SMALL REPRESENTATIONS AND
MINUSCULE RICHARDSON ORBITS

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May 2001

1. Introduction

(i) Let $G$ be a simple, adjoint algebraic group over $\mathbb{C}$, with maximal torus $T$ and Weyl group $W$. Let $V_\mu$ be an irreducible representation of the simply-connected cover $G^{sc}$ of $G$ with highest weight $\mu$. Assume $V_\mu$ is minuscule, that is, the weights of $V_\mu$ form a single $W$-orbit. Kumar [Ku] has found the decomposition of tensor products where one factor is a minuscule $V_\mu$. Specialized to the case $\text{End}(V_\mu) = V_\mu \otimes V_\mu^*$, which is now a representation of $G$, Kumar’s results give a bijection between the irreducible constituents of $\text{End}(V_\mu)$ and double cosets $W_\mu \backslash W / W_\mu$, where $W_\mu$ is the stabilizer of $\mu$ in $W$. Namely, to $W_\mu w W_\mu$ corresponds the representation with extreme weight $\mu - w \mu$.

(ii) An irreducible representation $V_\lambda$ of $G$ is called small if twice a root is not a weight in $V_\lambda$. Small representations are distinguished by having remarkable relations, some proven, some conjectural, between the multiplicities of $V_\lambda$ in certain natural $G$-representations, and the multiplicities of the zero weight space $V_\lambda^0$ in certain natural $W$-representations. See for example, [B1-3], [K,§5], [R1-3], [So], [ST].

For minuscule $V_\mu$, it is easy to see (§2 below) that any constituent of $\text{End}(V_\mu)$ is small, and that the zero weight space $\text{End}_T(V_\mu)$ is isomorphic to the permutation module $\text{Ind}^W_{W_\mu} \mathbb{C}$, whose constituents are in bijection with double cosets $W_\mu \backslash W / W_\mu$.

(iii) Let $\hat{G}$, $\hat{T}$, be the dual groups of $G$ and $T$. Then the weight $\mu$ may be viewed as a dominant co-weight of the adjoint group $\hat{G}^{ad}$ of $\hat{G}$. Let $\hat{L}_\mu$ be the centralizer of $\mu$ in $\hat{G}$ (under the conjugation action of $G$ on $\hat{G}^{ad}$) and let $\hat{P}_\mu = \hat{L}_\mu \hat{U}_\mu$ be a parabolic subgroup of $\hat{G}$ with Levi $\hat{L}_\mu$. Let $R_\mu$ be the Richardson class of $\hat{P}_\mu$ in $\hat{G}$. This is the unique unipotent class in $\hat{G}$ containing a dense subset of $\hat{U}_\mu$. Richardson, Rohlle and Steinberg [RRS] made a detailed study of the classes contained in the closure $\overline{R}_\mu$, and obtained, among others, the following results. Let $\hat{W}$ be the Weyl
group of \( \hat{T} \) in \( \hat{G} \), and let \( \hat{W}_\mu \) be the centralizer of \( \mu \). Then the orbits in \( \overline{R}_\mu \) are in bijection with the double cosets \( \hat{W}_\mu \backslash \hat{W} / \hat{W}_\mu \). In fact, the orbits and double coset representatives may be constructed in parallel ways, as follows [RRS, Thm 1.2]. Let \( \{ \beta_1, \ldots, \beta_r \} \) be a maximal orthogonal set of long roots of \( \hat{T} \) in \( \hat{U} \). Such sets are unique up to \( \hat{W}_\mu \)-conjugacy. Let \( w_\beta \in \hat{W} \) be the reflection for \( \beta \). Then

\[
w_{\beta_1} \cdots w_{\beta_k}, \quad k = 1, \ldots r
\]
is a complete system of nontrivial double coset representatives for \( \hat{W}_\mu \backslash \hat{W} / \hat{W}_\mu \). On the other hand, for each \( i \), let \( u_{\beta_i} \) be a non-trivial element in the \( \beta_i \)-root group in \( \hat{U}_\mu \). Then

\[
u_{\beta_1} \cdots u_{\beta_k}, \quad k = 1, \ldots r
\]
is a complete system of representatives for the nontrivial \( \hat{G} \)-orbits in \( \overline{R}_\mu \).

(iv) Let \( u \) be a unipotent element of \( \hat{G} \), and let \( \mathcal{B}_u \) denote the variety of Borel subgroups of \( \hat{G} \) containing \( u \). It is known that every irreducible component of \( \mathcal{B}_u \) has the same dimension \( d(u) \). Let \( H^*(\mathcal{B}_u) \) denote the singular cohomology of \( \mathcal{B}_u \) with complex coefficients. The cohomology is zero in odd degrees [CLP], and the highest nonvanishing degree is \( 2d(u) \). Springer has defined an action of \( \hat{W} \) on \( H^*(\mathcal{B}_u) \) [Spr]. If \( u \) is regular in \( \hat{L}_\mu \), then by a result of Lusztig we have an isomorphism of \( \hat{W} \)-modules

\[
H^*(\mathcal{B}_u) \simeq \text{Ind}^{\hat{W}}_{\hat{W}_\mu} \mathbb{C}.
\]

In particular there is a geometrically defined \( \hat{W} \)-grading on the permutation representation \( \text{Ind}^{\hat{W}}_{\hat{W}_\mu} \mathbb{C} \).

The first main result in this paper gives some connections between the facts recalled in (i-iv).

The central theme is the attachment of a representation of \( G \) to a unipotent class \( C \) in \( \hat{G} \). Given \( C \), let \( \varphi : SL_2 \to \hat{G} \) be a homomorphism mapping the nontrivial unipotent elements of \( SL_2 \) into \( C \). Let \( \lambda(t) = \varphi \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right) \), for \( t \in \mathbb{C}^\times \). After possibly conjugating \( \varphi \), we may assume that \( \lambda \) is a dominant co-weight of \( \hat{T} \). View \( \lambda \) as a weight of the dual torus \( T \) in \( G \), and let \( V_\lambda \) be the irreducible \( G \)-module with highest weight \( \lambda \). We say that \( V_\lambda \) is attached to \( C \). For example, the adjoint representation of \( G \) is attached to the orbit of a short root element in \( \hat{G} \).

Two natural Weyl-group representations arise. On the one hand, we have the \( \hat{W} \)-action on the zero-weight space \( V_\lambda^0 \). On the other, we have the Springer representations \( \chi_{u, \rho} \) of \( \hat{W} \) associated to \( u \in C \). Here \( \rho \) is an irreducible representation of the component group \( A_u \) of the centralizer of \( u \) in \( \hat{G}^{ad} \), and \( \chi_{u, \rho} = \text{Hom}_{A_u}(\rho, H^{2d(u)}(\mathcal{B}_u)) \).

The root datum defining \( \hat{G} \) in terms of \( G \) gives a canonical isomorphism \( \iota : W \to \hat{W} \) sending short reflections in \( W \) to long reflections in \( \hat{W} \), and vice-versa. Let \( \iota^* \) denote the pull-back map sending representations of \( \hat{W} \) to those of \( W \).
Theorem 1. Let $V_\mu$ be an irreducible minuscule representation of $G^{sc}$ with highest weight $\mu$. Let $\hat{L}_\mu$ be the centralizer of $\mu$ in $G$, let $\hat{P}_\mu$ be a parabolic subgroup of $G$ with Levi factor $\hat{L}_\mu$, and let $R_\mu$ be the Richardson class of $\hat{P}_\mu$. Then the following hold.

