,q^{-1},q^{-1}) \in \Gamma(E_2)$, that violate the L^2 condition. Hence $x = q^{-9}$, and we have only one square-integrable module in case (1) (up to twists by Z), namely $E = [A_1D_6, -1]$. This concludes case (1).

In the remaining cases, we choose $\tau = (x,y,z) \in \Gamma(E)$ with $|z| \geq 1$. For any such τ, note that $\tau^{r_2r_1} = (z,x,y) \notin \Gamma(E)$.

Suppose (2): $\tau^{r_2} = (x,z,y) \in \Gamma(E)$. Then $x = q^{-4}z$, and $|z| < q^2$. Suppose (2.1): $|y| \geq 1$. Then $\tau^{r_1} = (\tau^{r_2})^{r_2r_1} \notin \Gamma(E)$, so $z = yq^8$ or $z = y$. The former contradicts L^2, the latter forces $(z,q^{-4},z) \in \Gamma(E)$ by Lemma 3 and again contradicts L^2. Suppose (2.2): $z \neq y^{-1} \neq q$. Then one finds $(y^{-1},q^{-4}z,z) \in \Gamma(E)$, contradicting L^2. Suppose (2.3): $z = y^{-1} \neq q$. Then we find $(z,q^{-4}z,z) \in \Gamma(E)$, contradicting L^2. So we must have $y = q^{-1}$. Suppose (2.4): $z \neq q,-1$. Then get $(z^{-1},q^4z^{-1},q^{-1}) \in \Gamma(E)$, contradicting L^2. Suppose (2.5): $z = -1$. Then $E = [A_1A_3A_3, -1]$. Suppose (2.6): $z = q$. The weight (q^{-3},q^{-1},q) belongs to two non-L^2 modules as in case (1). This concludes case (2).

Suppose (3): For any $\tau \in \Gamma(E)$ with $|z| \geq 1$, we have $\tau^{r_2} = (x,z,y) \notin \Gamma(E)$. Then $z = q^4y \geq 1$, and $\tau = (x,y,q^4y)$. Suppose (3.1): $xy^{-1} \neq q^{\pm 4}$. Then $(y,x,q^4y) \in \Gamma(E)$, so $q^4y = q^4x$, leading to $(x,x^{-1},q^4x) \in \Gamma(E)$, contradicting L^2. Suppose (3.2): $xy^{-1} = q^4$. Then $|x| = |q^4y| \geq 1$, contradicting L^2. So $xy^{-1} = q^{-4}$ and $\tau = (x,q^4x,q^8x)$. Then

$$\tau^{r_3r_2r_1r_3r_2r_3} = (q^{-8}x^{-1},q^{-4}x^{-1},x^{-1}) \notin \Gamma(E),$$

so $x \in \{q^{-7}, \pm q^{-8}, \pm q^{-6}, q^{-5}\}$. Of these, $x = q^{-8}$ and $x = \pm q^{-6}$ contradict L^2, $x = q^{-7}$ gives $E = [A_1D_6(5,7), -1]$, $x = q^{-8}$ gives $E = [A_7, -1]$, and $x = q^{-5}$ gives $E = [A_1D_6(39), -1]$, completing the proof of exhaustion.

If V is one of the five square integrable representations of $E_7(F)$ containing the D_4 cuspidal unipotent, then the Euler-Poincaré formula simplifies, by Lemma (5.1) to

$$2\deg(V) = \frac{1}{2}\frac{\dim[V_{E_7} \oplus V_{E_7^*}]}{E_7(q)} + \frac{1}{2}\frac{\dim[V_{A_1} \otimes V_{D_6}^* \oplus V_{A_1}^* \otimes V_{D_6}]}{A_1D_6(q)},$$

where the numerators are dimensions of invariants under pro-unipotent radicals of the indicated parahorics, and the denominators are Poincaré polynomials of the indicated Weyl groups. The terms V_{E_7} are given in the degree tables (12.1). The A_1D_6 terms are expressed as representations of the Weyl group A_1B_2 as follows.

$\mathcal{H}(E_7/D_4)$, A_1D_6-TYPES

V	$[-,11]$	$[11,-]$	$[1,1]$	$[11,1]$	$[-,2]$
$[\pm A_1D_6, -1]$	$\bar{1}$				
$[A_7, -1]$	1				$\bar{1}$
$[\pm A_1D_6(39), -1]$					1
$[\pm A_1D_6(57), -1]$	$1, \bar{1}$		$\bar{1}$		
$[A_1A_3A_3, -1]$	$1, \bar{1}$	$\bar{1}$	$\bar{1}$		1

13. E_8

Here $G = E_8(F)$. For each Langlands parameter $(x, \rho) \in \Psi(G)$ of $\hat{G} = E_8(\mathbb{C})$, we describe a square integrable unipotent representation $V_{x,\rho}$ of $G = E_8$. We give formal degree of $V_{x,\rho}$ and the action of the hyperspecial parahoric K (in some cases just the leading K-type) on the invariants of its pro-unipotent radical.

(13.1) There are five relevant affine Hecke algebras: The Iwahori-spherical algebra of E_8 (with 53 L^2-representations), $\mathcal{H}(E_8/E_7)$ (with two L^2-representations π_s, π_0 for each of the two cuspidal unipotents for E_7), $\mathcal{H}(E_8/E_6)$ (five L^2-representations for each of the two cuspidal unipotent representations of E_6), $\mathcal{H}(E_8/D_4)$ (at least, and we expect, exactly 18 L^2 representations) and the algebra $\mathbb{C}$, for the 13 cuspidal unipotent representations of E_8. For the first algebra, we used (8.1a), the calculations of Beynon-Spaltenstein [BS], the induce/restrict tables of Alvis [A1], and (9.5). For the algebras $\mathcal{H}(E_8/E_6)$ and $\mathcal{H}(E_8/D_4)$ we used §9, and we give the weight diagrams of their square-integrable modules in (13.2) below.

