SMALL REPRESENTATIONS AND MINUSCULE RICHARDSON ORBITS

Mark Reeder

Boston College
Dept. of Mathematics
Chestnut Hill, MA, 02467 USA
e-mail: reederma@bc.edu

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1. Introduction

(i) Let G be a simple, adjoint algebraic group over $\mathbf{C}$, with maximal torus T and Weyl group W. Let V_μ be an irreducible representation of the simply-connected cover G^{sc} of G with highest weight μ. Assume V_μ is *minuscule,* that is, the weights of V_μ form a single W-orbit. Kumar [Ku] has found the decomposition of tensor products where one factor is a minuscule V_μ. Specialized to the case $\mathrm{End}(V_\mu) = V_\mu \otimes V_\mu^*$, which is now a representation of G, Kumar's results give a bijection between the irreducible constituents of $\mathrm{End}(V_\mu)$ and double cosets $W_\mu \backslash W / W_\mu$, where W_μ is the stabilizer of μ in W. Namely, to $W_\mu w W_\mu$ corresponds the representation with extreme weight $\mu - w\mu$.

(ii) An irreducible representation V_λ of G is called *small* if twice a root is not a weight in V_λ. Small representations are distinguished by having remarkable relations, some proven, some conjectural, between the multiplicities of V_λ in certain natural G-representations, and the multiplicities of the zero weight space V_λ^0 in certain natural W-representations. See for example, [B1-3], [K,§5], [R1-3], [So], [ST].

For minuscule V_μ, it is easy to see (§2 below) that any constituent of $\mathrm{End}(V_\mu)$ is small, and that the zero weight space $\mathrm{End}_T(V_\mu)$ is isomorphic to the permutation module $\mathrm{Ind}_{W_\mu}^W \mathbf{C}$, whose constituents are in bijection with double cosets $W_\mu \backslash W / W_\mu$.

(iii) Let $\hat{G}$, $\hat{T}$, be the dual groups of G and T. Then the weight μ may be viewed as a dominant co-weight of the adjoint group $\hat{G}^{ad}$ of $\hat{G}$. Let $\hat{L}_\mu$ be the centralizer of μ in $\hat{G}$ (under the conjugation action of $\hat{G}$ on $\hat{G}^{ad}$) and let $\hat{P}_\mu = \hat{L}_\mu \hat{U}_\mu$ be a parabolic subgroup of $\hat{G}$ with Levi $\hat{L}_\mu$. Let R_μ be the Richardson class of $\hat{P}_\mu$ in $\hat{G}$. This is the unique unipotent class in $\hat{G}$ containing a dense subset of $\hat{U}_\mu$. Richardson, Rohrle and Steinberg [RRS] made a detailed study of the classes contained in the closure $\overline{R}_\mu$, and obtained, among others, the following results. Let $\hat{W}$ be the Weyl

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group of $\hat{T}$ in $\hat{G}$, and let $\hat{W}_\mu$ be the centralizer of μ. Then the orbits in $\overline{R}_\mu$ are in bijection with the double cosets $\hat{W}_\mu\backslash\hat{W}/\hat{W}_\mu$. In fact, the orbits and double coset representatives may be constructed in parallel ways, as follows [RRS, Thm 1.2]. Let $\{\hat{\beta}_1, \ldots, \hat{\beta}_r\}$ be a maximal orthogonal set of long roots of $\hat{T}$ in $\hat{U}$. Such sets are unique up to $\hat{W}_\mu$-conjugacy. Let $w_\beta \in \hat{W}$ be the reflection for $\hat{\beta}$. Then

$$w_{\beta_1} \cdots w_{\beta_k}, \quad k = 1, \ldots r$$

is a complete system of nontrivial double coset representatives for $\hat{W}_\mu\backslash\hat{W}/\hat{W}_\mu$. On the other hand, for each i, let u_{β_i} be a non-trivial element in the β_i-root group in $\hat{U}_\mu$. Then

$$u_{\beta_1} \cdots u_{\beta_k}, \quad k = 1, \ldots r$$

is a complete system of representatives for the nontrivial $\hat{G}$-orbits in $\overline{R}_\mu$.

(iv) Let u be a unipotent element of $\hat{G}$, and let $\mathcal{B}_u$ denote the variety of Borel subgroups of $\hat{G}$ containing u. It is known that every irreducible component of $\mathcal{B}_u$ has the same dimension $d(u)$. Let $H^*(\mathcal{B}_u)$ denote the singular cohomology of $\mathcal{B}_u$ with complex coefficients. The cohomology is zero in odd degrees [CLP], and the highest nonvanishing degree is $2d(u)$. Springer has defined an action of $\hat{W}$ on $H^*(\mathcal{B}_u)$ [Spr]. If u is regular in $\hat{L}_\mu$, then by a result of Lusztig we have an isomorphism of $\hat{W}$-modules

$$H^*(\mathcal{B}_u) \simeq \mathrm{Ind}_{\hat{W}_\mu}^{\hat{W}} \mathbf{C}.$$

In particular there is a geometrically defined W-grading on the permutation representation $\mathrm{Ind}_{\hat{W}_\mu}^{\hat{W}} \mathbf{C}$.

The first main result in this paper gives some connections between the facts recalled in (i-iv).

The central theme is the attachment of a representation of G to a unipotent class C in $\hat{G}$. Given C, let $\varphi : SL_2 \longrightarrow \hat{G}$ be a homomorphism mapping the nontrivial unipotent elements of SL_2 into C. Let $\lambda(t) = \varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, for $t \in \mathbf{C}^\times$. After possibly conjugating φ, we may assume that λ is a dominant co-weight of $\hat{T}$. View λ as a weight of the dual torus T in G, and let V_λ be the irreducible G-module with highest weight λ. We say that V_λ is *attached* to C. For example, the adjoint representation of G is attached to the orbit of a short root element in $\hat{G}$.

Two natural Weyl-group representations arise. On the one hand, we have the W-action on the zero-weight space V_λ^0. On the other, we have the Springer representations $\chi_{u,\rho}$ of $\hat{W}$ associated to $u \in C$. Here ρ is an irreducible representation of the component group A_u of the centralizer of u in $\hat{G}^{ad}$, and $\chi_{u,\rho} = \mathrm{Hom}_{A_u}(\rho, H^{2d(u)}(\mathcal{B}_u))$.

The root datum defining $\hat{G}$ in terms of G gives a canonical isomorphism $\iota : W \longrightarrow \hat{W}$ sending short reflections in W to long reflections in $\hat{W}$, and vice-versa. Let ι^* denote the pull-back map sending representations of $\hat{W}$ to those of W.