(a) $\mathrm{End}(V_\mu)$ is multiplicity-free, and its constituents are exactly those representations of $G$ which are attached to the orbits in the closure of $R_\mu$. More precisely, we may, in (iii) above, choose the maximal orthogonal set of long roots $\{\beta_1, \ldots, \beta_r\}$ in such a way that $\beta_1 + \cdots + \beta_k$ is dominant for each $k = 1, \ldots, r$. Let $\beta_k$ be the corresponding co-roots, and set $\lambda_k = \beta_1 + \cdots + \beta_k$, viewed as a weight of $T$, and put $\lambda_0 = 0$. Then

$$\mathrm{End}(V_\mu) = V_{\lambda_0} \oplus V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r},$$

and $V_{\lambda_k}$ is attached to the class of $u_k := u_{\beta_1} \cdots u_{\beta_k}$.

(b) Each $V_{\lambda_k}$ is self-dual and small.

(c) The zero weight space $V_{\lambda_k}^0$ of each constituent of $\mathrm{End}(V_\mu)$ is irreducible under $W$, and $\mathrm{End}_T(V_\mu) \simeq \mathrm{Ind}_{W_\mu}^W C$. Thus, we have the irreducible decomposition

$$\mathrm{Ind}_{W_\mu}^W C = V_{\lambda_0}^0 \oplus V_{\lambda_1}^0 \oplus \cdots \oplus V_{\lambda_r}^0.$$

(d) In terms of Springer representations, we have

$$V_{\lambda_k}^0 \otimes \epsilon \simeq \iota^* \chi_{u_k, \lambda},$$

where $\epsilon$ is the sign character of $W$, and $u_k$ is the unipotent element in (a).

(e) Assume $G$ is simply laced. Then the grading in (c) is isomorphic to the grading on $H^*(B_x)$ for $x$ regular unipotent in $\hat{L}_\mu$. That is, we have $W$-module isomorphisms

$$V_{\lambda_k}^0 \simeq \iota^* H^{2k}(B_x), \quad k = 0, \ldots, r.$$

(This is false if $G$ is not simply laced.)

The original motivation behind Theorem 1 was to use the theory of minuscule representations to give more uniform proofs of certain facts about small representations for simply-laced groups observed in [R1,2], and to extend these results to the non-simply-laced case. This was partly successful, though item (e) in Theorem 1 is proved case-by-case here. More serious, however, is the unfortunate fact that not all small representations occur in some $\mathrm{End}(V_\mu)$. The rest of the paper therefore puts $\mathrm{End}(V_\mu)$ aside, to study all small representations for all groups. In particular, we reduce the classification of small representations for non-simply laced groups to the simply-laced case, using graph automorphisms and Langlands functoriality, and we describe their zero-weight spaces in terms of Springer representations. The results, which are partial generalizations of those in Theorem 1, have uniform statements, but some of the proofs, mainly those involving the Springer correspondence, still rely on case-by-case calculations.

The basic relation between smallness and being attached to a unipotent class is as follows.
**Proposition 2.** A small representation $V_\lambda$ of $G$ is attached to a unipotent class $C \subset \hat{G}$ if and only if $V_\lambda$ is self-dual.

The small representations and their zero weight spaces were analyzed for simply-laced groups in [R1,2]. To handle multiply-laced groups, we assume, for the rest of this introduction, that $\hat{G}$ is simply-laced, and that $\sigma$ is an automorphism of $\hat{G}$ induced by a symmetry of the Dynkin diagram of $\hat{G}$ (see section 4). Let $\hat{G}_\sigma$ denote the group of fixed points under $\sigma$, and let $G_\sigma$ be the dual group of $\hat{G}_\sigma$. Each non-simply-laced adjoint simple group is uniquely a $G_\sigma$. The individual cases are given as follows.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\hat{G}$</th>
<th>$\hat{G}_\sigma$</th>
<th>$G_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PGL_{2n}$</td>
<td>$SL_{2n}$</td>
<td>$Sp_{2n}$</td>
<td>$SO_{2n+1}$</td>
</tr>
<tr>
<td>$PSO_{2n+2}$</td>
<td>$Spin_{2n+2}$</td>
<td>$Spin_{2n+1}$</td>
<td>$PSp_{2n}$</td>
</tr>
<tr>
<td>$PSO_8$</td>
<td>$Spin_8$</td>
<td>$G_2$</td>
<td>$G_2$</td>
</tr>
<tr>
<td>$E_6^{ad}$</td>
<td>$E_6^c$</td>
<td>$F_4$</td>
<td>$F_4$</td>
</tr>
</tbody>
</table>

(Note that the case $\hat{G} = SL_{2n+1}$, $\hat{G}_\sigma = SO_{2n+1}$ is excluded by the requirement that $\hat{G}_\sigma$ be simply-connected.)

We say $V_\lambda$ is $\sigma$-stable if $\lambda$, viewed as a co-weight of $\hat{G}$, has image contained in $G_\sigma$.

**Theorem 3.** Let $G$ be a simply-laced adjoint group, and let $G_\sigma$ be the adjoint group arising from an automorphism $\sigma$ of $\hat{G}$ as above. Then the following hold.

(a) Each $\sigma$-stable small representation $V_\lambda$ of $G$ is self-dual, hence is attached to a unipotent class $C_\lambda \subset \hat{G}$.

(b) The intersection $C_\lambda^\sigma = C_\lambda \cap \hat{G}_\sigma$ is a single class in $\hat{G}_\sigma$, and the $G_\sigma$-representation $V_\lambda^\sigma$ attached to $C_\lambda^\sigma$ is small.

(c) The correspondence $V_\lambda \rightarrow V_\lambda^\sigma$ is a bijection between the isomorphism classes of small $\sigma$-stable $G$-representations and isomorphism classes of small $G_\sigma$ representations.

One may think of $V_\lambda$ as the “functorial lift” of $V_\lambda^\sigma$, under the inclusion $\hat{G}_\sigma \hookrightarrow \hat{G}$.

The zero weight spaces $V_\lambda^0$ and $[V_\lambda^\sigma]^0$ are related to $C$ by means of the Springer correspondence, although this relation is in general less obvious than it was for the small representations appearing in Theorem 1.

Recall that, given a unipotent class $C$ in a reductive group, the “associated special class” is the unipotent class $\hat{C}$ uniquely characterized as follows [Sp1]. First, $\hat{C}$ is special, and contains $C$ in its closure. Second, there are no special classes other than $\hat{C}$ whose closures contain $C$ and which are contained in the closure of $\hat{C}$.

For $u \in G_\sigma$, let $A_u^\sigma$, $B_u^\sigma$, $\chi_u^\sigma$, $\epsilon$ be the analogues for $\hat{G}_\sigma$ of $A_u$, $B_u$, $\chi_u$, $\epsilon$. We again denote by $i : W_\sigma \rightarrow \hat{W}_\sigma$ the canonical isomorphism between the Weyl groups of $G_\sigma$ and $\hat{G}_\sigma$.

**Theorem 4.** Let $V_\lambda$ be a $\sigma$-stable small representation of $G$, attached to the unipotent class $C_\lambda$ as in Theorem 3. Let $\hat{C}_\lambda$ be the special class associated to $C_\lambda$. Then $\hat{C}_\lambda \cap \hat{G}_\sigma$ is a single class in $\hat{G}_\sigma$, and for $u \in \hat{C}_\lambda \cap \hat{G}_\sigma$, we have

$$V_\lambda^0 \otimes \epsilon = \bigoplus i^* \chi_{u, \rho}, \quad [V_\lambda^\sigma]^0 \otimes \epsilon = \bigoplus i^* \chi_{u, \rho}^\sigma.$$
where the sum for \( V_\lambda^0 \) runs over all irreducible representations \( \rho \) of \( A_u \), and the sum for \( [V_\lambda^0] \) runs over just those irreducible representations \( \rho \) of \( A_u^\sigma \) which factor through the natural homomorphism \( A_u^\sigma \to A_u \).