The simplified degree formula (with $vol(\mathcal{I}) = 1$) is

$$\begin{aligned}\deg(V) = {} & \frac{\dim[V^*_{E_8}]}{E_8(q)} + \frac{\dim[V_{A_1} \otimes V^*_{E_7}]}{A_1E_7(q)} + \frac{\dim[V_{A_2} \otimes V^*_{E_6}]}{A_2E_6(q)} + \frac{\dim[V_{A_3} \otimes V^*_{D_5}]}{A_3D_5(q)} \\ & + \frac{\dim[V_{A_4} \otimes V^*_{A_4}]}{A_4A_4(q)} + \frac{\dim[V_{A_5} \otimes V^*_{A_1} \otimes V^*_{A_2}]}{A_1A_2A_5(q)} + \frac{\dim[V_{A_7} \otimes V^*_{A_1}]}{A_7A_1(q)} \\ & - \frac{\dim[V_{A_6} \otimes V^*_{A_1}]}{A_6A_1(q)} + \frac{\dim[V_{A_8}]}{A_8(q)} - \frac{\dim[V_{A_7}]}{A_7(q)} + \frac{\dim[V_{D_8}]}{D_8(q)}.\end{aligned}$$

$$\mathbf{u} = \mathbf{E_8}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{0}, \quad \mathbf{A_u} = \mathbf{1}$$

$[u \in \hat{G}_s, \rho]$	$K-$ types	Degree
$[E_8, 1]$	1_{120}	$\frac{\Phi_1^8\Phi_{11}\Phi_{13}\Phi_{17}\Phi_{19}\Phi_{23}\Phi_{29}}{\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}}$

$$\mathbf{u} = \mathbf{E_8(a_1)}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{1}, \quad \mathbf{A_u} = \mathbf{1}$$

$[E_8(a_1), 1]$	$1_{120} + 8_{91}$	$\frac{q\Phi_1^8\Phi_7\Phi_{11}\Phi_{13}\Phi_{17}\Phi_{19}\Phi_{23}}{\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}}$

$$\mathbf{u} = \mathbf{E_8(a_2)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{2}, \quad \mathbf{A_u} = \mathbf{1}$$

$[E_8(a_2), 1]$	$35_{74} + 8_{91} + 1_{120}$	$\frac{q^2\Phi_1^8\Phi_7\Phi_{11}^2\Phi_{13}\Phi_{17}\Phi_{19}}{\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{20}}$

$$\mathbf{u} = \mathbf{E_8(a_3)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{3}, \quad \mathbf{A_u} = \mathbf{S_2}$$

$[E_8(a_3), +]$	$112_{63} + 35_{74} + 8_{91} + 1_{120}$	$\frac{q^3\Phi_1^8\Phi_{11}\Phi_{13}^2\Phi_{17}}{2\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_{10}^2\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}}$
$[E_8(a_3), -]$	$28_{68} + 8_{91}$	same
$[A_1E_7, +]$	$84_{64} + 35_{74} + 1_{120}$	$\frac{q^3\Phi_1^8\Phi_{11}\Phi_{13}\Phi_{16}\Phi_{17}\Phi_{26}}{2\Phi_2^8\Phi_3^2\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{20}\Phi_{30}}$
$[A_1E_7, -]$	$D_4(1, 24)$	same

$$\mathbf{u} = \mathbf{E_8(a_4)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{4}, \quad \mathbf{A_u} = \mathbf{S_2}$$

$[E_8(a_4), +]$	$210_{52} + [E_8(a_3), +]$	$\frac{q^4\Phi_1^8\Phi_7^3\Phi_9\Phi_{11}^2\Phi_{13}}{2\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{15}}$
$[E_8(a_4), -]$	$160_{55} + 28_{68} + 35_{74} + 8_{91}$	same
$[D_8, +]$	$50_{56} + 84_{64} + 1_{120}$	$\frac{q^4\Phi_1^8\Phi_7\Phi_9\Phi_{11}\Phi_{13}\Phi_{22}\Phi_{28}}{2\Phi_2^8\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}^2\Phi_{20}\Phi_{24}\Phi_{30}}$
$[D_8, -]$	$D_4(2, 16)'' + D_4(1, 24)$	same

$$\mathbf{u} = \mathbf{E_8(b_4)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{5} \quad \mathbf{A_u} = \mathbf{S_2}$$

$[E_8(b_4), +]$	$560_{47} + 84_{64} + [E_8(a_4, +)]$	$\frac{q^5\Phi_1^8\Phi_5^3\Phi_{11}\Phi_{13}}{2\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_9\Phi_{12}^2\Phi_{14}}$
$[E_8(b_4), -]$	50_{56}	same
$[A_1E_7(a_1), +]$	$[E_8(b_4), 1] - 210_{52} + 160_{55}$	$\frac{q^5\Phi_1^8\Phi_5\Phi_{11}\Phi_{13}\Phi_{20}}{2\Phi_2^8\Phi_3^2\Phi_4^4\Phi_6^2\Phi_8^2\Phi_{12}\Phi_{14}\Phi_{18}\Phi_{24}}$
$[A_1E_7(a_1), -]$	$D_4(2, 16)''$	same

$\mathbf{u = E_8(a_5)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{6}, \quad \mathbf{A_u = S_2}$

$[E_8(a_5),+]$	$700_{42}+[E_8(b_4),+]$	$\frac{q^6\Phi_1^8\Phi_5^3\Phi_7\Phi_{11}}{2\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{12}^2}$
$[E_8(a_5),-]$	$300_{44}+160_{55}+35_{74}$	same

$[D_8(3,13),+]$	$400_{43}+560_{47}+50_{56}+112_{63}+84_{64}+8_{91}+1_{120}$	$\frac{q^6\Phi_1^8\Phi_5\Phi_7\Phi_9\Phi_{11}\Phi_{20}}{2\Phi_2^8\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{12}^2\Phi_{18}\Phi_{24}}$
$[D_8(3,13),-]$	$D_4(1,12)''+D_4(2,16)''$	same

$\mathbf{u = E_8(b_5)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{7}, \quad \mathbf{A_u = S_3}$

