SMALL MODULES, NILPOTENT ORBITS AND MOTIVES OF REDUCTIVE GROUPS

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1. Introduction

Among the finite dimensional representations of $PGL_n(\mathbb{C})$, those whose highest weight is a partition of n enjoy special properties. For example, the zero weight space in such a representation is the irreducible representation of the symmetric group corresponding to the dual partition. This was generalized in [R] to the other simply laced groups, as follows. Let $\mathfrak{g}$ be a simple complex Lie algebra of type A, D or E, with adjoint group G. A finite dimensional representation V of G is *small* if twice a root is not a weight of V. The small modules of $PGL_n(\mathbb{C})$ are, up to contragredient, exactly those whose highest weight is a partition of n. In [R] we found a bijection from small irreducible representations of G, modulo contragredients, to those nilpotent G-orbits in $\mathfrak{g}$ lying in the complement of the closure of the unique maximal non-special orbit (which exists). Let us call such orbits and their elements *big*. In type A all orbits are big. In D_n and $E_{6,7,8}$ there are $n+1, 6, 6, 5$ big orbits respectively.

The bijection is determined by zero-weight spaces and the Springer correspondence, as follows. For any nilpotent element $u \in \mathfrak{g}$, let $\mathcal{B}_u$ be the variety of Borel subalgebras of $\mathfrak{g}$ containing u. Let V be an irreducible small module. Then the zero-weight space V^T is isomorphic, as a Weyl group representation, to the Springer action on the entire top non-vanishing cohomology group of $\mathcal{B}_u$ (which may be reducible), for some nilpotent element $u = u(V)$. The orbits of the $u(V)$'s are exactly the big nilpotent orbits, and $u(V) = u(V')$ for different small modules V, V' if and only if V' is the linear dual of V.

The main purpose of the present paper is to give another connection between V and $u(V)$, in terms of polynomial functions on the "subdual orbit" of $u(V)$.

To explain "subdual", we first recall that the big nilpotent orbits are special. Let $\mathcal{O} \mapsto \mathcal{O}_d$ be the order-reversing duality on the special orbits. The boundary $\partial\mathcal{O}$ of a nilpotent orbit $\mathcal{O}$ is defined as the complement of $\mathcal{O}$ in its closure $\overline{\mathcal{O}}$. Given a big nilpotent orbit $\mathcal{O}$, the subdual orbit $\mathcal{O}_{sd}$ is defined as follows. If every maximal

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orbit in $\partial\mathcal{O}_d$ is special, we define $\mathcal{O}_{sd} = \mathcal{O}_d$. Otherwise, inspection of cases (§5,6) shows there is exactly one nonspecial maximal orbit in $\partial\mathcal{O}_d$, and then we take $\mathcal{O}_{sd}$ to be this nonspecial maximal orbit.

It turns out that $\mathcal{O}_d = \mathcal{O}_{sd}$ exactly when the centralizer of a point in $\mathcal{O}_d$ is connected, and the centralizers for $\mathcal{O}_{sd}$ are always connected. Let us call a nilpotent orbit *small* if it is the subdual of a big orbit. Distinct big orbits have distinct subduals. Thus we also have a bijection between small modules, modulo contragredients, and small orbits. For E_7, E_8 and D_{2m-1}, the union of the small orbits is closed in the null-cone, but it is not closed in E_6 or D_{2m}.

Main Theorem. *Assume G is adjoint of type ADE. Let V be an irreducible small G-module whose zero-weight space is the Springer representation on the big nilpotent orbit $\mathcal{O}$. Let $d(u) = \dim \mathcal{B}_u$, for $u \in \mathcal{O}$. Then V appears with multiplicity one in the space $R^{d(u)}(\mathcal{O}_{sd})$ of regular functions of degree $d(u)$ on the subdual orbit of $\mathcal{O}$. Moreover, V can appear in $R^{d(u)}(\mathcal{O}')$ for another nilpotent orbit $\mathcal{O}'$ only if $\mathcal{O}_{sd} \leq \mathcal{O}'$.*

For $PGL(n)$, a stronger version of this result follows easily from results of Borho, Kraft and Procesi (see §4). This was mentioned in the announcement [B2], which overlaps with our theorem for $PGL(n)$, since here all orbits are of Richardson type. In other groups small orbits are usually not Richardson.