Theorem 1. *Let V_μ be an irreducible minuscule representation of G^{sc} with highest weight μ. Let $\hat{L}_\mu$ be the centralizer of μ in $\hat{G}$, let $\hat{P}_\mu$ be a parabolic subgroup of $\hat{G}$ with Levi factor $\hat{L}_\mu$, and let R_μ be the Richardson class of $\hat{P}_\mu$. Then the following hold.*

(a) $\mathrm{End}(V_\mu)$ *is multiplicity-free, and its constituents are exactly those representations of G which are attached to the orbits in the closure of R_μ. More precisely, we may, in (iii) above, choose the maximal orthogonal set of long roots $\{\hat{\beta}_1, \dots, \hat{\beta}_r\}$ in such a way that $\hat{\beta}_1 + \cdots + \hat{\beta}_k$ is dominant for each $k = 1, \dots, r$. Let β_i be the corresponding co-roots, and set $\lambda_k = \beta_1 + \cdots + \beta_k$, viewed as a weight of T, and put $\lambda_0 = 0$. Then*

$$\mathrm{End}(V_\mu) = V_{\lambda_0} \oplus V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r},$$

and V_{λ_k} is attached to the class of $u_k := u_{\beta_1} \cdots u_{\beta_k}$.

(b) *Each V_{λ_k} is self-dual and small.*

(c) *The zero weight space $V^0_{\lambda_k}$ of each constituent of* $\mathrm{End}(V_\mu)$ *is irreducible under W, and* $\mathrm{End}_T(V_\mu) \simeq \mathrm{Ind}^W_{W_\mu} \mathbf{C}$. *Thus, we have the irreducible decomposition*

$$\mathrm{Ind}^W_{W_\mu} \mathbf{C} = V^0_{\lambda_0} \oplus V^0_{\lambda_1} \oplus \cdots \oplus V^0_{\lambda_r}.$$

(d) *In terms of Springer representations, we have*

$$V^0_{\lambda_k} \otimes \epsilon \simeq \iota^* \chi_{u_k, 1},$$

where ϵ is the sign character of W, and u_k is the unipotent element in (a).

(e) *Assume G is simply-laced. Then the grading in (c) is isomorphic to the grading on $H^*(\mathcal{B}_x)$ for x regular unipotent in $\hat{L}_\mu$. That is, we have W-module isomorphisms*

$$V^0_{\lambda_k} \simeq \iota^* H^{2k}(\mathcal{B}_x), \qquad k = 0, \dots, r.$$

(This is false if G is not simply-laced.)

The original motivation behind Theorem 1 was to use the theory of minuscule representations to give more uniform proofs of certain facts about small representations for simply-laced groups observed in [R1,2], and to extend these results to the non-simply-laced case. This was partly successful, though item (e) in Theorem 1 is proved case-by-case here. More serious, however, is the unfortunate fact that not all small representations occur in some $\mathrm{End}(V_\mu)$. The rest of the paper therefore puts $\mathrm{End}(V_\mu)$ aside, to study all small representations for all groups. In particular, we reduce the classification of small representations for non-simply laced groups to the simply-laced case, using graph automorphisms and Langlands functorality, and we describe their zero-weight spaces in terms of Springer representations. The results, which are partial generalizations of those in Theorem 1, have uniform statements, but some of the proofs, mainly those involving the Springer correspondence, still rely on case-by-case calculations.

The basic relation between smallness and being attached to a unipotent class is as follows.

Proposition 2. *A small representation V_λ of G is attached to a unipotent class $C \subset \hat{G}$ if and only if V_λ is self-dual.*

The small representations and their zero weight spaces were analyzed for simply-laced groups in [R1,2]. To handle multiply-laced groups, we assume, for the rest of this introduction, that $\hat{G}$ is simply-laced, and that σ is an automorphism of $\hat{G}$ induced by a symmetry of the Dynkin diagram of $\hat{G}$ (see section 4). Let $\hat{G}_\sigma$ denote the group of fixed points under σ, and let G_σ be the dual group of $\hat{G}_\sigma$. Each non-simply-laced adjoint simple group is uniquely a G_σ. The individual cases are given as follows.

G	$\hat{G}$	$\hat{G}_\sigma$	G_σ
PGL_{2n}	SL_{2n}	Sp_{2n}	SO_{2n+1}
PSO_{2n+2}	Spin_{2n+2}	Spin_{2n+1}	PSp_{2n}
PSO_8	Spin_8	$\hat{G}_2$	G_2
E_6^{ad}	E_6^{sc}	$\hat{F}_4$	F_4

(Note that the case $\hat{G} = SL_{2n+1}$, $\hat{G}_\sigma = SO_{2n+1}$ is excluded by the requirement that $\hat{G}_\sigma$ be simply-connected.)

We say V_λ is *σ-stable* if λ, viewed as a co-weight of $\hat{G}$, has image contained in $\hat{G}_\sigma$.

Theorem 3. *Let G be a simply-laced adjoint group, and let G_σ be the adjoint group arising from an automorphism σ of $\hat{G}$ as above. Then the following hold.*

(a) *Each σ-stable small representation V_λ of G is self-dual, hence is attached to a unipotent class $C_\lambda \subset \hat{G}$.*

(b) *The intersection $C_\lambda^\sigma = C_\lambda \cap \hat{G}_\sigma$ is a single class in $\hat{G}_\sigma$, and the G_σ-representation V_λ^σ attached to C_λ^σ is small.*

(c) *The correspondence $V_\lambda \to V_\lambda^\sigma$ is a bijection between the isomorphism classes of small σ-stable G-representations and isomorphism classes of small G_σ representations.*

One may think of V_λ as the "functorial lift" of V_λ^σ, under the inclusion $\hat{G}_\sigma \hookrightarrow \hat{G}$.

The zero weight spaces V_λ^0 and $[V_\lambda^\sigma]^0$ are related to C by means of the Springer correspondence, although this relation is in general less obvious than it was for the small representations appearing in Theorem 1.

Recall that, given a unipotent class C in a reductive group, the "associated special class" is the unipotent class $\dot{C}$ uniquely characterized as follows [Sp1]. First, $\dot{C}$ is special, and contains C in its closure. Second, there are no special classes other than $\dot{C}$ whose closures contain C and which are contained in the closure of $\dot{C}$.

For $u \in \hat{G}_\sigma$, let A_u^σ, $\mathcal{B}_u^\sigma$, $\chi_{u,\rho}^\sigma$, ϵ_σ be the analogues for $\hat{G}_\sigma$ of A_u, $\mathcal{B}_u$, $\chi_{u,\rho}$, ϵ. We again denote by $\iota : W_\sigma \longrightarrow \hat{W}_\sigma$ the canonical isomorphism between the Weyl groups of G_σ and $\hat{G}_\sigma$.

Theorem 4. *Let V_λ be a σ-stable small representation of G, attached to the unipotent class C_λ as in Theorem 3. Let $\dot{C}_\lambda$ be the special class associated to C_λ. Then $\dot{C}_\lambda \cap \hat{G}_\sigma$ is a single class in $\hat{G}_\sigma$, and for $u \in \dot{C}_\lambda \cap \hat{G}_\sigma$, we have*

$$V_\lambda^0 \otimes \epsilon = \bigoplus \iota^* \chi_{u,\rho}, \qquad [V_\lambda^\sigma]^0 \otimes \epsilon_\sigma = \bigoplus \iota^* \chi_{u,\rho}^\sigma,$$

where the sum for V_λ^0 runs over all irreducible representations ρ of A_u, and the sum for $[V_\lambda^\sigma]^0$ runs over just those irreducible representations ρ of A_u^σ which factor through the natural homomorphism $A_u^\sigma \longrightarrow A_u$.