**Remarks:** The description of \( V_\lambda^0 \) in Theorem 4 is essentially contained in [R2]; for self-dual \( V_\lambda \) (and simply laced \( G \)) the class \( C_\lambda \) turns out to be the “small orbit” associated to \( V_\lambda \) in [R2]. It is known [R4] that if any unipotent class \( C \) in \( G \) meets \( \hat{G}_\sigma \), then the intersection is a single class. The content of the first assertion of Theorem 4 is that \( \hat{C}_\lambda \cap \hat{G}_\sigma \) is non-empty. It is not always a special class in \( \hat{G}_\sigma \). For the unipotent elements \( u \) in Theorem 4, we always have \( \chi_{u, \rho} \neq 0 \), but there are a few cases where \( \chi_{u, \rho} = 0 \), meaning that \( \rho \) does not appear in \( H^{2d_u}(u) (B_u^\sigma) \).

I thank B. Gross for giving me a preprint of his paper [G], which helped me to understand the role of the dual group here. I was also influenced by discussions about small representations with V. Toledano Laredo. Finally, hearty thanks go to D. Vogan, for his comments an earlier version this paper.

**2. Proof of Theorem 1**

Since \( V_\mu \) is minuscule, we have \( \langle \mu, \beta \rangle \in \{ -1, 0, 1 \} \) for all \( \alpha \)-roots \( \beta \) of \( G \), and every weight in \( \text{End}(V_\mu) \) is \( W \)-conjugate to \( \mu - w\mu \) for some \( w \in W \). Let \(( \ , \ )\) be a \( W \)-invariant inner product on the real vector space spanned by the weight lattice of \( T \), so that \( \langle \mu, \beta \rangle = 2(\mu, \beta)/(\beta, \beta) \). Now if \( \beta \) is a root of \( G \) and \( 2\beta \) is a weight in \( \text{End}(V_\mu) \), say \( 2\beta = \mu - w\mu \) as above, we have

\[
(\mu, \mu) = (w\mu, w\mu) = (\mu - 2\beta, \mu - 2\beta) = (\mu, \mu) - 4(\mu, \beta) + 4(\beta, \beta),
\]

so

\[
\langle \mu, \beta \rangle = 2 \frac{(\mu, \beta)}{(\beta, \beta)} = 2,
\]

a contradiction. Hence every constituent of \( \text{End}(V_\mu) \) is small.

For \( wW_\mu \in W/W_\mu \), let \( v_\mu \), \( v_{-w\mu} \) denote weight vectors, with the indicated weights, in \( V_\mu \) and \( V_\mu^* \). If \( w \in W_\mu \) then \( w \) is represented by an element \( \hat{w} \) of the derived group of the parabolic subgroup stabilizing the line through \( v_\mu \), and \( \hat{w} \) fixes \( v_\mu \). Now \( W \) acts on the zero weight space in \( V_\mu \otimes V_\mu^* \), and \( w \) fixes \( v_\mu \otimes v_{-\mu} \). The set

\[
\{v_\mu \otimes v_{-\mu} : wW_\mu \in W/W_\mu \}
\]

is therefore a basis of the zero weight space in \( V_\mu \otimes V_\mu^* \). It follows that

\[
\text{End}_T(V_\mu) \simeq \text{Ind}_{W_\mu}^W C.
\]

Since \( W_\mu \backslash W/W_\mu \) is represented by the involutions \( w_{\beta_1} \cdots w_{\beta_h} \) [RRS, Thm. 1.2], it follows that \( \text{Ind}_{W_\mu}^W C \) is multiplicity-free, hence \( \text{End}(V_\mu) \) is also multiplicity-free. Both \( \text{End}_T(V_\mu) \) and \( \text{End}(V_\mu) \) have the same number \( |W_\mu \backslash W/W_\mu| \) of constituents, so the zero weight space of each constituent of \( \text{End}(V_\mu) \) must be irreducible. Since dual representations have isomorphic zero weight spaces, each constituent of \( \text{End}(V_\mu) \) must be self-dual.
We pass now to the dual group \( \hat{G} \). Every root \( \hat{\beta} \) in \( \hat{U} \) has the property \( \langle \mu, \hat{\beta} \rangle = 1 \). Let \( \hat{\beta}_1, \ldots, \hat{\beta}_r \) be a maximal orthogonal set of long roots in \( \hat{U} \), let \( \beta_i \) be the corresponding co-roots, and let \( w_{\beta_i} \) be the corresponding reflections in \( \hat{W} \). Then for \( w = w_{\beta_1} \cdots w_{\beta_k} \), we have
\[
\mu - w\mu = \beta_1 + \cdots + \beta_k.
\]
Set \( \lambda_k := \beta_1 + \cdots + \beta_k \). This is the extreme weight of a constituent \( V_k \) of \( \text{End}(V_{\mu}) \).

Each \( \beta_k \) extends, by Chevalley theory, to a homomorphism \( \varphi_{\beta_k} : \text{SL}_2 \to \hat{G} \) and since the \( \beta_k \)'s are orthogonal, the images of the \( \varphi_{\beta_k} \)'s commute with one another. The sum \( \beta_1 + \cdots + \beta_k \) is a co-weight which extends to the map \( g \mapsto \varphi_k(g) = \varphi_{\beta_1}(g) \cdots \varphi_{\beta_k}(g), g \in \text{SL}_2 \). Thus, \( V_k \) is attached to the class of \( u_k := \varphi_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

The proof of parts (a)-(c) of Theorem 1 will be complete if we can ensure dominance of \( \lambda_k = \beta_1 + \cdots + \beta_k \). The following construction appeared with a different purpose in the proof of [RRS, (2.8)]. Fix a simple system \( \hat{S} \) of the root system \( \hat{R} \) of \( \hat{T} \) in \( \hat{G} \), so that \( \hat{S} \) is the vertex set of the Dynkin graph \( \hat{D} \) of \( \hat{G} \). For every \( J \subseteq \hat{S} \), let \( \hat{R}(J) \) be the based subroot system of \( \hat{R} \), with basis \( J \). There is a unique “minuscule vertex” \( \hat{\alpha} \in \hat{S} \) such that \( \langle \mu, \hat{\alpha} \rangle = 1 \), and \( \mu \) vanishes on the remaining roots in \( \hat{S} \). Let \( J_1 = \hat{S} \), and let \( \hat{\beta}_1 \) be the highest root with respect to \( \hat{R} = \hat{R}(J_1) \). Let \( \hat{D}_1 \) be the subgraph of \( \hat{D} \) whose vertices are orthogonal to \( \hat{\beta}_1 \). If \( \langle \hat{\beta}_1, \hat{\alpha} \rangle = 0 \), let \( J_2 \) be the set of vertices in the component of \( \hat{D}_1 \) containing \( \hat{\alpha} \). Then \( \hat{\alpha} \) is again a minuscule vertex with respect to \( \hat{J}_2 \) [loc.cit.], and we let \( \hat{\beta}_2 \) be the highest root in \( \hat{R}(J_2) \). Repeat until \( \langle \hat{\beta}_r, \hat{\alpha} \rangle \neq 0 \).