In type D most, but not all, small orbits have normal closures, and the theorem can be sharpened in those cases, since, when $\overline{\mathcal{O}}_{sd}$ is normal, we actually have $V_\lambda \subset R^{d(u)}(\overline{\mathcal{O}}_{sd})$. This might be true in general, since we always have $V_\lambda \subset R^{d(u)}(\mathcal{N})$, where $\mathcal{N}$ is the variety of nilpotent elements in $\mathfrak{g}$ (see 3.3).

The theorem was already known for the adjoint representation, where $u = u(V)$ is subregular, and $\mathcal{B}_u$ is a union of rank G projective lines (in the simply-laced case). The dual of the subregular orbit is the minimal orbit $\mathcal{O}_{min}$, and the G-module $R(\mathcal{O}_{min})$ of regular functions on $\mathcal{O}_{min}$ is $V_{d\alpha_0}$ in degree d, where α_0 is the highest root and V_λ denotes the irreducible module with highest weight λ. Thus $\mathfrak{g} = V_{\alpha_0}$ occurs once in $R(\mathcal{O}_{min})$, in degree equal to the dimension of $\mathcal{B}_u$, and does not occur in the boundary of $\overline{\mathcal{O}}_{min}$ (which is just the zero orbit).

The proof of the main theorem is based on the following preliminary result, which may be of independent interest, since it extends to arbitrary nilpotent orbits certain results of Kostant on the regular orbit.

Let G be any simple complex Lie group, and take any nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$. Let (f, h, e) be the $\mathfrak{sl}(2)$- triple attached to $\mathcal{O}$. We choose h in a Cartan subalgebra $\mathfrak{t}$, such that a fixed set of positive roots are non-negative on h. Let G_e be the centralizer of $e \in \mathcal{O}$, let $L = G_h$ be the centralizer of h, and let $M = L_e$ be the centralizer of e in L. Let U be the unipotent group generated by root-groups for roots taking positive values on h. If λ is a weight of $\mathfrak{t}$, let ξ_λ be the corresponding character of the maximal torus $T = \exp \mathfrak{t}$, let V_λ be the simple G-module with highest weight λ, and let $\check{V}_\lambda$ be the linear dual of V_λ.

Proposition 1. *Let $\mathcal{O}$ be a nilpotent orbit, and V_λ a simple G-module, with notation as above. Then*

(1) $\operatorname{Hom}_G(\check{V}_\lambda, R^d(\mathcal{O})) \simeq V_\lambda^{G_e}(2d)$, *where $V_\lambda^{G_e}(2d)$ is the $2d$-eigenspace of h on the invariants of G_e in V_λ. In particular, $\check{V}_\lambda$ cannot appear in $R^d(\mathcal{O})$ when $2d > \lambda(h)$.*

(2) *If $\lambda(h) = 2d$, then $\operatorname{Hom}_G(\check{V}_\lambda, R^d(\mathcal{O}))$ is isomorphic to the M-invariants in the simple L-module V_λ^U.*

(3) *Suppose ξ_λ extends to L, and $\lambda(h) = 2d$. Then the multiplicity of $\check{V}_\lambda$ in $R^d(\mathcal{O})$ is at most one, with equality if and only if ξ_λ is trivial on M.*

When $\mathcal{O}$ is the regular-orbit, we have $L = T$, $M = 1$, and these are well-known results due to Kostant [K]. In general, M is reductive, but sometimes disconnected. Proposition 1 is proved in section 2 of this paper.

We prove one more result about multiplicities of small modules. To introduce it, we return to the adjoint representation. G is still simply-laced. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, and let R^d be the regular functions on $\mathcal{N}$ of degree d. For a G module V, we have the multiplicity polynomial

$$P(V, R, q) = \sum_{d \geq 0} \dim \operatorname{Hom}_G(V, R^d) q^d.$$

Kostant proved that

$$P(\mathfrak{g}, R, q) = q^{m_1} + \cdots + q^{m_\ell},$$

where the m_i's are the exponents of the Weyl group.