Remarks: The description of V_λ^0 in Theorem 4 is essentially contained in [R2]; for self-dual V_λ (and simply-laced G) the class C_λ turns out to be the "small orbit" associated to V_λ in [R2]. It is known [R4] that if any unipotent class C in $\hat{G}$ meets $\hat{G}_\sigma$, then the intersection is a single class. The content of the first assertion of Theorem 4 is that $\dot{C}_\lambda \cap \hat{G}_\sigma$ is non-empty. It is not always a special class in $\hat{G}_\sigma$. For the unipotent elements u in Theorem 4, we always have $\chi_{u,\rho} \neq 0$, but there are a few cases where $\chi_{u,\rho}^\sigma = 0$, meaning that ρ does not appear in $H^{2d_\sigma(u)}(\mathcal{B}_u^\sigma)$.

I thank B. Gross for giving me a preprint of his paper [G], which helped me to understand the role of the dual group here. I was also influenced by discussions about small representations with V. Toledano Laredo. Finally, hearty thanks go to D. Vogan, for his comments an earlier version this paper.

2. Proof of Theorem 1

Since V_μ is minuscule, we have $\langle \mu, \hat{\beta} \rangle \in \{-1, 0, 1\}$ for all co-roots $\hat{\beta}$ of G, and every weight in $\mathrm{End}(V_\mu)$ is W-conjugate to $\mu - w\mu$ for some $w \in W$. Let $(\ ,\ )$ be a W-invariant inner product on the real vector space spanned by the weight lattice of T, so that $\langle \mu, \hat{\beta} \rangle = 2(\mu, \beta)/(\beta, \beta)$. Now if β is a root of G and 2β is a weight in $\mathrm{End}(V_\mu)$, say $2\beta = \mu - w\mu$ as above, we have

$$(\mu, \mu) = (w\mu, w\mu) = (\mu - 2\beta, \mu - 2\beta) = (\mu, \mu) - 4(\mu, \beta) + 4(\beta, \beta),$$

so

$$\langle \mu, \hat{\beta} \rangle = 2\frac{(\mu, \beta)}{(\beta, \beta)} = 2,$$

a contradiction. Hence every constituent of $\mathrm{End}(V_\mu)$ is small.

For $wW_\mu \in W/W_\mu$, let $v_{w\mu}$, $v_{-w\mu}$ denote weight vectors, with the indicated weights, in V_μ and V_μ^*. If $w \in W_\mu$ then w is represented by an element $\dot{w}$ of the derived group of the parabolic subgroup stabilizing the line through v_μ, and $\dot{w}$ fixes v_μ. Now W acts on the zero weight space in $V_\mu \otimes V_\mu^*$, and w fixes $v_\mu \otimes v_{-\mu}$. The set

$$\{v_{w\mu} \otimes v_{-w\mu} :\ wW_\mu \in W/W_\mu\}$$

is therefore a basis of the zero weight space in $V_\mu \otimes V_\mu^*$. It follows that

$$\mathrm{End}_T(V_\mu) \simeq \mathrm{Ind}_{W_\mu}^W \mathbf{C}.$$

Since $W_\mu \backslash W / W_\mu$ is represented by the involutions $w_{\beta_1} \dots w_{\beta_k}$ [RRS,Thm. 1.2], it follows that $\mathrm{Ind}_{W_\mu}^W \mathbf{C}$ is multiplicity-free, hence $\mathrm{End}(V_\mu)$ is also multiplicity-free. Both $\mathrm{End}_T(V_\mu)$ and $\mathrm{End}(V_\mu)$ have the same number $|W_\mu \backslash W / W_\mu|$ of constituents, so the zero weight space of each constituent of $\mathrm{End}(V_\mu)$ must be irreducible. Since dual representations have isomorphic zero weight spaces, each constituent of $\mathrm{End}(V_\mu)$ must be self-dual.

We pass now to the dual group $\hat{G}$. Every root $\hat{\beta}$ in $\hat{U}$ has the property $\langle \mu, \hat{\beta} \rangle = 1$. Let $\hat{\beta}_1, \ldots, \hat{\beta}_r$ be a maximal orthogonal set of long roots in $\hat{U}$, let β_i be the corresponding co-roots, and let w_{β_i} be the corresponding reflections in $\hat{W}$. Then for $w = w_{\beta_1} \ldots w_{\beta_k}$, we have

$$\mu - w\mu = \beta_1 + \cdots + \beta_k.$$

Set $\lambda_k := \beta_1 + \cdots + \beta_k$. This is the extreme weight of a constituent V_k of $\mathrm{End}(V_\mu)$.

Each β_k extends, by Chevalley theory, to a homomorphism $\varphi_{\beta_k} : SL_2 \longrightarrow \hat{G}$ and since the β_k's are orthogonal, the images of the φ_{β_k}'s commute with one another. The sum $\beta_1 + \cdots + \beta_k$ is a co-weight which extends to the map $g \mapsto \varphi_k(g) = \varphi_{\beta_1}(g) \ldots \varphi_{\beta_k}(g)$, $g \in SL_2$. Thus, V_k is attached to the class of $u_k := \varphi_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The proof of parts (a)-(c) of Theorem 1 will be complete if we can ensure dominance of $\lambda_k = \beta_1 + \cdots + \beta_k$. The following construction appeared with a different purpose in the proof of [RRS, (2.8)]. Fix a simple system $\hat{S}$ of the root system $\hat{R}$ of $\hat{T}$ in $\hat{G}$, so that $\hat{S}$ is the vertex set of the Dynkin graph $\mathcal{D}$ of $\hat{G}$. For every $J \subseteq \hat{S}$, let $\hat{R}(J)$ be the based subroot system of $\hat{R}$, with basis J. There is a unique "minuscule vertex" $\hat{\alpha} \in \hat{S}$ such that $\langle \mu, \hat{\alpha} \rangle = 1$, and μ vanishes on the remaining roots in $\hat{S}$. Let $J_1 = \hat{S}$, and let $\hat{\beta}_1$ be the highest root with respect to $\hat{R} = \hat{R}(J_1)$. Let $\mathcal{D}_1$ be the subgraph of $\mathcal{D}$ whose vertices are orthogonal to $\hat{\beta}_1$. If $\langle \beta_1, \hat{\alpha} \rangle = 0$, let J_2 be the set of vertices in the component of $\mathcal{D}_1$ containing $\hat{\alpha}$. Then $\hat{\alpha}$ is again a minuscule vertex with respect to J_2 [loc.cit.], and we let $\hat{\beta}_2$ be the highest root in $\hat{R}(J_2)$. Repeat until $\langle \beta_r, \hat{\alpha} \rangle \neq 0$.