We must show that \( \langle \lambda_k, \hat{\gamma} \rangle \geq 0 \) for all \( \hat{\gamma} \in \hat{S} \). Suppose first that \( \hat{\gamma} \in \hat{S} \) is not orthogonal to the highest root \( \hat{\beta}_1 \). Considering each Dynkin diagram, we find a unique root \( \hat{\delta} \in J_2 \) not orthogonal to \( \hat{\gamma} \). Moreover, \( \hat{\delta} \) is minuscule for \( J_2 \) (hence has coefficient \( =1 \) in \( \hat{\beta}_2 \) and does not belong to \( J_3 \). In all but one case, \( \hat{\delta} \) and \( \hat{\beta}_1 \) are joined to \( \hat{\gamma} \) by single bonds, so that
\[
\langle \hat{\beta}_2, \hat{\gamma} \rangle = \langle \hat{\delta}, \hat{\gamma} \rangle = -1 = -\langle \hat{\beta}_1, \hat{\gamma} \rangle.
\]
The exception is \( C_n \), where -1 in the line above is replaced by -2.

Now if \( \hat{\nu} \in J_3 \), we must have \( \langle \nu, \hat{\gamma} \rangle = 0 \), else there would be a cycle in \( \hat{D} \), involving \( \hat{\gamma}, \hat{\delta}, \hat{\nu} \). Thus, in all cases, we have
\[
\langle \hat{\beta}_1, \hat{\gamma} \rangle > 0, \quad \langle \hat{\beta}_1 + \cdots + \hat{\beta}_k, \hat{\gamma} \rangle = 0, \quad \text{for } k > 1.
\]

Now let \( \hat{\gamma} \in \hat{S} \) be arbitrary. We may suppose there is a minimal \( m \) such that \( \langle \beta_m, \hat{\gamma} \rangle \neq 0 \). Clearly \( \langle \lambda_k, \hat{\gamma} \rangle = 0 \) if \( k < m \). It follows that \( \hat{\gamma} \) belongs to the \( \hat{\alpha} \)-component of \( \hat{D}_{m-1} \), i.e., \( \hat{\gamma} \in J_m \). The previous paragraph applied to \( S = J_m \) shows that \( \langle \lambda_k, \hat{\gamma} \rangle \geq 0 \) if \( k \geq m \), so \( \lambda_k \) is indeed dominant.

Part (d) of Theorem 1 is a special case of a general fact for all small representations, to be proved later. It remains to prove (e), so we now assume \( G \) is simply-laced. The small modules and their zero weight spaces were determined in [R1]. For \( E_6 \) and \( E_7 \), the assertion (e) is obtained by comparing [R1, p. 439] with the tables in [BS]. If \( G = SO_{2n} \) with \( \mu = \omega_1 \), then \( x \) is regular in \( SO_{2n-2} \),
and \( d(x) = 2 = r \). The calculation of \( H^2(\mathcal{B}_x) \) follows from [Sp2], and \( H^4(\mathcal{B}_x) \) is given by the Springer correspondence (c.f. [Ca]). Using the notation of [R1] for representations of \( W(D_n) \), we have

\[
\begin{align*}
H^0(\mathcal{B}_x) &= \chi(n; -) = V_0^0, \\
H^2(\mathcal{B}_x) &= \chi(n - 1; 1) = V_{\omega_2}^0, \\
H^4(\mathcal{B}_x) &= \chi(n - 1, 1; -) = V_{2\omega_1}^0.
\end{align*}
\]

Consider now the case \( SO_{2n} \) with \( \mu_+ = \omega_n \) or \( \mu_- = \omega_{n-1} \). The corresponding Levi subgroups are \( \hat{L}_{\mu_{\pm}} \simeq GL_n \). If \( n \) is odd, then these Levi's are conjugate, and the element \( x \) is regular unipotent in \( GL_n \), with partition \([n, n]\). If \( n = 2m \) is even, then \( x = x_{\pm} \) are regular unipotent in \( \hat{L}_{\mu_{\pm}} \), and represent distinct classes with partition \([n, n]\). Part (e) amounts to the formulas

\[
H^{2k}(\mathcal{B}_x) = \chi(n - k, k) \quad n \text{ odd}
\]

\[
H^{2k}(\mathcal{B}_{\mu_{\pm}}) = \begin{cases} 
\chi(n - k, k) & k < m \quad (n = 2m) \\
\chi(m, m)_{\pm} & k = m \quad (n = 2m)
\end{cases}
\]

For fixed \( n \), the representations on the right side are determined by their dimension, and we know they all occur in \( H^*(\mathcal{B}_x) \), by Lusztig's result recalled in paragraph (iii) of the introduction, and the fact that

\[
\text{Ind}_{W_{\mu_{\pm}}}^W \mathbf{C} = \left\{ \chi(n; -) \oplus \chi(n - 1; 1) \oplus \cdots \oplus \chi(m + 1; m - 1), \quad (n = 2m + 1) \right. \\
\left. \chi(n; -) \oplus \chi(n - 1; 1) \oplus \cdots \oplus \chi(m + 1; m - 1) \oplus \chi(m, m)_{\pm}, \quad (n = 2m) \right\}
\]

It suffices therefore to prove that the dimension \( h_n^k \) of \( H^{2k}(\mathcal{B}_x) \) is given by

\[
(*) \quad h_n^k = \begin{cases} 
\binom{n}{k} & \text{if } k < \frac{n}{2} \\
\frac{1}{2} \binom{2m}{m} & \text{if } k = \frac{n}{2} = m.
\end{cases}
\]

For \( k = \left\lfloor \frac{n}{2} \right\rfloor = \dim \mathcal{B}_x \), this follows from the Springer correspondence.

For \( p \leq q \), let \( \mathcal{B}(p, q) \) be the variety of Borel subgroups of \( SO_{2(p+q)} \) containing a fixed unipotent element of partition \([p, q]\). Let

\[
h_k(p, q) = \dim H_{2k}(\mathcal{B}(p, q)),
\]

where \( H_* \) denotes Borel-Moore homology. Recall that \( \dim H_*(X) = \dim H^*(X) \) if \( X \) is compact. In particular, \( h_k(n, n) = h_n^k \). We have \( h_k(p, q) = 0 \) for \( k > \dim \mathcal{B}(p, q) = \left\lfloor \frac{p}{2} \right\rfloor \). We will also prove, for \( p < q \) both odd, \( 2n = p + q \), and \( k \leq \frac{p-1}{2} \), that

\[
(**) \quad h_k(p, q) = \binom{n}{k}.
\]

The proofs of \((*)\) and \((**)\) use induction on \( n \). If \( n = 2 \), then \( \mathcal{B}_x \simeq \mathbf{P}^1 \times \text{point} \), so \((*)\) is true. The only possibility for \((**)\) is \( \mathcal{B}(1, 3) = \text{point} \), and \( h_0(1, 3) = 1 \), so \((**)\) is true.
Assume $n > 2$, let $V = \mathbb{C}^{2n}$ and let $Q$ be the symmetric bilinear form on $V$ defining $SO_{2n} = SO(V)$. Then $B$ is a connected component of the variety $X(V)$ of $Q$-isotropic flags $E = (E_1 \subset E_2 \subset \cdots E_n)$ in $V$ with $\dim E_d = d$. Let $\mathcal{P}$ be the variety of isotropic lines in the projective space $\mathbb{P}(V)$. We have a fibration $\pi : B \to \mathcal{P}$ sending $E$ to $E_1$. For any line $L \in \mathcal{P}$, let $L^\perp$ be the $Q$ orthocomplement of $L$. Then $Q$ induces a non-degenerate form on $L^\perp/L$. We identify $\pi^{-1}(L) = B \cap X(L^\perp/L) = \text{variety of Borel subgroups of } SO(L^\perp/L)$.