Let $\mathbb{F}_q$ be a finite field with q elements, where the prime divisor p of q does not divide the coefficients of the highest root. Let $\check{G}$ be the split simply-connected group over $\bar{\mathbb{F}}_q$, having root datum dual to that of G, with Frobenius F. Let $\rho(\mathfrak{t})$ be the reflection representation of $\check{G}^F$. It was observed by Kilmoyer that, for simply laced $\check{G}$, we have

$$\dim \rho(\mathfrak{t}) = q^{m_1} + \cdots + q^{m_\ell}.$$

We generalize this to small modules as follows. Let V be a small module for a simply-laced group G with Weyl group W. Let Π be the representation of $\check{G}^F$ induced from trivial on a Borel subgroup. The endomorphism ring of Π is isomorphic to the group algebra of W. Hence the W-module V^T corresponds to a $\check{G}^F$ representation $\rho(V^T)$ whose constituents occur in Π. We prove

$$P(V, R, q) = \dim \rho(V^T).$$

This could be checked case-by-case, but we give a uniform proof, based on Lusztig's theory of families of unipotent representations. In fact, we have a more general equality of characters:

Proposition 2. *For small V and $g \in \check{G}^F$, we have*

$$tr(g, \rho(V^T)) = tr(F, [H(\check{\mathcal{B}}_g) \otimes V^T]^W).$$

Here $H(\check{\mathcal{B}}_g)$ is the ℓ-adic cohomology of the fixed points of g in the flag variety $\check{\mathcal{B}}$ of $\check{G}$. The W-action on $H(\check{\mathcal{B}}_g)$ was constructed by Lusztig, generalizing that of Springer, and the right side of proposition 2 is a linear combination of Green polynomials.

Inspired by Gan's formula for the trace of a semisimple element on $\rho(\mathfrak{t})$ [Ga], we reformulate the right side of Proposition 2 in terms of the *motive* of $\check{G}$, defined by Gross [G]. Actually, we need to generalize Gross' motive, to make the connection with Green polynomials. See §8. I am grateful to Gan and Gross for instruction on motives.

For interesting interpretations of multiplicities of small modules in other orbits, see [B2], [B3], [Ri] and [ST].

Lusztig has shown me how the result in [R] may have an intersection cohomological proof, based on the last sentence of [L4], and the method of [L2]. Though there remain some obstructions to carrying this out, it seems likely that such ideas may also be applied to this paper. I thank Lusztig for these insights, along with Gross and Kostant for their comments on an earlier version of this paper.

2. Functions on nilpotent orbits

Fix a Borel subgroup and maximal torus $\mathfrak{b} \supset \mathfrak{t}$, with corresponding roots, positive roots and simple roots $\Delta \supset \Delta^+ \supset \Sigma$. Let W be the Weyl group of $\mathfrak{t}$.

(2.1)

Let $e \in \mathfrak{g}$ be nilpotent, and let $\mathcal{O}$ be the adjoint orbit of e. There is a homomorphism $\phi : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}$, with $\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e$, and $h := \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{t}$, such that $\alpha(h) \in \{0, 1, 2\}$ for all simple roots α. Let $\mathfrak{s}$ be the image of ϕ.

Set

$$\mathfrak{g}_i = \{x \in \mathfrak{g} : \ [h, x] = ix\}, \qquad \text{and} \qquad \mathfrak{g}_{\geq m} = \bigoplus_{i \geq m} \mathfrak{g}_i$$

for any m. Then $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ and the subalgebras

$$\mathfrak{p} = \mathfrak{g}_{\geq 0}, \quad \mathfrak{u} = \mathfrak{g}_{\geq 1}, \quad \mathfrak{l} = \mathfrak{g}_0$$

are respectively parabolic, nilradical therein, and Levi component. We let P, U, L be the corresponding subgroups.

Let $M = L_e$ be the centralizer of e in L. Then M is also the centralizer in G of $\mathfrak{s}$. In particular, the identity component of M is reductive. We have [BV]

$$G_e = MU_e, \qquad \text{(semidirect)}$$

and U_e is the unipotent radical of G_e.

The centralizer G_e is contained in the normalizer of e, defined by

$$N = \{g \in G : \ Ad(g)e \in \mathbb{C}^\times e\}.$$

We have an exact sequence

$$1 \to G_e \to N \xrightarrow{\chi} \mathbb{C}^\times \to 1,$$

where $Ad(n)e = \chi(n)e$, for $n \in N$.

(2.2) For any variety X, let $R(X)$ be the ring of everywhere-defined rational functions on X. When X has a $\mathbb{C}^\times$-action, we let $R^d(X)$ denote functions of degree d.

The h-part of the $\mathfrak{sl}(2)$-triple (f, h, e) shows that any nilpotent orbit $\mathcal{O}$ is closed under scalar multiplication by $\mathbb{C}^\times$ in $\mathfrak{g}$. Hence the same is true of the closure $\overline{\mathcal{O}}$, and we may consider $R^d(\mathcal{O})$ and $R^d(\overline{\mathcal{O}})$. The multiplicities in $R^d(\mathcal{O})$ can be computed, in principal, from "graded Frobenius reciprocity", as follows.