We must show that $\langle \lambda_k, \hat{\gamma} \rangle \geq 0$ for all $\hat{\gamma} \in \hat{S}$. Suppose first that $\hat{\gamma} \in \hat{S}$ is not orthogonal to the highest root $\hat{\beta}_1$. Considering each Dynkin diagram, we find a unique root $\hat{\delta} \in J_2$ not orthogonal to $\hat{\gamma}$. Moreover, $\hat{\delta}$ is minuscule for J_2 (hence has coefficient =1 in $\hat{\beta}_2$) and does not belong to J_3. In all but one case, $\hat{\delta}$ and $\hat{\beta}_1$ are joined to $\hat{\gamma}$ by single bonds, so that

$$\langle \beta_2, \hat{\gamma} \rangle = \langle \delta, \hat{\gamma} \rangle = -1 = -\langle \beta_1, \hat{\gamma} \rangle.$$

The exception is C_n, where -1 in the line above is replaced by -2.

Now if $\hat{\nu} \in J_3$, we must have $\langle \nu, \hat{\gamma} \rangle = 0$, else there would be a cycle in $\mathcal{D}$, involving $\hat{\gamma}, \hat{\delta}, \hat{\nu}$. Thus, in all cases, we have

$$\langle \beta_1, \hat{\gamma} \rangle > 0, \qquad \langle \beta_1 + \cdots + \beta_k, \hat{\gamma} \rangle = 0, \quad \text{for } k > 1.$$

Now let $\hat{\gamma} \in \hat{S}$ be arbitrary. We may suppose there is a minimal m such that $\langle \beta_m, \hat{\gamma} \rangle \neq 0$. Clearly $\langle \lambda_k, \hat{\gamma} \rangle = 0$ if $k < m$. It follows that $\hat{\gamma}$ belongs to the $\hat{\alpha}$-component of $\mathcal{D}_{m-1}$, i.e., $\hat{\gamma} \in J_m$. The previous paragraph applied to $S = J_m$ shows that $\langle \lambda_k, \hat{\gamma} \rangle \geq 0$ if $k \geq m$, so λ_k is indeed dominant.

Part (d) of Theorem 1 is a special case of a general fact for all small representations, to be proved later. It remains to prove (e), so we now assume G is simply-laced. The small modules and their zero weight spaces were determined in [R1]. For E_6 and E_7, the assertion (e) is obtained by comparing [R1, p. 439] with the tables in [BS]. If $G = SO_{2n}$ with $\mu = \omega_1$, then x is regular in SO_{2n-2}, and $d(x) = 2 = r$. The calculation of $H^2(\mathcal{B}_x)$ follows from [Sp2], and $H^4(\mathcal{B}_x)$ is given by the Springer correspondence (c.f. [Ca]). Using the notation of [R1] for representations of $W(D_n)$, we have

$$\begin{aligned} H^0(\mathcal{B}_x) &= \chi(n; -) = V_0^0, \\ H^2(\mathcal{B}_x) &= \chi(n-1; 1) = V_{\omega_2}^0, \\ H^4(\mathcal{B}_x) &= \chi(n-1, 1; -) = V_{2\omega_1}^0. \end{aligned}$$

Consider now the case SO_{2n} with $\mu_+ = \omega_n$ or $\mu_- = \omega_{n-1}$. The corresponding Levi subgroups are $\hat{L}_{\mu_\pm} \simeq GL_n$. If n is odd, then these Levi's are conjugate, and the element x is regular unipotent in GL_n, with partition $[n, n]$. If $n = 2m$ is even, then $x = x_\pm$ are regular unipotent in $\hat{L}_{\mu_\pm}$, and represent distinct classes with partition $[n, n]$. Part (e) amounts to the formulas

$$H^{2k}(\mathcal{B}_x) = \chi(n-k, k) \quad n \text{ odd}$$

$$H^{2k}(\mathcal{B}_{x_\pm}) = \begin{cases} \chi(n-k, k) & k < m \quad (n = 2m) \\ \chi(m, m)_\pm & k = m \quad (n = 2m) \end{cases}$$

For fixed n, the representations on the right side are determined by their dimension, and we know they all occur in $H^*(\mathcal{B}_x)$, by Lusztig's result recalled in paragraph (iii) of the introduction, and the fact that

$$\mathrm{Ind}_{W_{\mu_\pm}}^{W} \mathbf{C} = \begin{cases} \chi(n; -) \oplus \chi(n-1; 1) \oplus \cdots \oplus \chi(m+1; m-1), & (n = 2m+1) \\ \chi(n; -) \oplus \chi(n-1; 1) \oplus \cdots \oplus \chi(m+1; m-1) \oplus \chi(m, m)_\pm, & (n = 2m) \end{cases}$$

It suffices therefore to prove that the dimension h_n^k of $H^{2k}(\mathcal{B}_x)$ is given by

(*)
$$h_n^k = \begin{cases} \binom{n}{k} & \text{if } k < \frac{n}{2} \\ \frac{1}{2}\binom{2m}{m} & \text{if } k = \frac{n}{2} = m. \end{cases}$$

For $k = \left[\frac{n}{2}\right] = \dim \mathcal{B}_x$, this follows from the Springer correspondence.

For $p \le q$, let $\mathcal{B}(p, q)$ be the variety of Borel subgroups of $SO_{2(p+q)}$ containing a fixed unipotent element of partition $[p, q]$. Let

$$h_k(p, q) = \dim H_{2k}(\mathcal{B}(p, q)),$$

where H_* denotes Borel-Moore homology. Recall that $\dim H_*(X) = \dim H^*(X)$ if X is compact. In particular, $h_k(n, n) = h_n^k$. We have $h_k(p, q) = 0$ for $k > \dim \mathcal{B}(p, q) = \left[\frac{p}{2}\right]$. We will also prove, for $p < q$ both odd, $2n = p + q$, and $k \le \frac{p-1}{2}$, that

(**)
$$h_k(p, q) = \binom{n}{k}.$$

The proofs of $(*)$ and $(**)$ use induction on n. If $n = 2$, then $\mathcal{B}_x \simeq \mathbf{P}^1 \times \text{point}$, so $(*)$ is true. The only possibility for $(**)$ is $\mathcal{B}(1, 3) = \text{point}$, and $h_0(1, 3) = 1$, so $(**)$ is true.

Assume $n > 2$, let $V = \mathbf{C}^{2n}$ and let Q be the symmetric bilinear form on V defining $SO_{2n} = SO(V)$. Then $\mathcal{B}$ is a connected component of the variety $X(V)$ of Q-isotropic flags $E = (E_1 \subset E_2 \subset \cdots E_n)$ in V with $\dim E_d = d$. Let $\mathcal{P}$ be the variety of isotropic lines in the projective space $\mathbf{P}(V)$. We have a fibration $\pi : \mathcal{B} \longrightarrow \mathcal{P}$ sending E to E_1. For any line $L \in \mathcal{P}$, let $L^\perp$ be the Q ortho-complement of L. Then Q induces a non-degenerate form on $L^\perp/L$. We identify $\pi^{-1}(L) = \mathcal{B} \cap X(L^\perp/L)$ = variety of Borel subgroups of $SO(L^\perp/L)$.