$[E_8(b_5),3]$	$1400_{37}+300_{44}+567_{46}+160_{55}+112_{63}+35_{74}+[E_8(a_5),+]$	$\frac{q^7\Phi_1^8\Phi_7^3\Phi_{11}}{6\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}}$
$[E_8(b_5),21]$	$1008_{39}+567_{46}+210_{52}+112_{63}+[E_8(a_4),-]$	$\frac{q^7\Phi_1^8\Phi_7^3\Phi_{11}}{3\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}}$
$[E_8(b_5),111]$	$56_{49}+28_{68}$	$\frac{q^7\Phi_1^8\Phi_7^3\Phi_{11}}{6\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_9\Phi_{10}^2\Phi_{12}}$

$[A_1E_7(a_2),+]$	$1344_{38}-112_{63}+84_{64}-35_{74}+[E_8(b_5),3]$	$\frac{q^7\Phi_1^8\Phi_7^2\Phi_{11}\Phi_{14}\Phi_{16}}{2\Phi_2^8\Phi_3^2\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2\Phi_{18}\Phi_{20}}$
$[A_1E_7(a_2),-]$	$D_4(4,13)+D_4(1,24)$	same

$[A_2E_6,1]$	$448_{39}+560_{47}+112_{63}+700_{42}+300_{44}+84_{64}+35_{74}+1_{120}$	$\frac{q^7\Phi_1^8\Phi_7\Phi_{11}\Phi_{21}}{3\Phi_2^2\Phi_3^4\Phi_6^4\Phi_9\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{30}}$
$[A_2E_6,\theta]$	$E_6[\theta](1,6)$	same
$[A_2E_6,\theta^2]$	$E_6[\theta^2](1,6)$	same

$\mathbf{u = E_8(a_6)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{8}, \quad \mathbf{A_u = S_3}$

$[E_8(a_6),3]$	$1400_{32}+1400_{37}+400_{43}+567_{46}+112_{63}+[E_8(a_5),+]$	$\frac{q^8\Phi_1^8\Phi_3^5\Phi_9}{6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_8^2\Phi_{10}^2}$
$[E_8(a_6),21]$	$1575_{34}+560_{47}+[E_8(b_5),21]$	$\frac{q^8\Phi_1^8\Phi_3^5\Phi_9}{3\Phi_2^8\Phi_4^4\Phi_5^2\Phi_8^2\Phi_{10}^2}$
$[E_8(a_6),111]$	$350_{38}+56_{49}+160_{55}+28_{68}$	$\frac{q^8\Phi_1^8\Phi_3^5\Phi_9}{6\Phi_2^8\Phi_4^4\Phi_5^2\Phi_8^2\Phi_{10}^2}$

$[D_8(5,11),+]$	$\mathrm{Ind}_{D_8}^{E_8}([1^8,\cdot]+[1^7,1]+[1^6,11]) = 1050_{34}+\cdots$	$\frac{q^8\Phi_1^8\Phi_3^3\Phi_7\Phi_9\Phi_{12}}{2\Phi_2^8\Phi_4^4\Phi_8^2\Phi_{10}^2\Phi_{14}\Phi_{20}}$
$[D_8(5,11),-]$	$D_4(8,9)''+D_4(4,13)+D_4(2,16)''+D_4(1,24)$	same

$[A_8,1]$	$175_{36}+448_{39}+700_{42}+400_{43}+112_{63}+84_{64}+1_{120}$	$\frac{q^8\Phi_1^8\Phi_9\Phi_{27}}{3\Phi_3^4\Phi_6^4\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}}$
$[A_8,\theta]$	$E_6[\theta](1,3)''+E_6[\theta](1,6)$	same
$[A_8,\theta^2]$	$E_6[\theta^2](1,3)''+E_6[\theta^2](1,6)$	same

$\mathbf{u = D_7(a_1)}, \quad \dim \mathcal{B}_\mathbf{u} = \mathbf{9}, \quad \mathbf{A_{su} = S_2^2}$

$[A_1E_7(a_3),++]$	$3240_{31}+1344_{38}+400_{43}+560_{47}+84_{64}+[E_8(b_5),3]$	$\frac{q^9\Phi_1^8\Phi_5\Phi_7^2}{2\Phi_2^8\Phi_3^2\Phi_4^4\Phi_6^4\Phi_{12}^2\Phi_{18}}$
$[A_1E_7(a_3),-+]$	$1575_{34}+350_{38}+567_{46}+[A_1E_7(a_1),1]-[A_1E_7,1]$	same
$[A_1E_7(a_3),+-]$	$D_4(1,12)'$	same
$[A_1E_7(a_3),-+]$	$D_4(8,9)''+D_4(2,16)''$	same

$\mathbf{u=E_8(b_6)}$, $\dim\mathcal{B}_\mathbf{u}=\mathbf{10}$, $\mathbf{A_u=S_3}$

$[E_8(b_6),3]$	$2240_{28}+3240_{31}+1344_{38}+448_{39}+700_{42}+560_{47}+84_{64}+[E_8(a_6),3]$	$\frac{q^{10}\Phi_1^8\Phi_5^3\Phi_7}{6\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_8^2\Phi_9}$
$[E_8(b_6),21]$	175_{36}	$\frac{q^{10}\Phi_1^8\Phi_5^3\Phi_7}{3\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_8^2\Phi_9}$
$[E_8(b_6),111]$	$840_{31}+350_{38}+300_{44}+160_{55}$	$\frac{q^{10}\Phi_1^8\Phi_5^3\Phi_7}{6\Phi_2^8\Phi_3^4\Phi_4^4\Phi_6^4\Phi_8^2\Phi_9}$
$[D_8(7,9),+]$	$\mathrm{Ind}_{D_8}^{E_8}([1^8,\cdot]+[1^7,1]+[1^6,11]+[1^5,111])\ =\ 1400_{29}+\cdots$	$\frac{q^{10}\Phi_1^8\Phi_5^2\Phi_7\Phi_{10}}{2\Phi_2^8\Phi_4^4\Phi_6^4\Phi_8^2\Phi_{12}^2\Phi_{18}}$
$[D_8(7,9),-]$	$D_4(4,7)''+D_4(8,9)''+D_4(1,12)''+D_4(2,16)''$	same
$[A_2E_6(a_1),1]$	$2240_{28}+\cdots$	$\frac{q^{10}\Phi_1^8\Phi_5\Phi_7\Phi_{15}}{3\Phi_2^2\Phi_3^4\Phi_6^4\Phi_9\Phi_{12}^2\Phi_{18}\Phi_{24}}$
$[A_2E_6(a_1),\theta]$	$E_6[\theta](1,3)''$	same
$[A_2E_6(a_1),\theta^2]$	$E_6[\theta^2](1,3)''$	same