Proposition. *For any G-module V, we have*

$$\mathrm{Hom}_G(\check{V}, R^d(\mathcal{O})) \simeq V^{G_e}(2d),$$

where $\check{V}$ is the linear dual of V, and $V^{G_e}(2d)$ is the $2d$ eigenspace of h in the invariants of G_e in V.

proof. We give two proofs. First, view $R(\mathcal{O})$ as right G_e-invariant functions on G. The ungraded Frobenius reciprocity isomorphism is

$$V^{G_e} \longrightarrow \mathrm{Hom}_G(\check{V}, R(\mathcal{O})),$$

via $v \mapsto \phi_v$, and $\phi_v(\epsilon)(g) = \epsilon(gv)$, for $\epsilon \in \check{V}$, $g \in G$.

Note that N normalizes G_e, and acts semisimply on V^{G_e}. Also N has a right action on $\mathcal{O} = G/G_e$, commuting with G, hence $G \times N$ acts on $R(\mathcal{O})$, and the Frobenius reciprocity isomorphism is N-equivariant. It remains to check that $V^{G_e}(2d) = \{v \in V : \; nv = \chi(n)^d v\}$, and $R^d(\mathcal{O}) = \{f \in R(\mathcal{O}) : \; f(gn) = \chi(n)^d f(g)\}$ are the corresponding N-eigenspaces.

The second proof is longer but perhaps more descriptive. For any character μ of N, let $E_\mu = G \times_N \mathbb{C}_\mu$ be the homogeneous line bundle over G/N on which N acts by μ on the fiber over the identity coset. So E_μ is the set of pairs $[g, z]$, $g \in G$, $z \in \mathbb{C}$, with identifications $[gn, z] = [g, \mu(n)z]$ for $n \in N$. Let E°_χ be the complement of the zero section. It admits a $\mathbb{C}^\times$-action by multiplication in the fibers: $t[g, z] = [g, tz]$. The map

$$E^\circ_\chi \longrightarrow \mathcal{O}, \qquad [g, z] \mapsto zAd(g)e$$

is a $G \times \mathbb{C}^\times$ equivariant isomorphism, so

$$R^d(\mathcal{O}) \simeq R^d(E^\circ_\chi).$$

Let $\Gamma(E_{\chi^{-d}})$ denote the global sections of the bundle $E_{\chi^{-d}} \longrightarrow G/N$, sections viewed as regular functions $s : G \longrightarrow \mathbb{C}$ satisfying $s(gn) = \chi(n)^d s(g)$. If $f \in R^d(E^\circ_\chi)$, we define the section $s_f \in \Gamma(E_{\chi^{-d}})$ by $s_f(g) = f([g, 1])$, and if s is a section, we define $f_s \in R^d(E^\circ_\chi)$ by $f_s([g, z]) = z^d s(g)$. Thus,

$$R^d(\mathcal{O}) \simeq \Gamma(E_{\chi^{-d}})$$

as G-modules. By ordinary Frobenius reciprocity, we have

$$\mathrm{Hom}_G(\check{V}, \Gamma(E_{\chi^{-d}})) = \mathrm{Hom}_N(\check{V}, \chi^{-d}) = \mathrm{Hom}_N(\chi^d, V) = V^{G_e}(2d).$$

□

Corollary 1.

(1) *If $\lambda(h) < 2d$ then $\check{V}_\lambda$ does not appear in $R^d(\mathcal{O})$.*
(2) *$R^d(\mathcal{O}) = 0$ for $d < 0$.*
(3) *$R^0(\mathcal{O})$ contains only constant functions.*

proof. Since the positive roots are non-negative on h, we have $V_\lambda(2d) = 0$, unless $\lambda(h) \geq 2d$. For (2), note that $V^{G_e} \subseteq V^e$, and the h-eigenvalues in V^e are ≥ 0. For (3) we have $V^{G_e}(0) \subseteq V^e(0) = V^{\mathfrak{s}}$. Hence $V^{G_e}(0)$ is killed by the subalgebra generated by $\mathfrak{g}^e$ and $\mathfrak{s}$. This subalgebra is all of $\mathfrak{g}$, since any $\mathfrak{s}$-module is generated by its highest weight vectors. □

Note that assertions (2) and (3) are false for general group actions (take $G = \mathbb{C}^\times \times \mathbb{C}^\times$ acting on itself by multiplication, with diagonal $\mathbb{C}^\times$-action).