Let π_x be the restriction to x-fixed points,

$$\pi_x : \mathcal{B}_x \longrightarrow \mathcal{P}_x.$$

Since x has partition $[n, n]$, we have $\mathcal{P}_x = \mathbf{P}(V_x) \simeq \mathbf{P}^1$, and $\pi_x^{-1}(L) = \mathcal{B}_x \cap X(L^\perp/L)$ for $L \in \mathcal{P}_x$, so we consider the action of x on $L^\perp/L$.

Write V as a direct sum $V = V_1 \oplus V_2$ of x-stable isotropic subspaces V_i, on each of which x has a single Jordan block of size n, and let $N = x - I$. There are bases $\{v_1, \ldots, v_n\}$ of V_1 and $\{v_{-n,}, \ldots, v_{-1}\}$ of V_2 such that $Q(v_i, v_{-j}) = \delta_{ij}$, and $Nv_i = v_{i-1}$ for $2 \le i \le n$, $Nv_{-j} = -v_{-j-1}$ for $1 \le j \le n-1$ and $\ker N$ is spanned by $\{v_1, v_{-n}\}$. Then L is spanned by a vector $\ell = av_1 + bv_{-n}$, and we choose $\ell' = cv_n + dv_{-1}$ nonzero such that $Q(\ell, \ell') = 0$. One checks that the kernel $[L^\perp/L]_{\bar{N}}$ of the map $\bar{N}$ on $L^\perp/L$ induced by N is spanned by the images of $\{v_1, v_{-n}, av_2 - bv_{1-n}\}$, hence $\dim[L^\perp/L]_{\bar{N}} = 2$, so $\bar{N}$ has Jordan partition $[p, q]$, with $p \le q \le n$ and $p + q = 2n - 2$. Moreover, $N^{n-1}\ell' = ce_1 + (-1)^{n-1} de_{-n}$, which belongs to L iff n is even or $ab = 0$.

Thus, the map $\pi_x : \mathcal{B}_x \longrightarrow \mathcal{P}_x$ has the following structure: If n is even, and $x = x_\pm$, then $\pi_x : \mathcal{B}_x \longrightarrow \mathbf{P}^1$ is a fibration with fiber $\mathcal{B}(n-1, n-1)$. It follows that

$$\text{(A)} \qquad h_k(n, n) = h_k(n-1, n-1) + h_{k-1}(n-1, n-1) \qquad (n \text{ even}).$$

If n is odd, let L_1, L_{-n} denote the lines through v_1, v_{-n}, so that $\mathbf{P}^1 - \{L_1, L_{-n}\} = \mathbf{C}^\times$. Define

$$Z = \pi_x^{-1}(L_1) \cup \pi_x^{-1}(L_{-n}), \qquad U = \mathcal{B}_x - Z = \pi_x^{-1}(\mathbf{C}^\times).$$

Then Z consists of two disjoint copies of $\mathcal{B}(n-1, n-1)$, and $\pi_x : U \longrightarrow \mathbf{C}^\times$ is a fibration with fiber $\mathcal{B}(n-2, n)$, so

$$\dim H_{2k}(U) = h_{k-1}(n-2, n), \qquad \dim H_{2k+1}(U) = h_k(n-2, n).$$

Since $H_{odd}(\mathcal{B}(p, q)) = 0$, the long exact sequence arising from the inclusion $Z \hookrightarrow \mathcal{B}(n, n)$ gives an exact sequence

$$0 \longrightarrow H_{2k+1}(U) \longrightarrow H_{2k}(Z) \longrightarrow H_{2k}(\mathcal{B}(n, n)) \longrightarrow H_{2k}(U) \longrightarrow 0.$$

It follows that

$$\text{(B)} \quad h_k(n, n) = 2h_k(n-1, n-1) + h_{k-1}(n-2, n) - h_k(n-2, n) \qquad (n \text{ odd}).$$

Suppose $y \in SO_{2(p+q)}$ is unipotent with partition $[p, q]$, and $p < q$ are both odd. We again have $\mathcal{P}_y = \mathbf{P}(V_y) = \mathbf{P}^1$, and a similar calculation shows that the map $\pi_y : \mathcal{B}(p, q) \longrightarrow \mathbf{P}^1$ (restriction of π) is a fibration above $\mathbf{C}$ with fiber $\mathcal{B}(p-2, q)$, while $\pi_y^{-1}(\infty) = \mathcal{B}(p, q-2)$ (where $\infty \in \mathbf{P}^1$ is the kernel of $y-1$ in the q-block). The long exact sequence now gives

$$\text{(C)} \qquad h_k(p,q) = h_k(p, q-2) + h_k(p-2, q) \qquad (p < q \text{ both odd}).$$

Finally, assuming $(*)$ and $(**)$ hold for all $n' < n$, formulas (A-C) show that $(*)$ and $(**)$ hold for n, using the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

The proof of Theorem 1(e) for SL_n goes along the same lines, using an analogue of formula (C). We omit these details.

3. Representations attached to unipotent orbits.

Let V_λ be a representation of G of highest weight λ, and let C be a unipotent class in $\hat{G}$.

3.1 Definition. *We say that V_λ is* attached *to C if the co-weight $\lambda : \mathbf{C}^\times \longrightarrow \hat{G}$ is of the form $\lambda(t) = \varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$, $t \in \mathbf{C}^\times$, where $\varphi : SL_2 \longrightarrow \hat{G}$ is a homomorphism mapping the nontrivial unipotent elements of SL_2 into C.*

Concretely, this means that if α is a simple root in $\hat{G}$, then the integer attached to the α-node in the weighted Dynkin diagram of C is equal to $\langle\lambda, \check{\alpha}\rangle$, where $\check{\alpha}$ is the corresponding co-root of G.

Assume for the time being that G is simply-laced. Then we can identify unipotent classes in G and $\hat{G}$ having the same weighted Dynkin diagram. For every small G-module V_λ, there is a unique unipotent class $C_\lambda \subset G$ and an integer $d(\lambda)$ such that C_λ is minimal among all orbits whose ring of functions contains V_λ in degree $d(\lambda)$. Moreover, if u belongs to the dual of the special class $\dot{C}_\lambda$ associated to C_λ, then $d(\lambda) = \dim \mathcal{B}_u$ and $V_\lambda^0 \simeq H^{2d(u)}(\mathcal{B}_u)$. All of this is proved in [R2].

3.2 Proposition. *Assume G is simply-laced, and V_λ is small. Then V_λ is attached to some unipotent class C, in the sense of 3.1, if and only if V_λ is self-dual. In this case, $C = C_\lambda$.*

Proof. We make some remarks on the individual groups, ignoring the trivial representation, and complete the proof with a table.
$G = PGL_n$: The weighted Dynkin diagram $\mathcal{D}(C)$ of a unipotent class C is invariant under the graph automorphism, so V_λ can only be attached to a unipotent class if it is self-dual. Now every small module, or its contragredient, has highest weight a partition λ of n, and $V_\lambda \simeq V_\lambda^*$ if and only if $\lambda = [2^k 1^{n-2k}]$, for $1 \le k \le \frac{n}{2}$. In terms of fundamental weights, we have $\lambda = \lambda_k + \lambda_{n-k}$. On the other hand, λ is also the partition giving the Jordan blocks of elements in C_λ.