Let $\pi_x$ be the restriction to $x$-fixed points,

$$\pi_x : B_x \to \mathcal{P}_x.$$  

Since $x$ has partition $[n, n]$, we have $\mathcal{P}_x = \mathbb{P}(V_x) \simeq \mathbb{P}^1$, and $\pi_x^{-1}(L) = B_x \cap X(L^\perp/L)$ for $L \in \mathcal{P}_x$, so we consider the action of $x$ on $L^\perp/L$.

Write $V$ as a direct sum $V = V_1 \oplus V_2$ of $x$-stable isotropic subspaces $V_i$, on each of which $x$ has a single Jordan block of size $n$, and let $N = x - I$. There are bases $\{v_1, \ldots, v_n\}$ of $V_1$ and $\{v_{-n}, \ldots, v_{-1}\}$ of $V_2$ such that $Q(v_i, v_{-j}) = \delta_{ij}$, and $N v_i = v_{i-1}$ for $2 \leq i \leq n$, $N v_{-j} = -v_{-j-1}$ for $1 \leq j \leq n - 1$ and $\ker N$ is spanned by $\{v_1, v_{-n}\}$. Then $L$ is spanned by a vector $\ell = av_1 + bv_{-n}$, and we choose $\ell' = cv_n + dv_{-1}$ nonzero such that $Q(\ell, \ell') = 0$. One checks that the kernel $[L^\perp/L]_N$ of the map $N$ on $L^\perp/L$ induced by $N$ is spanned by the images of $\{v_1, v_{-n}, av_2 - bv_{-1}\}$, hence $\dim [L^\perp/L]_N = 2$, so $\tilde{N}$ has Jordan partition $[p, q]$, with $p \leq q \leq n$ and $p + q = 2n - 2$. Moreover, $N^{n-1}\ell' = cc_1 + (-1)^{n-1}de_n$, which belongs to $L$ iff $n$ is even or $ab = 0$.

Thus, the map $\pi_x : B_x \to \mathcal{P}_x$ has the following structure: If $n$ is even, and $x = x_\pm$, then $\pi_x : B_x \to \mathbb{P}^1$ is a fibration with fiber $B(n-1, n-1)$. It follows that

(A) \[ h_k(n, n) = h_k(n-1, n-1) + h_{k-1}(n-1, n-1) \quad (n \text{ even}). \]

If $n$ is odd, let $L_1, L_{-n}$ denote the lines through $v_1, v_{-n}$, so that $\mathbb{P}^1 - \{L_1, L_{-n}\} = \mathbb{C}^\times$. Define

$$Z = \pi_x^{-1}(L_1) \cup \pi_x^{-1}(L_{-n}), \quad U = B_x - Z = \pi_x^{-1}(\mathbb{C}^\times).$$

Then $Z$ consists of two disjoint copies of $B(n-1, n-1)$, and $\pi_x : U \to \mathbb{C}^\times$ is a fibration with fiber $B(n-2, n)$, so

$$\dim H_{2k}(U) = h_{k-1}(n-2, n), \quad \dim H_{2k+1}(U) = h_k(n-2, n).$$

Since $H_{odd}(\mathcal{B}(p, q)) = 0$, the long exact sequence arising from the inclusion $Z \hookrightarrow B(n, n)$ gives an exact sequence

$$0 \to H_{2k+1}(U) \to H_{2k}(Z) \to H_{2k}(B(n, n)) \to H_{2k}(U) \to 0.$$  

It follows that

(B) \[ h_k(n, n) = 2h_k(n-1, n-1) + h_{k-1}(n-2, n) - h_k(n-2, n) \quad (n \text{ odd}). \]

Suppose $y \in SO_{2(p+q)}$ is unipotent with partition $[p, q]$, and $p < q$ are both odd. We again have $\mathcal{P}_y = \mathbb{P}(V_y) = \mathbb{P}^1$, and a similar calculation shows that the map $\pi_y : \mathcal{B}(p, q) \to \mathbb{P}^1$ (restriction of $\pi$) is a fibration above $\mathbb{C}$ with fiber $B(p-2, q)$,
while $\pi_y^{-1}(\infty) = B(p, q - 2)$ (where $\infty \in \mathbb{P}^1$ is the kernel of $y - 1$ in the $q$-block). The long exact sequence now gives

\[(C) \quad h_k(p, q) = h_k(p, q - 2) + h_k(p - 2, q) \quad (p < q \text{ both odd}).\]

Finally, assuming (*) and (**) hold for all $n' < n$, formulas (A-C) show that (*) and (**) hold for $n$, using the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

The proof of Theorem 1(e) for $SL_n$ goes along the same lines, using an analogue of formula (C). We omit these details.
3. Representations attached to unipotent orbits.

Let $V_\lambda$ be a representation of $G$ of highest weight $\lambda$, and let $C$ be a unipotent class in $\hat{G}$.

3.1 Definition. We say that $V_\lambda$ is attached to $C$ if the co-weight $\lambda : \mathbb{C}^\times \rightarrow \hat{G}$ is of the form $\lambda(t) = \varphi \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right)$, $t \in \mathbb{C}^\times$, where $\varphi : SL_2 \rightarrow \hat{G}$ is a homomorphism mapping the nontrivial unipotent elements of $SL_2$ into $C$.

Concretely, this means that if $\alpha$ is a simple root in $\hat{G}$, then the integer attached to the $\alpha$-node in the weighted Dynkin diagram of $C$ is equal to $\langle \lambda, \bar{\alpha} \rangle$, where $\bar{\alpha}$ is the corresponding co-root of $G$.

Assume for the time being that $G$ is simply-laced. Then we can identify unipotent classes in $G$ and $\hat{G}$ having the same weighted Dynkin diagram. For every small $G$-module $V_\lambda$, there is a unique unipotent class $C_\lambda \subset G$ and an integer $d(\lambda)$ such that $C_\lambda$ is minimal among all orbits whose ring of functions contains $V_\lambda$ in degree $d(\lambda)$. Moreover, if $u$ belongs to the dual of the special class $\hat{C}_\lambda$ associated to $C_\lambda$, then $d(\lambda) = \dim B_u$ and $V^0_\lambda \simeq H^{2d(u)}(B_u)$. All of this is proved in [R2].

3.2 Proposition. Assume $G$ is simply-laced, and $V_\lambda$ is small. Then $V_\lambda$ is attached to some unipotent class $C$, in the sense of 3.1, if and only if $V_\lambda$ is self-dual. In this case, $C = \hat{C}_\lambda$.

Proof. We make some remarks on the individual groups, ignoring the trivial representation, and complete the proof with a table.

$G = PGL_n$: The weighted Dynkin diagram $\mathcal{D}(C)$ of a unipotent class $C$ is invariant under the graph automorphism, so $V_\lambda$ can only be attached to a unipotent class if it is self-dual. Now every small module, or its contragredient, has highest weight a partition $\lambda$ of $n$, and $V_\lambda \simeq V^*_\lambda$ if and only if $\lambda = [2^k1^{n-2k}]$, for $1 \leq k \leq \frac{n}{2}$. In terms of fundamental weights, we have $\lambda = \lambda_k + \lambda_{n-k}$. On the other hand, $\lambda$ is also the partition giving the Jordan blocks of elements in $C_\lambda$.