$\mathbf{u=A_1E_6(a_1)}$, $\dim\mathcal{B}_\mathbf{u}=\mathbf{11}$, $\mathbf{A_{su}=Z_4}$

$[A_1A_7,1]$	$1400_{29}+\cdots$	$\frac{q^{11}\Phi_1^8\Phi_4\Phi_8\Phi_{16}}{2\Phi_2^8\Phi_6^4\Phi_{10}^2\Phi_{14}\Phi_{18}\Phi_{30}}$
$[A_1A_7,-1]$	$D_4(4,7)''+D_4(8,9)''+D_4(9,10)$ $+D_4(2,16)''+D_4(1,24)$	same
$[A_1A_7,\pm i]$	$E_7[\pm\xi](-)$	same

$$\mathbf{u = D_7(a_2)}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{12}, \quad \mathbf{A_{su} = Z_4}$$

$[A_3D_5, 1]$	$840_{26} + \cdots$	$\frac{q^{12}\Phi_1^8\Phi_7}{2\Phi_2^8\Phi_6^4\Phi_{10}^2\Phi_{18}\Phi_{30}}$
$[A_3D_5, -1]$	$D_4(9,10) + D_4(2,16)' + D_4(1,24)$	same
$[A_3D_5, \pm i]$	$E_7[\pm\xi](-)$	same

$$\mathbf{u = D_5A_2}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{13}, \quad \mathbf{A_{su} = S_2^2}$$

$[A_1E_7(a_4), ++]$	$4536_{23} + \cdots$	$\frac{q^{13}\Phi_1^8\Phi_5^2}{2\Phi_2^8\Phi_3^2\Phi_4^4\Phi_6^4\Phi_{12}^2\Phi_{14}}$
$[A_1E_7(a_4), -+]$	$1400_{29} + 400_{43}$	same
$[A_1E_7(a_4), +-]$	$D_4(4,7)'' + D_4(1,12)''$	same
$[A_1E_7(a_4), --]$	$D_4(2,16)'$	same

$$\mathbf{u = E_8(a_7)}, \quad \dim \mathcal{B}_{\mathbf{u}} = \mathbf{16}, \quad \mathbf{A_u = S_5}$$

$[E_8(a_7),5]$	$4480_{16}+\cdots$	$\frac{q^{16}\Phi_1^8}{120\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),41]$	$5670_{18}+\cdots$	$\frac{q^{16}\Phi_1^8}{30\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),32]$	$4536_{18}+5600_{21}+2800_{25}+2100_{28}+\cdots$	$\frac{q^{16}\Phi_1^8}{24\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),311]$	$1680_{22}+2400_{23}+2100_{28}+2\cdot 1296_{33}+\cdots$	$\frac{q^{16}\Phi_1^8}{20\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),221]$	$1400_{20}+2800_{25}+2268_{30}+1008_{39}+210_{52}$	$\frac{q^{16}\Phi_1^8}{24\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),2111]$	$70_{32}+56_{49}$	$\frac{q^{16}\Phi_1^8}{30\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$
$[E_8(a_7),1^5]$	$E_8^{II}[1]$	$\frac{q^{16}\Phi_1^8}{120\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4}$

$[A_1E_7(a_5),3]$	$7168_{17}+\cdots$	$\frac{q^{16}\Phi_1^8}{12\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$
$[A_1E_7(a_5),21]$	$5600_{19}+\cdots$	$\frac{q^{16}\Phi_1^8}{6\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$
$[A_1E_7(a_5),111]$	$2400_{23}+449_{25}+1296_{33}+1575_{34}+\cdots$	$\frac{q^{16}\Phi_1^8}{12\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$
$[A_1E_7(a_5),-3]$	$D_4(12,4)+D_4(8,9)'+D_4(8,9)''$ $+D_4(9,10)+D_4(4,13)+D_4(1,24)$	$\frac{q^{16}\Phi_1^8}{12\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$
$[A_1E_7(a_5),-21]$	$D_4(6,6)''+D_4(4,13)$	$\frac{q^{16}\Phi_1^8}{6\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$
$[A_1E_7(a_5),-111]$	$E_8[-1]$	$\frac{q^{16}\Phi_1^8}{12\Phi_2^8\Phi_3^2\Phi_6^4\Phi_8^2\Phi_{10}^2\Phi_{12}}$