We will later need the following consequence of Corollary 1.

Corollary 2. *Suppose $\lambda(h) \leq 2d$ and V_λ appears in $R^d(\mathcal{O}_1)$ for some other nilpotent orbit $\mathcal{O}_1$ with $\mathfrak{sl}(2)$ triple (f_1, h_1, e_1), with all positive roots non-negative on h_1. Then*

$$\lambda(h_1) \geq 2d \geq \lambda(h).$$

(2.3) At the borderline $\lambda(h) = 2d$, we have

$$V_\lambda = \bigoplus_{j=-2d}^{2d} V_\lambda(j).$$

Recall that $G_e = MU_e$. Since acting by $\mathfrak{u}$ increases h-weight, $\mathfrak{u}$ must kill $V_\lambda(2d)$. Hence $V_\lambda^U \supseteq V_\lambda(2d) \neq 0$. But V_λ^U is an irreducible L-module, and L centralizes h, so

$$V_\lambda^U = V_\lambda(2d) \supseteq V_\lambda^{G_e}(2d) = [V_\lambda^U]^M.$$

We have proved

Corollary. *If $\lambda(h) = 2d$, then the multiplicity of $\check{V}_\lambda$ in $R^d(\mathcal{O})$ is the dimension of M-invariants in the irreducible L-module V_λ^U.*

For example, if $\mathfrak{g}_2$ embeds in V_λ^U as M-modules, then e itself provides an M-invariant in V_λ^U, so $\check{V}_\lambda$ appears in $R^{\lambda(h)/2}(\mathcal{O})$.

(2.4) If λ and h are related in a certain way, it is easy to find the M invariants in V_λ^U.

Definition. *We say $\lambda \prec h$, if whenever α is a simple root with $\alpha(h) = 0$, then also $\langle \lambda, \check{\alpha} \rangle = 0$. Equivalently, $\lambda \prec h$ if ξ_λ extends to L.*

Later, we shall find $\lambda \prec h$ in the following situation. G will be simply laced, with Killing form normalized to give all roots squared-length two. Viewing $\lambda \in \mathfrak{t}$ via this inner product, we will have $\lambda = h$, which implies $\lambda \prec h$.

We now complete the proof of Proposition 1 in the introduction.

Proposition. *Suppose $\lambda(h) = 2d$, and $\lambda \prec h$. Then $\check{V}_\lambda$ has multiplicity at most one in $R^d(\mathcal{O})$. The multiplicity is one if and only if the character ξ_λ is trivial on M. Finally, $\check{V}_\lambda$ does not appear in $R^{d'}(\mathcal{O})$ for $d' > d$.*

proof. The last assertion is a reformulation of (2.2), Corollary 1. Since $\lambda(h) = 2d$, we have $\mathrm{Hom}_G(\check{V}, R^d(\mathcal{O})) \simeq V^{UM}$, by Corollary (2.3). Since $\lambda \prec h$, the L-module V^U is one-dimensional, affording ξ_λ. $\square$

Remark. It can happen that ξ_λ is nontrivial on M, though this can sometimes be ruled out *a priori.* If it is known, for example, that M belongs to the derived group L', or that M is itself connected semisimple, then ξ_λ is automatically trivial on M.

(2.5) If we consider the orbit closure, we sometimes get vanishing in low degrees as follows. The *birthday* of a simple module V_λ is the lowest degree d such that V_λ appears in the space $R^d = R^d(\mathcal{N})$ of degree d functions on the null-cone $\mathcal{N}$. This is a more subtle invariant than the *deathday*, which Kostant showed to be the height of λ, and which we have just found for smaller orbits and certain modules. A combinatorial formula for the birthday of self-dual modules was independently proved by Joseph [J, §4] and Kostant (unpublished). For small modules (self-dual or not) there is a geometric interpretation of the birthday (see section 3 below).

Let $\overline{\mathcal{O}}$ be the closure of a nilpotent orbit $\mathcal{O}$, with $\mathfrak{sl}(2)$ triple (f, h, e).

Lemma. *Let d be the birthday of a simple G-module V_λ. Suppose*

(1) $\lambda(h) = 2d$, *and*
(2) V_λ *has multiplicity one in its birthplace* $R^d(\mathcal{N})$, *and*
(3) *this incarnation of* V_λ *does not vanish on* $\mathcal{O}$.