$G = SO_{2n}$: All representations are self-dual if n is even. The classes C_λ are determined by their partitions except $C_\lambda = [2^n]_\pm$, which are the Richardson classes in $\hat{P}_{\mu\pm}$ (section 2). If n is odd, the only small modules which are not self-dual are $V_{\lambda_1+\lambda_n}$ and its dual. For these λ, the element h of the $\mathfrak{sl}_2$-triple for C_λ is given in [R2, 5], and we do not have $\alpha(h) = \langle\lambda, \check{\alpha}\rangle$ for all simple roots α.

$G = E_n$: The non-self-dual small modules occur only in E_6, where all diagrams $\mathcal{D}(C_\lambda)$ are invariant under the graph automorphism, so these modules are not attached to unipotent classes. The self-dual small-modules V_λ are listed below, along with orbit C_λ to which they are attached.

G	λ	C_λ	$\mathcal{D}(C_\lambda)$
PGL_n	$\lambda_q + \lambda_{n-q},\ (1 \le q < \frac{n}{2})$	$[2^q 1^{n-2q}]$	$0 \cdots 0\ \overset{q}{1}\ 0 \cdots 0\ \overset{n-q}{1}\ 0 \cdots 0$
PGL_n	$2\lambda_m,\ (n = 2m)$	$[2^m]$	$0 \cdots 0\ \overset{m}{2}\ 0 \cdots 0$
PSO_{2n}	$\lambda_{2q},\ (1 \le q < \frac{n}{2})$	$[2^{2q} 1^{4m-4q}]$	$0\ \cdots\ 0\,\overset{2q}{1}\,0\ \cdots\ \underset{0}{0}\ 0$
PSO_{2n}	$\lambda_1 + \lambda_{2q-1},\ (1 < q \le \frac{n}{2})$	$[32^{2q-2} 1^{4m-4q+1}]$	$1\ 0 \cdots\ 0\ \overset{2q-1}{1}\ 0\ \cdots\ \underset{0}{0}\ 0$
PSO_{2n}	$2\lambda_1$	$[31^{4m-3}]$	$2\ 0 \cdots\ \ 0\ \cdots\ \underset{0}{0}\ 0$
PSO_{4m}	$2\lambda_{2m}$	$[2^{2m}]_+$	$0\ \cdots\ 0\ \ \underset{2}{0}\ 0$
PSO_{4m}	$2\lambda_{2m-1}$	$[2^{2m}]_-$	$0\ \cdots\ 0\ \ \underset{0}{0}\ 2$
E_6	λ_3	$3A_1$	$0\ 0\ \underset{0}{1}\ 0\ 0$
E_6	$\lambda_1 + \lambda_5$	$2A_1$	$1\ 0\ \underset{0}{0}\ 0\ 1$
E_6	λ_6	A_1	$0\ 0\ \underset{1}{0}\ 0\ 0$
E_7	$\lambda_6 + \lambda_7$	$4A_1$	$0\ 0\ \underset{1}{0}\ 0\ 0\ 1$
E_7	$2\lambda_6$	$(3A_1)''$	$0\ 0\ \underset{0}{0}\ 0\ 0\ 2$
E_7	λ_2	$(3A_1)'$	$0\ 1\ \underset{0}{0}\ 0\ 0\ 0$
E_7	λ_5	$2A_1$	$0\ 0\ \underset{0}{0}\ 0\ 1\ 0$
E_7	λ_1	A_1	$1\ 0\ \underset{0}{0}\ 0\ 0\ 0$
E_8	λ_8	$4A_1$	$0\ 0\ \underset{1}{0}\ 0\ 0\ 0\ 0$
E_8	λ_6	$3A_1$	$0\ 0\ \underset{0}{0}\ 0\ 0\ 1\ 0$
E_8	λ_1	$2A_1$	$1\ 0\ \underset{0}{0}\ 0\ 0\ 0\ 0$
E_8	λ_7	A_1	$0\ 0\ \underset{0}{0}\ 0\ 0\ 0\ 1$

4. Multiply-laced groups

In this section we prove Theorem 3. We continue to assume G simply-laced, simple, and adjoint. The dual group $\hat{G}$ is simply-connected. A "graph automorphism" of $\hat{G}$ is defined as follows [S, §8]. Fix a maximal torus and Borel subgroup $\hat{T} \subset \hat{B}$, and non-trivial root elements $u_\alpha \in \hat{B}$, for each simple root α of $\hat{T}$ in $\hat{B}$. Any symmetry σ of the Dynkin diagram of $\hat{G}$ extends to a unique automorphism of $\hat{G}$, again denoted by σ, which preserves $\hat{T}$ and $\hat{B}$, and permutes the elements u_α according to the original diagram symmetry.

Let $R \subset Y$ denote the co-roots and co-weights of $\hat{T}$. Then σ permutes the elements of R and Y, as well as the set $S \subset R$ of simple co-roots. We have $Y = \mathbf{Z}S$, since $\hat{G}$ is simply-connected.

Now assume that σ is chosen so that the fixed point group $\hat{G}_\sigma$ is also simply-connected. For any co-root $\alpha \in R$, let $\bar{\alpha} = \alpha + \sigma\alpha + \cdots$ be the sum of the distinct σ-translates of α. Our assumption implies that distinct σ-translates are orthogonal to one another. Hence, if $(\ ,\ )$ is a W-invariant inner product on Y, we have

$$(\bar{\alpha}, \bar{\alpha}) = m(\alpha, \alpha),$$

where $m \in \{1, 2, 3\}$ is the number of distinct σ-translates of α.

Let S' be a set of representatives of the σ-orbits in S, and let $S_\sigma = \{\bar{\alpha} :\ \alpha \in S'\}$. The short elements of S_σ are those for which $\bar{\alpha} = \alpha$.

The fixed point group $\hat{T}_\sigma$ is a maximal torus of $\hat{G}_\sigma$. Its co-weight lattice is Y_σ, and S_σ is a set of simple co-roots of $\hat{T}_\sigma$. Since $\hat{G}_\sigma$ is simply-connected, we have $Y_\sigma = \mathbf{Z}S_\sigma$, and moreover $Y_\sigma \cap \mathbf{Z}_{\geq 0}S = \mathbf{Z}_{\geq 0}S_\sigma$.

Now the highest co-root β of $\hat{G}$ (with respect to S) is unique, hence is fixed by σ, and β is the highest short co-root of $\hat{G}_\sigma$. If $\lambda \in Y_\sigma$, then $\lambda - 2\beta \in Y_\sigma$, so we have

$$\lambda - 2\beta \in \mathbf{Z}_{\geq 0}S \quad \Leftrightarrow \quad \lambda - 2\beta \in \mathbf{Z}_{\geq 0}S_\sigma. \tag{4a}$$

Let G_σ be the adjoint group whose dual is $\hat{G}_\sigma$. Then G_σ has a maximal torus T_σ whose character group is identified with Y_σ, and S_σ is a set of simple roots of T_σ in G_σ. Note that G_σ is not necessarily a subgroup of G.