$G = SO_{2n}$: All representations are self-dual if $n$ is even. The classes $C_\lambda$ are determined by their partitions except $C_\lambda = [2^n]_\pm$, which are the Richardson classes in $\hat{P}_\mu^\pm$ (section 2). If $n$ is odd, the only small modules which are not self-dual are $V_{\lambda_1, \lambda_n}$ and its dual. For these $\lambda$, the element $h$ of the $\mathfrak{sl}_2$-triple for $C_\lambda$ is given in [R2, 5], and we do not have $\alpha(h) = \langle \lambda, \bar{\alpha} \rangle$ for all simple roots $\alpha$.

$G = E_n$: The non-self-dual small modules occur only in $E_6$, where all diagrams $\mathcal{D}(C_\lambda)$ are invariant under the graph automorphism, so these modules are not attached to unipotent classes. The self-dual small-modules $V_\lambda$ are listed below, along with orbit $C_\lambda$ to which they are attached.
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$\lambda$</td>
<td>$C_\lambda$</td>
<td>$D(C_\lambda)$</td>
</tr>
<tr>
<td>$PGL_n$ $\lambda_q + \lambda_{n-q}$, $(1 \leq q &lt; \frac{n}{2})$</td>
<td>$[2^{q1^{n-2q}}]$</td>
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<td></td>
</tr>
<tr>
<td>$PGL_n$ $2\lambda_m$, $(n = 2m)$</td>
<td>$[2^m]$</td>
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<td></td>
</tr>
<tr>
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<td>$[2^{2q1^{4m-4q}}]$</td>
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<td></td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>$PSO_{2n}$ $2\lambda_1$</td>
<td>$[31^{4m-3}]$</td>
<td>$2 \ 0 \ldots 0 \ 0 \ 0$</td>
<td></td>
</tr>
<tr>
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<td>$0 \ldots 0 \ 2$</td>
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<tr>
<td>$PSO_{4m}$ $2\lambda_{2m-1}$</td>
<td>$[2^{2m}]_-$</td>
<td>$0 \ldots 0 \ 2$</td>
<td></td>
</tr>
<tr>
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<td>$3A_1$</td>
<td>$0 \ 0 \ 1 \ 0 \ 0$</td>
<td></td>
</tr>
<tr>
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<td>$2A_1$</td>
<td>$1 \ 0 \ 0 \ 0 \ 1$</td>
<td></td>
</tr>
<tr>
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<td>$A_1$</td>
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<td></td>
</tr>
<tr>
<td>$E_7$ $\lambda_6 + \lambda_7$</td>
<td>$4A_1$</td>
<td>$0 \ 0 \ 0 \ 0 \ 1$</td>
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</tr>
<tr>
<td>$E_7$ $2\lambda_6$</td>
<td>$(3A_1)^\prime$</td>
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</tr>
<tr>
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<td>$(3A_1)^\prime$</td>
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</tr>
<tr>
<td>$E_7$ $\lambda_5$</td>
<td>$2A_1$</td>
<td>$0 \ 0 \ 0 \ 1 \ 0$</td>
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</tr>
<tr>
<td>$E_7$ $\lambda_1$</td>
<td>$A_1$</td>
<td>$1 \ 0 \ 0 \ 0 \ 0$</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>$3A_1$</td>
<td>$0 \ 0 \ 0 \ 0 \ 1 \ 0$</td>
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</tr>
<tr>
<td>$E_8$ $\lambda_1$</td>
<td>$2A_1$</td>
<td>$1 \ 0 \ 0 \ 0 \ 0$</td>
<td></td>
</tr>
<tr>
<td>$E_8$ $\lambda_7$</td>
<td>$A_1$</td>
<td>$0 \ 0 \ 0 \ 0 \ 0 \ 1$</td>
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</table>
4. Multiply-laced groups

In this section we prove Theorem 3. We continue to assume $G$ simply-laced, simple, and adjoint. The dual group $\hat{G}$ is simply-connected. A “graph automorphism” of $\hat{G}$ is defined as follows [S, §6]. Fix a maximal torus and Borel subgroup $T \subset \hat{B}$, and non-trivial root elements $u_\alpha \in \hat{B}$, for each simple root $\alpha$ of $\hat{T}$ in $\hat{B}$. Any symmetry $\sigma$ of the Dynkin diagram of $\hat{G}$ extends to a unique automorphism of $\hat{G}$, again denoted by $\sigma$, which preserves $\hat{T}$ and $\hat{B}$, and permutes the elements $u_\alpha$ according to the original diagram symmetry.

Let $R \subset Y$ denote the co-roots and co-weights of $\hat{T}$. Then $\sigma$ permutes the elements of $R$ and $Y$, as well as the set $S \subset R$ of simple co-roots. We have $Y = \mathbb{Z}S$, since $\hat{G}$ is simply-connected.

Now assume that $\sigma$ is chosen so that the fixed point group $\hat{G}_\sigma$ is also simply-connected. For any co-root $\alpha \in R$, let $\bar{\alpha} = \alpha + \sigma \alpha + \cdots$ be the sum of the distinct $\sigma$-translates of $\alpha$. Our assumption implies that distinct $\sigma$-translates are orthogonal to one another. Hence, if $(\ , \ )$ is a $W$-invariant inner product on $Y$, we have

$$(\bar{\alpha}, \bar{\alpha}) = m(\alpha, \alpha),$$

where $m \in \{1, 2, 3\}$ is the number of distinct $\sigma$-translates of $\alpha$.

Let $S'$ be a set of representatives of the $\sigma$-orbits in $S$, and let $S_\sigma = \{\bar{\alpha} : \alpha \in S'\}$. The short elements of $S_\sigma$ are those for which $\bar{\alpha} = \alpha$.

The fixed point group $\hat{T}_\sigma$ is a maximal torus of $\hat{G}_\sigma$. Its co-weight lattice is $Y_\sigma$, and $S_\sigma$ is a set of simple co-roots of $\hat{T}_\sigma$. Since $\hat{G}_\sigma$ is simply-connected, we have $Y_\sigma = \mathbb{Z}S_\sigma$, and moreover $Y_\sigma \cap \mathbb{Z}_{\geq 0}S = \mathbb{Z}_{\geq 0}S_\sigma$.

Now the highest co-root $\beta$ of $\hat{G}$ (with respect to $S$) is unique, hence is fixed by $\sigma$, and $\beta$ is the highest short co-root of $\hat{G}_\sigma$. If $\lambda \in Y_\sigma$, then $\lambda - 2\beta \in Y_\sigma$, so we have

\begin{equation}
\lambda - 2\beta \in \mathbb{Z}_{\geq 0}S \quad \iff \quad \lambda - 2\beta \in \mathbb{Z}_{\geq 0}S_\sigma.
\end{equation}

Let $G_\sigma$ be the adjoint group whose dual is $\hat{G}_\sigma$. Then $G_\sigma$ has a maximal torus $T_\sigma$ whose character group is identified with $Y_\sigma$, and $S_\sigma$ is a set of simple roots of $T_\sigma$ in $G_\sigma$. Note that $G_\sigma$ is not necessarily a subgroup of $G$.

We say the irreducible $G$-representation $V_\lambda$ is $\sigma$-stable, if $\lambda \in Y_\sigma$. If $\lambda$ is dominant with respect to $S$, then $\lambda$ is also dominant with respect to $S_\sigma$, and is the highest weight of an irreducible $G_\sigma$-representation $V_\lambda^\sigma$.