$\mathbf{u}=\mathbf{E_8(a_7)}$ continued

$[D_8(1357),1]$	$4200_{18}+\cdots$	$\frac{q^{16}\Phi_1^8}{8\Phi_2^8\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2}$
$[D_8(1357),\epsilon'']$	$2688_{20}+\cdots$	$\frac{q^{16}\Phi_1^8}{8\Phi_2^8\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2}$
$[D_8(1357),\epsilon']$	$168_{24}+\ldots$	$\frac{q^{16}\Phi_1^8}{8\Phi_2^8\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2}$
$[D_8(1357),r]$	$D_4(16,5)+D_4(8,9)'+D_4(8,9)''+2D_4(9,10)$	$\frac{q^{16}\Phi_1^8}{4\Phi_2^8\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2}$
	$+D_4(4,13)+D_4(9,6)''+D_4(2,16)''+D_4(1,24)$	
$[D_8(1357),\epsilon]$	$E_8^I[1]$	$\frac{q^{16}\Phi_1^8}{8\Phi_2^8\Phi_4^4\Phi_6^4\Phi_{10}^2\Phi_{12}^2}$
$[A_2E_6(a_3),1]$	$3150_{18}+\cdots$	$\frac{q^{16}\Phi_1^8}{6\Phi_2^8\Phi_3^4\Phi_6^4\Phi_{12}^2\Phi_{15}\Phi_{18}}$
$[A_2E_6(a_3),-1]$	$1134_{20}+\cdots$	same
$[A_2E_6(a_3),\theta]$	$E_6[\theta](2,1)+E_6[\theta](1,6)$	same
$[A_2E_6(a_3),\theta^2]$	$E_6[\theta^2](2,1)+E_6[\theta^2](1,6)$	same
$[A_2E_6(a_3),-\theta]$	$E_8[\theta]$	same
$[A_2E_6(a_3),-\theta^2]$	$E_8[\theta^2]$	same
$[A_1A_2A_5,1]$	$2016_{19}+\cdots$	$\frac{q^{16}\Phi_1^8}{6\Phi_2^2\Phi_3^2\Phi_6^4\Phi_{12}\Phi_{18}\Phi_{24}\Phi_{30}}$
$[A_1A_2A_5,-1]$	$D_4(6,6)'+D_4(8,9)'+D_4(8,9)''+D_4(9,10)+D_4(1,24)$	same
$[A_1A_2A_5,\theta]$	$E_6[\theta](2,2)+E_6[\theta](1,6)$	same
$[A_1A_2A_5,\theta^2]$	$E_6[\theta^2](2,2)+E_6[\theta^2](1,6)$	same
$[A_1A_2A_5,-\theta]$	$E_8[-\theta]$	same
$[A_1A_2A_5,-\theta^2]$	$E_8[-\theta^2]$	same
$[A_3D_5(37),1]$	$1344_{19}+\cdots$	$\frac{q^{16}\Phi_1^8}{4\Phi_4^4\Phi_8^2\Phi_{12}^2\Phi_{20}\Phi_{24}}$
$[A_3D_5(37),-1]$	$D_4(9,6)''+D_4(4,8)+D_4(9,10)+D_4(2,16)'$	same
$[A_3D_5,\pm i]$	$E_8[\pm i]$	same
$[A_4A_4,1]$	$420_{20}+\cdots$	$\frac{q^{16}\Phi_1^8}{5\Phi_5^2\Phi_{10}^2\Phi_{15}\Phi_{20}\Phi_{30}}$
$[A_4A_4,\zeta^j]$	$E_8[\zeta^j]$	same

(13.2) Here discuss the square integrable representations of the Hecke algebras arising in E_8 from non-Iwahori parahoric subgroups. according to their weight diagrams, as discussed in §9. We omit the restrictions to non-hyperspecial parahorics, which are easy to find, as in the examples in (9.7).

HECKE ALGEBRA $\mathcal{H}(E_8/E_7)$: 1—15

Here $\mathbf{G} = PSL_2(\mathbb{C})$ and we identify $\mathbf{T} = \mathbb{C}^\times$ via the positive root α. There are two square integrable $\mathcal{H}(E_8/E_7)$-modules, both one dimensional, with weight diagrams as follows.

$$[A_1A_7, \pm i]: \quad q^{-8}$$
$$[A_3D_5, \pm i]: \quad -q^{-7}$$

HECKE ALGEBRA $\mathcal{H}(E_8/E_6)$: 1—1 ⇛ 9

Here $\mathbf{G} = G_2(\mathbb{C})$, with simple roots α_1, α_2, and α_1 is short. Use (α_1, α_2) to identify $\mathbf{T} = (\mathbb{C}^\times)^2$. The weight diagrams of square integrable representations of $\mathcal{H}(P, \sigma)$ and corresponding Langlands parameters of E_8 are as follows (same diagrams with θ^2).

$$[A_2E_6, \theta]: \quad (q^{-1}, q^{-9})$$
$$[A_8, \theta]: \quad (\theta, q^{-9}) \overset{1}{—} (\theta^2, q^{-9})$$
$$[A_2E_6(a_1), \theta]: \quad (q, q^{-9})$$
$$[A_2E_6(a_3), \theta]: \quad (q^4, q^{-9}) \overset{1}{—} (q^{-4}, q^3) \overset{2}{—} (q^{-1}, q^{-3})$$
$$[A_1A_2A_5, \theta]: \quad (-q^4, -q^{-9}) \overset{1}{—} (-q^{-4}, -q^3) \overset{2}{—} (q^{-1}, -q^{-3})$$

Arguing as in (12.2), one checks that these exhaust all square-integrable modules over $\mathcal{H}(E_8/E_6)$.

HECKE ALGEBRA $\mathcal{H}(E_8/D_4)$: 1—1—1 ⇒ 4—4

Here $\mathbf{G} = F_4(\mathbb{C})$, with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_1, α_2 are short. Use $(e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, e_{\alpha_4})$ to identify $\mathbf{T} = (\mathbb{C}^\times)^4$. We have $e_{\alpha_i}(\tau) = \epsilon_i q^{-t_i}$ with $|\epsilon_i| = 1$, and $t_i \in \mathbb{R}$. If all $\epsilon_i = 1$, then we write $\tau = [t_1, t_2, t_3, t_4]$. If some $\epsilon_i = -1$, then write $\bar{t}_i$. A parameter c in the weight diagram indicates that the module is obtained by deformation from the parameter $c = 1$ to $c = 4$, remaining L^2 all the while.

$[A_1E_7, -]:$ $[1, 1, c, c]$

$[D_8, -1]:$ $[-\bar{1}, 1, c, c] \overset{1}{—} [\bar{1}, \bar{0}, c, c] \overset{2}{—} [1, \bar{0}, c, c]$

$[A_1E_7(a_1), -]:$ $[2, -1, c, c] \overset{1}{—} [-2, 1, c, c]$

$[D_8(3, 13), -]:$ $[\bar{1}, -1, c, c] \overset{1}{—} [-\bar{1}, \bar{0}, c, c] \overset{2}{—} [-1, \bar{0}, c, c]$

$[A_1E_7(a_2), -]:$ $[-c, 1, c, c] \overset{1}{—} [c, 1-c, c, c] \overset{2}{—} [1, c-1, 2-c, c] \overset{3}{—} [1, 1, c-2, -2] \overset{4}{—} [1, 1, c, 2]$

$[A_1E_7(a_3), +-]:$ $[-1, -1, c, c]$

$[A_1E_7(a_4), +-]:$

$[-4, -1, 4, 4] \overset{1}{—} [4, -4-1, 4, 4] \overset{2}{—} [-1, 4+1, -4-2, 4] \overset{3}{—} [-1, -1, 4+2, -2] \overset{4}{—} [-1, -1, 4, 2]$

$[A_1E_7(a_4), --]:$ $[1, 1, -c, 2c] \overset{4}{—} [1, 1, c, -2c]$

$[A_1A_2A_5, -1]:$ Deformation of $[A_2A_2, 1]$, graph omitted, see (9.8)

$[A_1E_7(a_3), -+]:$
(The upper right component has multiplicity two, the others have multiplicity one.)