Then V_λ has multiplicity one in all of $R(\overline{\mathcal{O}})$. If $\mathcal{O}_1$ is another orbit with triple (f_1, h_1, e_1) such that V_λ appears in $R(\overline{\mathcal{O}}_1)$, then $\lambda(h_1) \geq \lambda(h)$.

proof. Conditions (2) and (3) force V_λ to have multiplicity one in $R^d(\overline{\mathcal{O}})$. Suppose d' is any degree and $V_\lambda \hookrightarrow R^{d'}(\overline{\mathcal{O}})$. Since $R^{d'}(\overline{\mathcal{O}}) \subseteq R^{d'}(\mathcal{O})$, proposition 1 says $2d' \leq \lambda(h) = 2d$. Since $R^{d'}(\mathcal{N})$ surjects onto $R^{d'}(\overline{\mathcal{O}})$, we have $2d' \geq 2d$, by the definition of birthday. Hence $d' = d$.

Now suppose $V_\lambda \hookrightarrow R^{d_1}(\overline{\mathcal{O}}_1)$. Then with the same arguments, we have $\lambda(h_1) \geq 2d_1 \geq 2d = \lambda(h)$. □

The conditions of this lemma will be seen to hold for those small orbits in type D and E having normal closures.

3. Small modules

In this section we recall some results of Broer [B], combine them with [R] and [BM], and determine the birthday of small modules in the simply laced case.

(3.1) Recall that $R^d = R^d(\mathcal{N})$ is the space of degree-d regular functions on the nilpotent variety of G. Let $\mathcal{H}^d$ be the W-harmonic polynomials on $\mathfrak{t}$ of degree d. We will need the following theorem of Broer.

Theorem [B]. *V is small if and only if*

$$\mathrm{Hom}_G(V, R^d) \simeq \mathrm{Hom}_W(V^T, \mathcal{H}^d).$$

One might ask if Broer's theorem generalizes to say the multiplicity of a small V in $R^d(\mathcal{O})$ equals that of V^T in the scheme theoretic intersection $R^d(\mathfrak{t} \cap \overline{\mathcal{O}})$. This at least holds for PGL_n (see §4 below) and Broer's theorem itself implies one direction in general.

Corollary [B, p.388]. *Assume V is small, and let $\mathcal{O}$ be any nilpotent orbit. Then in each degree d, we have*

$$\dim \mathrm{Hom}_G(V, R^d(\overline{\mathcal{O}})) \geq \dim \mathrm{Hom}_W(V^T, R^d(\mathfrak{t} \cap \overline{\mathcal{O}})).$$

Another distinguishing feature of small modules is contained in Broer's proof of the previous corollary, but we give a more direct proof here.

Lemma. *Let L be a Levi subgroup of G containing T. Let W_L be the subgroup of W generated by reflections from L. Then for small V, we have*

$$V^L = [V^T]^{W_L}.$$

proof. Clearly the left side is contained in the right, for any V. Since L is generated by rank one Levi subgroups L_α for simple roots α in L, we have $V^L = \cap V^{L_\alpha}$, intersection over simple α in L. A similar intersection holds on the Weyl group side, so we may assume $L = L_\alpha$. Let e_α be a simple root vector for α. Then $e_\alpha^2 V^T = 0$, since V is small. Thus the L_α module $V(L_\alpha)$ generated by V^T contains only the trivial and adjoint representations of $SL(2) \subset L_\alpha$. The reflection s_α is $+1$ on the zero weight space of the trivial part of $V(L_\alpha)$, and -1 on the zero weight space in the adjoint part. □

Remark. If $\mathcal{O}$ is a Richardson orbit in a parabolic with Levi L, then the lemma, combined with Borho-Kraft [BK] or McGovern's formula [M], says

$$\dim \mathrm{Hom}_G(\check{V}, R(\mathcal{O})) = \dim V^L = [V^T]^{W_L}.$$

Broer [B2] has conjectured a graded version of this.