We say the irreducible G-representation V_λ is σ-stable, if $\lambda \in Y_\sigma$. If λ is dominant with respect to S, then λ is also dominant with respect to S_σ, and is the highest weight of an irreducible G_σ representation V_λ^σ.

A representation of an adjoint group is small exactly when twice the highest short root β is not a weight, which by saturation of weights is equivalent to $2\beta \not\leq \lambda$, in the usual partial order on weights. from (4a), we see that V_λ is small for G if and only if V_λ^σ is small for G_σ.

If G has type $A_n,\ D_{2m+1},\ E_6$, then V_λ being σ-stable is equivalent to its being self-dual, and if G has type D_{2m} then V_λ is always self-dual. By 3.2, every σ-stable small V_λ is therefore attached to a unipotent class $C_\lambda \subset \hat{G}$. Since λ is fixed by σ, we can choose $\varphi : SL_2 \longrightarrow \hat{G}$ to have image in $\hat{G}_\sigma$, hence V_λ^σ is attached to the unipotent class $C_\lambda^\sigma \subset \hat{G}_\sigma$ containing the image under φ of the nontrivial unipotent elements in SL_2. By [R4, Cor. 4.3] we in fact have $C_\lambda^\sigma = \hat{G}_\sigma \cap C_\lambda$. This proves Theorem 3.

We tabulate the bijection of Theorem 3 below. For $\hat{G} = SL_{2n}$ the relevant unipotent classes are

$$u_k := [2^k 1^{2n-2k}],$$

each of which meets $\hat{G}_\sigma = Sp_{2n}$ in a single class, denoted u_k^σ.

For $\hat{G} = \mathrm{Spin}_{2n+2}$, the relevant unipotent classes are given by their partitions in SO_{2n+2} as follows

$$x_p = [2^{2p}1^{2n-4p+2}] \ (0 \le 2p \le n), \quad y_q = [32^{2q-2}1^{2n-4q+3}] \ (1 \le 2q-1 \le n).$$

Each of these classes meets Spin_{2n+1} in a single class, denoted x_p^σ, y_q^σ, respectively, whose partition is obtained by removing a 1 from the corresponding partition in Spin_{2n+2}.

For the exceptional groups unipotent classes are denoted as in [Ca]. In the table below, the columns are respectively G, the highest weight of the small module V_λ, the Dynkin diagram $\mathcal{D}(C_\lambda)$, the Dynkin diagram of $\mathcal{D}(C_\lambda^\sigma)$, the name or partition of the class $C_\lambda^\sigma \subset \hat{G}_\sigma$, the highest weight of the G_σ-module V_λ^σ, and the group G_σ. We denote by λ_i the fundamental weights of G^{sc}, and by ω_i the fundamental weights of G_σ^{sc}, using the indexing given in section 3, along with $1 \Rrightarrow 2$ for $G_\sigma = G_2$ and $12 \Rightarrow 34$ for $G_\sigma = F_4$.

G	V_λ	C_λ	$\mathcal{D}(C_\lambda)$	$\mathcal{D}(C_\lambda^\sigma)$	C_λ^σ	V_λ^σ	G_σ
PGL_{2n}	$\lambda_k+\lambda_{2n-k}$	u_k	$0\cdots0\overset{k}{1}0\cdots010\cdots0$	$0\cdots0\overset{k}{1}0\cdots0\Leftarrow0$	u_k^σ	ω_k	SO_{2n+1}
PGL_{2n}	$2\lambda_n$	u_n	$0\cdots020\cdots0$	$0\cdots0\Leftarrow2$	u_n^σ	$2\omega_n$	SO_{2n+1}
PSO_{2n+2}	λ_j (j even,$<n$)	$x_{j/2}$	$0\ \cdots\ 0\overset{j}{1}0\ \cdots\ 0\ 0$ 0	$0\cdots0\overset{j}{1}0\cdots0\Rightarrow0$	$x_{j/2}^\sigma$	ω_j	PSp_{2n}
PSO_{4m+2}	$\lambda_n+\lambda_{n+1}$ ($n=2m$)	x_m	$0\ \cdots\ 0\ \ 0\ 1$ 1	$0\cdots0\Rightarrow1$	x_m^σ	ω_n	PSp_{2n}
PSO_{2n+2}	$\lambda_1+\lambda_k$ (k odd,$<n$)	$y_{\frac{k+1}{2}}$	$1\ \cdots\ 0\ \overset{k}{1}\ 0\ \cdots\ 0\ 0$ 0	$10\cdots0\overset{k}{1}0\cdots0\Rightarrow0$	$y_{\frac{k+1}{2}}^\sigma$	$\omega_1+\omega_k$	PSp_{2n}
PSO_{4m}	$\lambda_1+\lambda_n+\lambda_{n+1}$ ($n=2m-1$)	y_m	$1\ \cdots\ 0\ \ 0\ 1$ 1	$10\cdots0\Rightarrow1$	y_m^σ	$\omega_1+\omega_n$	PSp_{2n}
PSO_8	λ_2	x_1	$0\ 1\ 0$ 0	$0\Lleftarrow1$	A_1	ω_2	G_2
PSO_8	$\lambda_1+\lambda_3+\lambda_4$	y_2	$1\ 0\ 1$ 1	$1\Lleftarrow0$	$\widetilde{A}_1$	ω_1	G_2
E_6	λ_3	$3A_1$	$0\ 0\ 1\ 0\ 0$ 0	$00\Leftarrow10$	$A_1+\widetilde{A}_1$	ω_3	F_4
E_6	$\lambda_1+\lambda_5$	$2A_1$	$1\ 0\ 0\ 0\ 1$ 0	$10\Leftarrow00$	$\widetilde{A}_1$	ω_1	F_4
E_6	λ_6	A_1	$0\ 0\ 0\ 0\ 0$ 1	$00\Leftarrow01$	A_1	ω_4	F_4

5. Zero Weight Spaces

In this section we prove Theorem 4. Recall that G is simply-laced. Let V_λ be a small σ-stable G-module, attached to the unipotent class C_λ. Let $\dot{C}_\lambda$ be the special class attached to C_λ by Spaltenstein. By [R1,2] we have

$$V_\lambda^0 \otimes \epsilon = \bigoplus \iota^* \chi_{u,\rho}$$

for $u \in \dot{C}_\lambda$, summing over all characters ρ of A_u. We verify an analogue of this for the G_σ-module V_λ^σ, using case-by-case calculations. The zero-weight spaces of the non-simply laced small representations are calculated as in [R1], and we omit these details. We refer to [Ca] for the Springer correspondence.