$[-1,-1,4,4]$						$[1,-2,4,4]$	—2—	$[-1,2,0,4]$
$[-7,1,4,4]$	—1—	$[7,-6,4,4]$	—2—	$[1,6,-8,4]$	—3—	$[1,-2,8,-4]$	—2—	$[-1,2,4,-4]$

$[A_1E_7(a_5), -21]:$

$[1,1,c-2,-c]$	—3—	$[1,c-1,2-c,-2]$	—2—	$[c,1-c,c,-2]$	—1—	$[-c,1,c,-2]$
		|4		|4		|4
		$[1,c-1,-c,2]$	—2—	$[c,1-c,c-2,2]$	—1—	$[-c,1,c-2,2]$
				|3		|3
				$[c,-1,2-c,c]$	—1—	$[-c,c-1,2-c,c]$
						|2
						$[-1,1-c,c,c]$

$[A_1E_7(a_5), -3]$: (Weights in the first row have multiplicity four, those in the second have multiplicity two, and those in the remaining component have multiplicity one.)

		$[0,1,0,c-2]$	—4—	$[0,1,c-2,2-c]$	—3—	$[0,c-1,2-c,0]$	—2—	$[c-1,1-c,c,0]$	—1—	$[1-c,0,c,0]$
$[1,1,-2,c]$	—3—	$[1,-1,2,c-2]$	—4—	$[1,-1,c,2-c]$		$[c-1,1,-c,c]$	—1—	$[1-c,c,-c,c]$	—2—	$[1,-c,c,c]$
				$[-1,1-c,c,c]$	—2—	$[-c,c-1,2-c,c]$	—3—	$[-c,1,c-2,2]$	—4—	$[-c,1,c,-2]$
						|1		|1		|1
						$[c,-1,2-c,c]$	—3—	$[c,1-c,c-2,2]$	—4—	$[c,1-c,c,-2]$
								|2		|2
								$[1,c-1,-c,2]$	—4—	$[1,c-1,2-c,-2]$
										|3
										$[1,1,c-2,-c]$

$[A_3D_5,-1]:$

(deformation of $[A_1C_3,+]$)

$$
\begin{array}{ccccc}
[1,1,c,\overline{-3-c}] & & & & \\
\Big|4 & & & & \\
[1,1,-\bar{3},\overline{3+c}] & & & & \\
\Big|3 & & & & \\
[1,-\bar{2},\bar{3},c] & & & & \\
\Big|2 & & & & \\
[-\bar{1},\bar{2},-\bar{1},c] & \overset{1}{—} & [\bar{1},1,-\bar{1},c] & & [1,1,-\bar{1},c] \\
\Big|3 & & \Big|3 & & \Big|3 \\
[-\bar{1},1,\bar{1},\overline{c-1}] & \overset{1}{—} & [\bar{1},\bar{0},\bar{1},\overline{c-1}] & \overset{2}{—} & [1,\bar{0},\bar{1},\overline{c-1}] \\
\Big|4 & & \Big|4 & & \Big|4 \\
[-\bar{1},1,c,\overline{1-c}] & \overset{1}{—} & [\bar{1},\bar{0},c,\overline{1-c}] & \overset{2}{—} & [1,\bar{0},c,\overline{1-c}]
\end{array}
$$

$[D_8(5,11),-]:$

$$
\begin{array}{ccccc}
[-\bar{3},\bar{0},4,4] & \overset{2}{—} & [-3,\bar{0},4,4] & & \\
\Big|1 & & \Big|1 & & \\
[\bar{3},-3,4,4] & & [3,-\bar{3},4,4] & & \\
\Big|2 & & \Big|2 & & \\
[\bar{0},3,-2,4] & \overset{1}{—} & [\bar{0},\bar{3},-2,4] & & \\
\Big|3 & & \Big|3 & & \\
[\bar{0},1,2,2] & \overset{1}{—} & [\bar{0},\bar{1},2,2] & \overset{2}{—} & [1,-\bar{1},4,2] \\
\Big|4 & & \Big|4 & & \Big|4 \\
[\bar{0},1,4,-2] & \overset{1}{—} & [\bar{0},\bar{1},4,-2] & \overset{2}{—} & [1,-\bar{1},6,-2] \\
 & & & & \Big|3 \\
[-\bar{6},1,4,4] & \overset{1}{—} & [\bar{6},-\bar{5},4,4] & \overset{2}{—} & [1,\bar{5},-6,4]
\end{array}
$$

$[D_8(7,9),-]:$

$$
\begin{array}{ccccc}
[-\bar{5},\bar{0},4,4] & \overset{2}{—} & [-5,\bar{0},4,4] & & \\
\Big|1 & & \Big|1 & & \\
[\bar{5},-5,4,4] & & [5,-\bar{5},4,4] & & \\
\Big|2 & & \Big|2 & & \\
[\bar{0},5,-6,4] & \overset{1}{—} & [\bar{0},\bar{5},-6,4] & & \\
\Big|3 & & \Big|3 & & \\
[\bar{0},-1,6,-2] & \overset{1}{—} & [\bar{0},-\bar{1},6,-2] & \overset{2}{—} & [-1,\bar{1},4,-2] \\
\Big|4 & & \Big|4 & & \Big|4 \\
[\bar{0},-1,4,2] & \overset{1}{—} & [\bar{0},-\bar{1},4,2] & \overset{2}{—} & [-1,\bar{1},2,2] \\
 & & & & \Big|3 \\
[-\bar{2},-1,4,4] & \overset{1}{—} & [\bar{2},-\bar{3},4,4] & \overset{2}{—} & [-1,\bar{3},-2,4]
\end{array}
$$

$[A_1A_7,-1]$: (Deformation of $[A_1A_3,1]$. Here, $\tau = (q^{-1}, iq^{-\frac{3c+1}{2}}, q^{-c}, q^{-c}) \in \mathbf{T}$, and the other vertices τ^w are labelled by w. There is only one component.)