(3.2) From now on, we consider only simply-laced groups. Let V be an irreducible small G-module. By [R], there is a big nilpotent element $u = u(V) \in \mathfrak{g}$, unique up to conjugacy, such that $V^T \simeq H^{2d(u)}(\mathcal{B}_u)$ as W-modules, where $d(u) = \dim \mathcal{B}_u$. Combining this with Broer's theorem, we get, for any d,

$$\mathrm{Hom}_G(V, R^d) \simeq \mathrm{Hom}_W(H^{2d(u)}(\mathcal{B}_u), \mathcal{H}^d).$$

These multiplicities can be found in [C]. From this and [BM, cor. 4] we conclude

Proposition. *The birthday of the small module V is $d(u) = \dim \mathcal{B}_u$, and V appears in $R^{d(u)}$ with multiplicity one.*

4. Type A

In this section, $G = PGL(n)$, and we show how a result stronger than the main theorem follows from work of DeConcini and Procesi. It was already stated in [B2] as evidence for a conjecture on multiplicities of small modules in functions on covers of Richardson orbits. We give a proof here.

Let $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots]$ be a partition of n, with dual partition λ', and let u be a nilpotent matrix with Jordan blocks of size λ'. We write $\mathcal{B}_{\lambda'}$ for $\mathcal{B}_u$, and then $d(u) = d(\lambda') = \sum \lambda_i^2 - n$. The sub-dual (i.e. dual) orbit of u has blocks of size λ, and we write $\mathcal{O}_\lambda = \mathcal{O}_{sd}$. It is the Richardson orbit for the parabolic subgroup $P_{\lambda'}$ whose Levi subgroup has blocks λ'. The G-module V_λ has zero weight space $V_\lambda^T = H^{d(\lambda')}(\mathcal{B}_{\lambda'})$, classically known as the irreducible representation $\chi_{\lambda'}$ of $W = S_n$ corresponding to λ'.

Proposition [B2]**.**

$$\mathrm{Hom}_G(\check{V}_\lambda, R^d(\overline{\mathcal{O}}_\mu)) \simeq \mathrm{Hom}_W(\chi_{\lambda'}, H^{2d}(\mathcal{B}_{\mu'})).$$

This immediately shows that $\check{V}_\lambda$ has multiplicity one in $R^{d(\lambda')}(\overline{\mathcal{O}}_\lambda)$. By [Kr], we have

$$H(\mathcal{B}_{\mu'}) \simeq \psi_{\mu'},$$

where $\psi_{\mu'}$ is the induction to S_n of the trivial representation on the Young subgroup of type μ'. It is well-known that $\psi_{\mu'}$ contains $\chi_{\mu'}$ with multiplicity one, and contains $\chi_{\nu'}$ only if $\nu' \geq \mu'$. This last is equivalent to $\mathcal{O}_\mu \geq \mathcal{O}_\nu$, so the proposition says V_λ can only appear in functions on orbits whose closure contains $\mathcal{O}_\lambda$. This proves the main theorem in type A.

proof of the proposition. de Concini and Procesi [CP] found an isomorphism of graded rings

$$R^\bullet(\mathfrak{t} \cap \overline{\mathcal{O}}_\mu) \simeq H^{2\bullet}(\mathcal{B}_{\mu'}),$$

so by lemma (3.2), we have

$$\dim \mathrm{Hom}_G(V, R^d(\overline{\mathcal{O}}_\mu)) \geq \dim \mathrm{Hom}_W(V^T, R^d(\mathfrak{t} \cap \overline{\mathcal{O}}_\mu)) = \dim \mathrm{Hom}_W(V^T, H^{2d}(\mathcal{B}_{\mu'})).$$

It suffices to prove equality after summing over all d. By [BK] (see remark (3.2)), we have

$$\mathrm{Hom}_G(\check{V}_\lambda, R(\overline{\mathcal{O}}_\mu)) \simeq V_\lambda^{L_{\mu'}}.$$

The latter space is $[V_\lambda^T]^{S_{\mu'}}$ by lemma (3.1), and

$$[V_\lambda^T]^{S_{\mu'}} = \mathrm{Hom}_W(V^T, \psi_{\mu'}) = \mathrm{Hom}_W(V^T, H(\mathcal{B}_{\mu'})).$$

□

5. Type D

In this section, $V = \mathbb{C}^{2n}$, $n \geq 4$, is a nondegenerate orthogonal space, and $G = PSO(2n)$ is the adjoint group of the special orthogonal group of V. We work in $SO(2n)$, and expect the reader to divide by the center when necessary. Let m be the greatest integer $\leq \frac{n}{2}$. For $0 \leq q < m$, we define nilpotent orbits $x_q,\ y_q,\ z_q$ by their partitions of elementary divisors as follows.

$$\begin{aligned} x_q &= [2^{2q}, 1^{2n-4q}] \quad 0 \leq q < m \\ y_q &= [3, 2^{2q-2}, 1^{2n-4q+1}] \quad 1 \leq q < m \\ z_q &= [3^2, 2^{2q-4}, 1^{2n-4q+2}] \quad 2 \leq q \leq m. \end{aligned}$$