A representation of an adjoint group is small exactly when twice the highest short root $\beta$ is not a weight, which by saturation of weights is equivalent to $2\beta \not\leq \lambda$, in the usual partial order on weights. From (4a), we see that $V_\lambda$ is small for $G$ if and only if $V_\lambda^\sigma$ is small for $G_\sigma$.

If $G$ has type $A_n$, $D_{2m+1}$, $E_6$, then $V_\lambda$ being $\sigma$-stable is equivalent to its being self-dual, and if $G$ has type $D_{2m}$ then $V_\lambda$ is always self-dual. By 3.2, every $\sigma$-stable small $V_\lambda$ is therefore attached to a unipotent class $C_\lambda \subset \hat{G}$. Since $\lambda$ is fixed by $\sigma$, we can choose $\phi : SL_2 \to \hat{G}$ to have image in $\hat{G}_\sigma$, hence $V_\lambda^\sigma$ is attached to the unipotent class $C_\lambda^\sigma \subset \hat{G}_\sigma$ containing the image under $\phi$ of the nontrivial unipotent elements in $SL_2$. By [R4, Cor. 4.3] we in fact have $C_\lambda^\sigma = \hat{G}_\sigma \cap C_\lambda$. This proves Theorem 3.

We tabulate the bijection of Theorem 3 below. For $\hat{G} = SL_{2n}$ the relevant unipotent classes are

$$u_k := [2k, 1^{2n-2k}],$$
each of which meets $\hat{G}_\sigma = S_{p2n}$ in a single class, denoted $u^\sigma_k$.

For $\hat{G} = \text{Spin}_{2n+2}$, the relevant unipotent classes are given by their partitions in $SO_{2n+2}$ as follows

$$x_p = [2^{2p}1^{2n-4p+2}] \ (0 \leq 2p \leq n), \quad y_q = [32^{2q-2}1^{2n-4q+3}] \ (1 \leq 2q - 1 \leq n).$$

Each of these classes meets $\text{Spin}_{2n+1}$ in a single class, denoted $x^\sigma_p$, $y^\sigma_q$, respectively, whose partition is obtained by removing a 1 from the corresponding partition in $\text{Spin}_{2n+2}$.

For the exceptional groups unipotent classes are denoted as in [Ca]. In the table below, the columns are respectively $G$, the highest weight of the small module $V_\lambda$, the Dynkin diagram $D(C_\lambda)$, the Dynkin diagram of $D(C^\sigma_\lambda)$, the name or partition of the class $C^\sigma_\lambda \subseteq \hat{G}_\sigma$, the highest weight of the $G_\sigma$-module $V^\sigma_\lambda$, and the group $G_\sigma$. We denote by $\lambda_i$ the fundamental weights of $G_{sc}^\sigma$, and by $\omega_i$ the fundamental weights of $G_{sc}^\sigma$, using the indexing given in section 3, along with $1 \Rightarrow 2$ for $G_\sigma = G_2$ and $12 \Rightarrow 34$ for $G_\sigma = F_4$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$V_\lambda$</th>
<th>$C_\lambda$</th>
<th>$D(C_\lambda)$</th>
<th>$D(C^\sigma_\lambda)$</th>
<th>$C^\sigma_\lambda$</th>
<th>$V^\sigma_\lambda$</th>
<th>$G_\sigma$</th>
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<td>$PGL_{2n}$</td>
<td>$\lambda_k + \lambda_{2n-k}$</td>
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<td>$0 \leftarrow 0$</td>
<td>$u^\sigma_k$</td>
</tr>
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<td>$2\lambda_n$</td>
<td>$u_n$</td>
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<td>$0 \leftarrow 2$</td>
<td>$u^\sigma_n$</td>
<td>$2\omega_n$</td>
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<td>$0 \ldots 010 \ldots 0$</td>
<td>$0 \leftarrow 0$</td>
<td>$x^\sigma_{j/2}$</td>
<td>$\omega_j$</td>
</tr>
<tr>
<td>$PSO_{4m+2}$</td>
<td>$\lambda_n + \lambda_{n+1}$</td>
<td>$x_m$</td>
<td>$0 \ldots 001 \ldots 1$</td>
<td>$0 \leftarrow 1$</td>
<td>$x^\sigma_m$</td>
<td>$\omega_n$</td>
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</tr>
<tr>
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<td>$y_{k+1}$</td>
<td>$1 \ldots 010 \ldots 00$</td>
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<td>$y^\sigma_{k+1}$</td>
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<td>$y^\sigma_m$</td>
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<td>$G_2$</td>
</tr>
<tr>
<td>$PSO_8$</td>
<td>$\lambda_1 + \lambda_3 + \lambda_4$</td>
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<td>$\bar{A}_1$</td>
<td>$\omega_1$</td>
<td>$G_2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\lambda_3$</td>
<td>$3A_1$</td>
<td>$0 \ldots 010 \ldots 0$</td>
<td>$0 \leftarrow 10$</td>
<td>$A_1 + \bar{A}_1$</td>
<td>$\omega_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\lambda_1 + \lambda_5$</td>
<td>$2A_1$</td>
<td>$1 \ldots 000 \ldots 1$</td>
<td>$10 \leftarrow 00$</td>
<td>$\bar{A}_1$</td>
<td>$\omega_1$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$E_6$</td>
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<td>$0 \leftarrow 01$</td>
<td>$A_1$</td>
<td>$\omega_4$</td>
<td>$F_4$</td>
</tr>
</tbody>
</table>
5. Zero Weight Spaces

In this section we prove Theorem 4. Recall that $G$ is simply-laced. Let $V_\lambda$ be a small $\sigma$-stable $G$-module, attached to the unipotent class $C_\lambda$. Let $\hat{C}_\lambda$ be the special class attached to $C_\lambda$ by Spaltenstein. By [R1,2] we have

\[ V_\lambda^0 \otimes \epsilon = \bigoplus \iota^* \chi_{u,\rho} \]

for $u \in \hat{C}_\lambda$, summing over all characters $\rho$ of $A_u$. We verify an analogue of this for the $G_\sigma$-module $V_\lambda^\sigma$, using case-by-case calculations. The zero-weight spaces of the non-simply laced small representations are calculated as in [R1], and we omit these details. We refer to [Ca] for the Springer correspondence.

4.1 Theorem. The class $\hat{C}_\lambda$ meets $\hat{G}_\sigma$ in a single unipotent class $\hat{C}_\lambda^\sigma$ of $\hat{G}_\sigma$. For $u \in \hat{C}_\lambda^\sigma$, we have

\[ [V_\lambda^\sigma]^0 \otimes \epsilon_\sigma \simeq \bigoplus \iota^* \chi_{u,\rho}, \]

where $\rho$ runs over those irreducible representations of $A_u^\sigma$ which factor through the natural map $A_u^\sigma \hookrightarrow A_u$.

Proof. In the table below, we denote representations of a Weyl group of type $B_n$ by a pair $(\alpha, \beta)$ of partitions with sum $|\alpha| + |\beta| = n$. For exceptional Weyl groups we use the notation of [Ca]. For the class $y_q = [3^2 2^{q-2} 1^{2n-4q+3}]$ in $SO_{2n+2}$, the associated special class is $z_q = [3^2 2^{2q-4} 1^{2n-4q+2}]$, and $z_q^\sigma = [3^2 2^{2q-4} 1^{2n-4q+3}]$ in $SO_{2n+1}$.