τ

2

23 232

234 2342

21 232432 23243

321 2134 2324321 232431 23241 2321

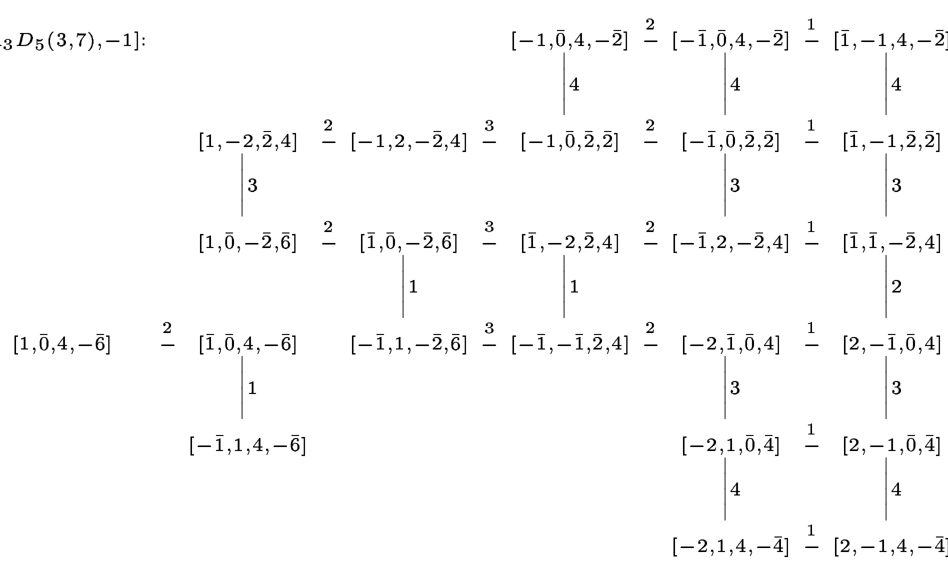
23243212 2324312

21342 213421

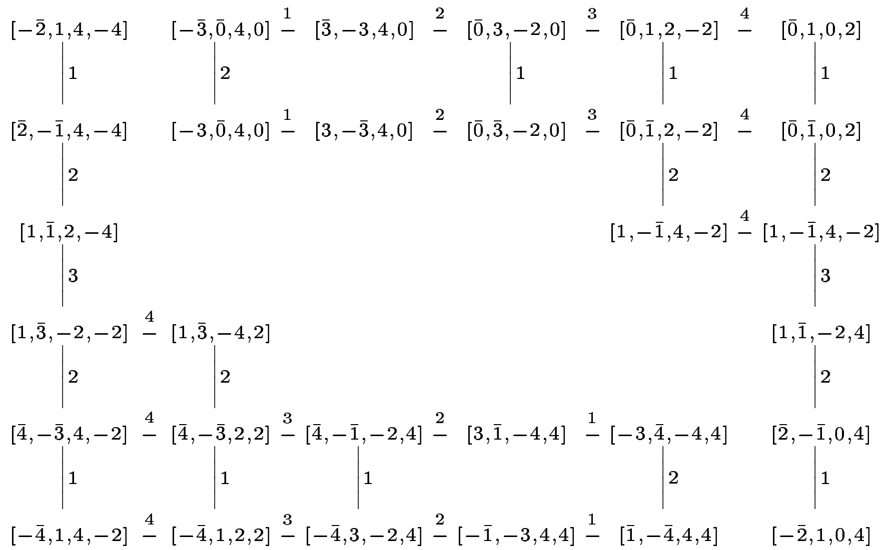
213424 2134214

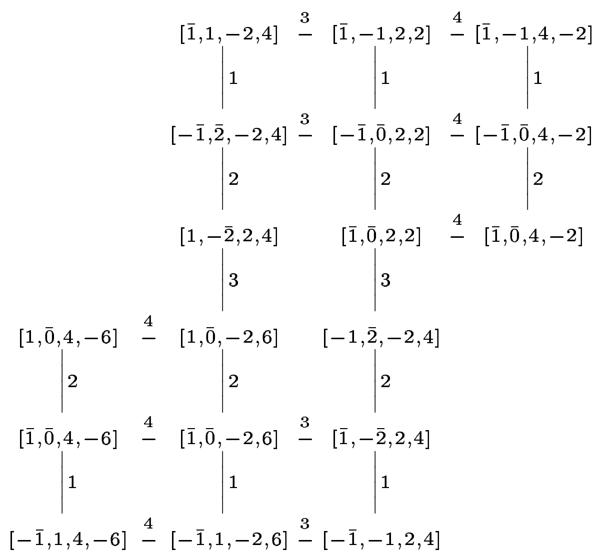
2134243 21342143 213421432

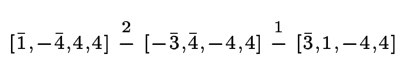
$[A_3D_5(3,7),-1]$:



$[D_8(1357), r]$ (There are four components, the upper right has weights of multiplicity two, the other three components have multiplicity one.)







14. Leading K-types

The above calculations lead us to observe a relation between Langlands parameters and Lusztig's families of unipotent representations.

(14.1) First, some remarks on truncated induction [C,11.2]. Let W be a finite Coxeter group, and let R denote its reflection representation. For any finite dimensional W module χ, let $b(\chi)$ be the minimum degree in which some constituent of χ appears in the symmetric algebra $S^\bullet R$. Let $\hat{W}$ be the set of irreducible representations of W, let $\langle\ ,\ \rangle_W$ denote multiplicities, and define

$$j(\chi) := \sum_{\substack{\psi\in\hat{W}\\ b(\psi)=b(\chi)}} \langle\psi,\chi\rangle_W\,\psi.$$

Let W_s be a subgroup of W generated by reflections. For simplicity, since it is the case in our applications, assume that W_s has the same rank as W. Then R is also the reflection representation of W_s. Let φ be a representation of W_s. Then

$$b(\operatorname{Ind}_{W_s}^W \varphi) = b(\varphi). \tag{14.1a}$$

Suppose $\varphi = \varphi_b \oplus \varphi'$, where $b = b(\varphi) < b(\varphi')$. If $\psi \in \hat{W}$ has $b(\psi) = b$, then $\langle\varphi',\psi\rangle_{W_s} = 0$. It follows that

$$j(\operatorname{Ind}_{W_s}^W \varphi) = j(\operatorname{Ind}_{W_s}^W \varphi_b) \tag{14.1b}$$

If we suppose in addition that φ_b is irreducible and has multiplicity one in S^bR, then the right side of (14.1b) is irreducible for W, and has multiplicity one in $\operatorname{Ind}_{W_s}^W \varphi$.