4.1 Theorem. *The class $\dot{C}_\lambda$ meets $\hat{G}_\sigma$ in a single unipotent class $\dot{C}_\lambda^\sigma$ of $\hat{G}_\sigma$. For $u \in \dot{C}_\lambda^\sigma$, we have*

$$[V_\lambda^\sigma]^0 \otimes \epsilon_\sigma \simeq \bigoplus \iota^* \chi_{u,\rho},$$

where ρ runs over those irreducible representations of A_u^σ which factor through the natural map $A_u^\sigma \longrightarrow A_u$.

Proof. In the table below, we denote representations of a Weyl group of type B_n by a pair (α, β) of partitions with sum $|\alpha| + |\beta| = n$. For exceptional Weyl groups we use the notation of [Ca]. For the class $y_q = [32^{2q-2}1^{2n-4q+3}]$ in SO_{2n+2}, the associated special class is $z_q = [3^2 2^{2q-4} 1^{2n-4q+4}]$, and $z_q^\sigma = [3^2 2^{2q-4} 1^{2n-4q+3}]$ in SO_{2n+1}.

For $u \in \dot{C}_\lambda^\sigma$ we give the Springer representation $\chi_{u,\rho}$, with $\chi_{u,1}$ listed first. This is a representation of the Weyl group of $\hat{G}_\sigma$. In the next column we give the representation of the Weyl group of G_σ on $[V_\lambda^\sigma]^0 \otimes \epsilon_\sigma$. An entry $\leftarrow$ means that entry is the same as the one immediately to the left (ι^* is the identity in the standard partition notation for classical Weyl groups).

G	λ	$\dot{C}_\lambda$	A_u	A_u^σ	$\dot{C}_\lambda^\sigma$	$H^{2d_\sigma(u)}(\mathcal{B}_u^\sigma)$	$[V_\lambda^\sigma]^0\otimes\epsilon_\sigma$	V_λ^σ	G_σ
PGL_{2n}	$\lambda_k+\lambda_{2n-k}$ $(2\lambda_n$ if $k{=}n)$	u_k	1	1	u_k^σ	$(1^p,1^{n-p})$ $(k{=}2p)$ $(1^{n-p},1^p)$ $(k{=}2p{+}1)$	$\leftarrow$ $\leftarrow$	ω_k $(2\omega_n$ if $k{=}n)$	SO_{2n+1}
PSO_{2n+2}	λ_{2p} $(2p{\leq}n)$	x_p	1	1	x_p^σ	$(\cdot,2^p1^{n-2p})$	$\leftarrow$	ω_{2p}	PSp_{2n}
PSO_{2n+2}	$\lambda_1+\lambda_{2q-1}$ $(1{<}2q{-}1{<}n)$	z_q	S_2	S_2	z_q^σ	$(1,2^{q-1}1^{n-2q+1})$ $\oplus$ $(\cdot,32^{q-2}1^{n-2q+1})$	$\leftarrow$ $\oplus$ $\leftarrow$	$\omega_1+\omega_{2q-1}$	PSp_{2n}
PSO_{2n+2}	$2\lambda_1$	y_1	1	S_2	y_1^σ	$(1,1^{n-1})$	$\leftarrow$	$2\omega_1$	PSp_{2n}
PSO_{4m}	$\lambda_1+\lambda_n+\lambda_{n+1}$ $(n{=}2m{-}1)$	z_m	S_2	S_2	z_m^σ	$(1,2^{m-1})$ $\oplus$ $(\cdot,32^{m-2})$	$\leftarrow$ $\oplus$ $\leftarrow$	$\omega_1+\omega_{n-1}$	PSp_{2n}
PSO_8	λ_2	x_1	1	1	A_1	$\phi''_{1,3}$	$\phi'_{1,3}$	ω_2	G_2
PSO_8	$\lambda_1+\lambda_3+\lambda_4 3$	z_2	S_2	S_3	$G_2(a_1)$	$\phi_{2,1}\ \oplus\ \phi'_{1,3}$	$\phi_{2,1}$	ω_1	G_2
E_6	λ_3	A_2	S_2	S_2	A_2	$\phi''_{8,9}\ \oplus\ \phi''_{1,12}$	$\phi'_{8,9}\ \oplus\ \phi'_{1,12}$	ω_3	F_4
E_6	$\lambda_1+\lambda_5$	$2A_1$	1	1	$\widetilde{A}_1$	$\phi_{4,13}\ \oplus\ \phi'_{2,16}$	$\phi_{4,13}$	ω_1	F_4
E_6	λ_6	A_1	1	1	A_1	$\phi''_{2,16}$	$\phi'_{2,16}$	ω_4	F_4

Theorem 4 is clear from the table in those cases where $A_u = 1$. For $u = z_q$ in SO_{2n+2} the map $A_u^\sigma \longrightarrow A_u$ is induced by the inclusion $O_{2n-4q+3} \hookrightarrow O_{2n-4q+4}$ which is an isomorphism on component groups.

Suppose $u = z_2$ in PSO_8. We have $A_u^\sigma = S_3$, and $d_\sigma(u) = 1$. The sign character sgn of S_3 does not appear in $H^2(\mathcal{B}_u^\sigma)$, so $\chi_{u,sgn} = 0$. The reflection representation of S_3 does not factor through any map $S_3 \longrightarrow S_2$, so Theorem 4 is verified in this case.

Suppose that u has type A_2 in E_6. Let W be the Weyl group of E_6. Let w_0, α_0 denote the longest element of W and the highest root in E_6, respectively. Then we may take $u = \exp(e_{\alpha_6} + e_{\alpha_0-\alpha_6})$, where e_β is the root vector for a root β. Let $w = w_0 s_{\alpha_0} \in W$. Then w switches α_6 and $\alpha_0 - \alpha_6$. Let $\hat{T}$ be a σ-stable maximal torus in $\hat{G}$, with normalizer $\hat{N}$. Since w is σ-invariant, we may, by [S, (8.2)], choose a representative n for w in $\hat{N}_\sigma$. Multiplying n by an element of $\hat{T}_\sigma$ if necessary, we may assume n switches e_{α_6} and $e_{\alpha_0-\alpha_6}$, so n belongs to the centralizer of u in $\hat{G}_\sigma$. It remains to see that n represents the non-trivial class in $A_u = S_2$. Let $\{e,h,f\}$ be an $\mathfrak{sl}_2$-triple for $e = e_{\alpha_6} + e_{\alpha_0-\alpha_6}$, and let $\mathfrak{g}_i$ be the i-eigenspace of $ad(h)$ on $\mathfrak{g} = \mathfrak{e}_6$. Then $\mathfrak{g}_0 = \mathfrak{gl}_6$ and e belongs to the dense orbit of GL_6 on $\mathfrak{g}_2 = \bigwedge^3 \mathbf{C}^6$. In terms of a basis $\{v_i\}$ of $\mathbf{C}^6$, we have $e = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$, where one wedge product corresponds to e_{α_6}, the other to $e_{\alpha_0-\alpha_6}$. The connected centralizer of e is the obvious $SL_3 \times SL_3$, and the component group of the centralizer is generated by the element of GL_6 which switches the two wedge products in e. This proves the claim, and completes the proof of Theorem 4.

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