For $u \in \hat{C}_\lambda^\sigma$ we give the Springer representation $\chi_{u,\rho}$, with $\chi_{u,1}$ listed first. This is a representation of the Weyl group of $\hat{G}_\sigma$. In the next column we give the representation of the Weyl group of $G_\sigma$ on $[V_\lambda^\sigma]^0 \otimes \epsilon_\sigma$. An entry $\leftarrow$ means that entry is the same as the one immediately to the left ($\iota^*$ is the identity in the standard partition notation for classical Weyl groups).
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\lambda$</th>
<th>$\tilde{C}_\lambda$</th>
<th>$A_u$</th>
<th>$A_u^\sigma$</th>
<th>$\tilde{C}_\lambda^\sigma$</th>
<th>$H^2 d_\sigma(u)(B_u^\sigma)$</th>
<th>$[V^\sigma_u]^0 \otimes \epsilon_\sigma$</th>
<th>$V^\sigma_u$</th>
<th>$G_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$PGL_2$</td>
<td>$\lambda_k \cdot \lambda_{2n-k}$</td>
<td>$u_k$</td>
<td>1</td>
<td>1</td>
<td>$u_k^\sigma$</td>
<td>$(1^p, 1^{n-p})$ (if $k = 2p$)</td>
<td>$\omega_k$</td>
<td>$SO_{2n+1}$</td>
<td></td>
</tr>
<tr>
<td>$PSO_{2n+2}$</td>
<td>$\lambda_2$</td>
<td>$x_p$</td>
<td>1</td>
<td>1</td>
<td>$x_p^\sigma$</td>
<td>$(1^{2p} 1^{n-2p})$</td>
<td>$\omega_2$</td>
<td>$PSp_{2n}$</td>
<td></td>
</tr>
<tr>
<td>$PSO_{2n+2}$</td>
<td>$\lambda_1 + \lambda_{2q-1}$</td>
<td>$z_q$</td>
<td>2</td>
<td>2</td>
<td>$z_q^\sigma$</td>
<td>$(1, 2^{2q-1}, 1^{n-2q-1})$</td>
<td>$\omega_1 + \omega_{2q-1}$</td>
<td>$PSp_{2n}$</td>
<td></td>
</tr>
<tr>
<td>$PSO_{2n+2}$</td>
<td>$2\lambda_1$</td>
<td>$y_n$</td>
<td>1</td>
<td>1</td>
<td>$y_n^\sigma$</td>
<td>$(1, 1^{n-1})$</td>
<td>$2\omega_1$</td>
<td>$PSp_{2n}$</td>
<td></td>
</tr>
<tr>
<td>$PSO_{4m}$</td>
<td>$\lambda_1 + \lambda_n + \lambda_{n+1}$</td>
<td>$z_m$</td>
<td>2</td>
<td>2</td>
<td>$z_m^\sigma$</td>
<td>$(1, 2^m-1)$</td>
<td>$\omega_1 + \omega_{n-1}$</td>
<td>$PSp_{2n}$</td>
<td></td>
</tr>
<tr>
<td>$PSO_8$</td>
<td>$\lambda_2$</td>
<td>$x_1$</td>
<td>1</td>
<td>1</td>
<td>$A_1$</td>
<td>$\phi^\prime_{1,3}$</td>
<td>$\phi^\prime_{1,3}$</td>
<td>$\omega_2$</td>
<td>$G_2$</td>
</tr>
<tr>
<td>$PSO_8$</td>
<td>$\lambda_1 + \lambda_3 + \lambda_4$</td>
<td>$z_2$</td>
<td>2</td>
<td>2</td>
<td>$S_2$</td>
<td>$G_2(a_1)$</td>
<td>$\phi^\prime_{2,1} \oplus \phi^\prime_{1,3}$</td>
<td>$\phi^\prime_{2,1}$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\lambda_3$</td>
<td>$A_2$</td>
<td>2</td>
<td>2</td>
<td>$S_2$</td>
<td>$A_2$</td>
<td>$\phi^\prime_{6,9} \oplus \phi^\prime_{1,12}$</td>
<td>$\phi^\prime_{6,9} \oplus \phi^\prime_{1,12}$</td>
<td>$\omega_3$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\lambda_1 + \lambda_5$</td>
<td>$2A_1$</td>
<td>1</td>
<td>1</td>
<td>$\tilde{A}_1$</td>
<td>$\phi_{4,13} \oplus \phi^\prime_{2,16}$</td>
<td>$\phi_{4,13}$</td>
<td>$\omega_1$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\lambda_6$</td>
<td>$A_1$</td>
<td>1</td>
<td>1</td>
<td>$A_1$</td>
<td>$\phi^\prime_{2,10}$</td>
<td>$\phi^\prime_{2,10}$</td>
<td>$\omega_4$</td>
<td>$F_4$</td>
</tr>
</tbody>
</table>

Theorem 4 is clear from the table in those cases where $A_u = 1$. For $u = z_q$ in $SO_{2n+2}$ the map $A_u^\sigma \mapsto A_u$ is induced by the inclusion $O_{2n-4q+3} \leftrightarrow O_{2n-4q+4}$ which is an isomorphism on component groups.

Suppose $u = z_2$ in $PSO_8$. We have $A_u^\sigma = S_3$, and $d_\sigma(u) = 1$. The sign character $sgn$ of $S_3$ does not appear in $H^2(B_u^\sigma)$, so $\chi_{u, sgn} = 0$. The reflection representation of $S_3$ does not factor through any map $S_3 \mapsto S_2$, so Theorem 4 is verified in this case.

Suppose that $u$ has type $A_2$ in $E_6$. Let $W$ be the Weyl group of $E_6$. Let $w_0$, $\alpha_0$ denote the longest element of $W$ and the highest root in $E_6$, respectively. Then we may take $u = \exp(e_\alpha + e_{\alpha_0 - \alpha_0})$, where $e_\beta$ is the root vector for a root $\beta$. Let $w = w_0 s_{\alpha_0} \in W$. Then $w$ switches $\alpha_6$ and $\alpha_0 - \alpha_6$. Let $T$ be a $\sigma$-stable maximal torus in $G$, with normalizer $\tilde{N}$. Since $w$ is $\sigma$-invariant, we may, by $[S, (8.2)]$, choose a representative $n$ for $w$ in $\tilde{N}_\sigma$. Multiplying $n$ by an element of $T_\sigma$ if necessary, we may assume $n$ switches $e_\alpha$ and $e_{\alpha_0 - \alpha_0}$, so $n$ belongs to the centralizer of $u$ in $G_\sigma$. It remains to see that $n$ represents the non-trivial class in $A_u = S_2$. Let $\{ e, h, f \}$ be an $sl_2$-triple for $e = e_\alpha + e_{\alpha_0 - \alpha_0}$, and let $g_i$ be the $i$-eigenspace of $ad(h)$ on $g = e_6$. Then $g_0 = g_6$ and $e$ belongs to the dense orbit of $GL_6$ on $g_2 = \bigwedge^3 C^6$. In terms of a basis $\{ v_i \}$ of $C^6$, we have $e = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$, where one wedge product corresponds to $e_\alpha$, the other to $e_{\alpha_0 - \alpha_0}$. The connected centralizer of $e$ is the obvious $SL_3 \times SL_3$, and the component group of the centralizer is generated by the element of $GL_6$ which switches the two wedge products in $e$. This proves the claim, and completes the proof of Theorem 4.
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