(14.2) Now suppose V is a square-integrable unipotent representation of our p-adic group G, containing the cuspidal unipotent pair (P,σ). Let $\widetilde{W} = WX$ be the affine Weyl group underlying $\mathcal{H}(G/P)$, as in (5.2). Then we define

$$J(V) := j\big([V^\sigma]_{q=1}\big),$$

where $[V^\sigma]_{q=1}$ is viewed as a W-module by restricting from $\widetilde{W}$. If $P < K$, then the constituents of $J(V)$ correspond to unipotent representations of K contained in $\operatorname{Ind}_P^K \sigma$, and we call $J(V)$, or its corresponding K-module, the "leading K-type" of V.

From our tables, we observe the following

Proposition. *If G is exceptional with V unipotent square integrable, then the leading K-type is irreducible with multiplicity one in V. If $V^{\mathcal{I}} \neq 0$, so that W is the Weyl group of G, then*

$$J(V) = j(\epsilon\otimes\varphi_{s,u,\rho}),$$

where $V = V_{su,\rho}$ and $\varphi_{s,u,\rho}$ is the Springer representation of W_s on $H^{top}(\mathcal{B}_s^u)^\rho$.

(14.3) We must point out that Proposition (14.2) does not always hold for classical groups. In the Iwahori-spherical case, it can fail only when W_s has a direct factor of type D_{2n}, $n \geq 4$, and then it does not always fail. Rather than try to sort this out here, we merely observe that Proposition (14.2) is true for those representations admitting a Whittaker model. In fact, we have the

Proposition. *Suppose G is split adjoint, and V is a unipotent, square-integrable, generic irreducible representation of G. Then V is Iwahori spherical, its leading K-type is irreducible with multiplicity one in V, and*

$$J(V) = j(\varphi_{s,u',1}), \tag{14.3a}$$

where u' belongs to the orbit dual to that of a unipotent $u \in \hat{G}_s$ such that su is elliptic in $\hat{G}$. Moreover, V is the unique constituent of $\mathrm{Ind}_B^G(\tau)$ admitting a Whittaker model, where τ is constructed from $x = su$ as in (7.2). In particular, the conjugacy class of x in $\hat{G}$ is canonically determined by $J(V)$. Finally, we have $b[J(V)] = \dim \mathcal{B}_s^{u'}$.

Proof. The first assertion is [R3]. Equation (14.3a) and the last assertion follow from Lusztig's theorem (8.1) along with [BM, 10.4], and the fact that an elliptic $u \in \hat{G}_s$ is special, and is thus subject to the duality involution on special unipotent orbits in $\hat{G}_s$, which the Springer correspondence translates to tensoring by the sign character. The assertion about $\mathrm{Ind}_B^G(\tau)$ is [R6,10.1]. □

The next section is a partial generalization of this to the other representations occuring in Proposition (14.2).

(14.4) We now compare the leading K-types in the p-adic L-packets with Lusztig's classification of unipotent representations of the finite group $K/K_1 = G(\mathbb{F}_q)$. The latter are partitioned into families $\mathcal{F}_u$, one family for each special unipotent class $[u]$ in $\hat{G}$. According to [L3, (13.1)], there is a canonical quotient Γ_u of A_u, with the following property: Let $\mathcal{M}_u$ be the set of pairs (s, ϱ), taken up to conjugacy in Γ_u, where $s \in \Gamma_u$ and ϱ is a representation of the centralizer Γ_u^s of s in Γ_u. Then we have a family

$$\mathcal{F}_u := \{\chi_{s,\varrho}^u : \ (s, \varrho) \in \mathcal{M}_u\},$$

of unipotent representations of $G(\mathbb{F}_q)$, such that the multiplicity of $\chi_{s,\varrho}^u$ in the "almost characters" R_φ, $\varphi \in \hat{W}$, (cf. [C, (12.3)]) is given by a natural pairing on the set $\mathcal{M}_u$ [L3, Thm 4.23]. Lusztig's explicit calculations of the $\chi_{s,\varrho}^u$ are summarized in [C, §13].

Let $\gamma : A_u \longrightarrow \Gamma_u$ be the canonical quotient mapping. Suppose $x = su$ is elliptic in $\hat{G}$. Then $A_x = M^s/Z$ (notation as in §7) and $A_u = M/ZM_0$, so we have a natural map $f : A_x \longrightarrow A_u$ whose image is contained in the centralizer A_u^s of $f(s)$ in A_u. Let $\bar{s} = \gamma(f(s))$, let $\gamma_s : A_x \longrightarrow \Gamma_u^s$ be the composition $\gamma_s = \gamma \circ f$, and let $\hat{\gamma}_s : (\Gamma_u^s)\hat{} \longrightarrow \hat{A}_x$ be the transposed map on irreducible representations. From our tables, we observe the

Proposition. *Assume G is exceptional and A_x is not cyclic of order four. Then the correspondence $(x, \rho) \mapsto V_{x,\rho}$ has the property*

$$J(V_{su,\hat{\gamma}_s\varrho}) = \chi_{\bar{s},\varrho}^u, \tag{14.4a}$$

for all $\varrho \in (\Gamma_u^s)\hat{}$.

Property (14.4a) fails in all three instances of $A_x \simeq \mathbb{Z}_4$, which occur for $G = E_7$ with $u = A_4A_1$, $G = E_8$ with $u = A_1E_6(a_1)$, and $G = E_8$ with $u = D_7(a_2)$. These are precisely the cases where G contains a parahoric P of type E_7 and the packet Π_x has a representation containing a pair $(P, E_7[\pm\xi])$.

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