For $n = 2m$ even we define $x_m,\ y_m$ by the same formulas, but x_m is a union of two $SO(2n)$-orbits. When n is odd there is no x_m or y_m. The union of these orbits is a closed subvariety of the nilpotent variety of G, and the closure relations are as shown

n even			n odd		
z_m			z_m		
	y_m		z_{m-1}		
z_{m-1}		x_m		y_{m-1}	
	y_{m-1}		z_{m-2}		x_{m-1}
z_{m-2}		x_{m-1}		y_{m-2}	
	y_{m-2}		z_{m-3}		x_{m-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z_2		x_3	z_2		x_3
	y_2			y_2	
	y_1	x_2		y_1	x_2
		x_1			x_1
		x_0			x_0

The classes x_q and z_q are special. They are the duals of the big nilpotent orbits, which are

$$\begin{aligned} u_q &= [2n-2q-1, 2q+1] \quad \text{for} \quad 0 \leq q < m \\ u_m &= [n, n] \quad \text{for} \quad n = 2m \\ v_q &= [2n-2q-1, 2q-1, 1, 1] \quad \text{for} \quad 1 \leq q \leq m \end{aligned}$$

and under the duality involution on special orbits, we have

$$x_q = (u_q)_d, \qquad z_q = (v_q)_d.$$

When $n = 2m$, each orbit in x_m is the dual of an orbit in $u_m = [n,n]$. The classes y_q are not special. From the above closure relations, we see that y_q is the subdual of v_q and x_q is the subdual of u_q. Thus, the small orbits are x_q, y_q when defined, and also z_m when $n = 2m-1$. Using the criterion given in [KP,16.2)], we find that all small orbits have normal closures except x_m, $(n = 2m)$ and z_m, $(n = 2m-1)$.

We now list the h-part of an $\mathfrak{sl}(2)$ triple (f,h,e) for each nilpotent orbit, x_q, y_q, z_q, identifying the Cartan subalgebra of $\mathfrak{g}$ with $\mathbb{C}^n$.

$$\begin{aligned}
h_{x_q} &= (\underbrace{1,\ldots,1}_{2q},\underbrace{0,\ldots,0}_{n-2q}) \quad (0 \le q < m)\\
h_{y_q} &= (2,\underbrace{1,\ldots,1}_{2q-2},\underbrace{0,\ldots,0}_{n-2q+1}) \quad (1 \le q \le m)\\
h^{\pm}_{x_m} &= (1,\cdots,\pm 1) \quad (n = 2m)\\
h_{z_q} &= (2,2,\underbrace{1,\ldots,1}_{2q-4},\underbrace{0,\ldots,0}_{n-2q+2}) \quad (2 \le q \le m).
\end{aligned}$$

The small modules, with their highest weights λ are as follows.

$\Lambda^{2q}V$	$\lambda = \epsilon_1 + \cdots + \epsilon_{2q}$	$0 \le q < \frac{n}{2}$
V_{2q-1}	$\lambda = 2\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2q-1}$	$1 \le q < m$
$\Lambda^{\frac{n}{2}}_{\pm}V$	$\lambda = \epsilon_1 + \cdots + \epsilon_{n-1} \pm \epsilon_n$	$n = 2m$
$V_{n,\pm}$	$\lambda = 2\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} \pm \epsilon_n$	$n = 2m-1$

Here V_{2q-1} is the kernel of the map

$$V \otimes \Lambda^{2q-1}V \longrightarrow \Lambda^{2q}V \oplus \Lambda^{2q-2}V$$

given by wedging into the first component, contracting into the second. $V_{n,\pm}$ are the two irreducible components of V_n when n is odd, and $\Lambda^m_{\pm}V$ are the two irreducible components of $\Lambda^m V$ when $n = 2m$.

For each big orbit $\mathcal{O}$, we give the small module V_λ whose zero weight space is $H^{2d(v)}(\mathcal{B}_v)$ for $v \in \mathcal{O}$, the value $\lambda(h)$ for the h attached to $e \in \mathcal{O}_{sd}$ as listed above, the group $L = G_h$, its module $\mathfrak{g}_2$, and the generic stabilizer $M = L_e$.

e	V_λ	$\lambda(h)$	L	$\mathfrak{g}_2$	M
x_q	$\Lambda^{2q}V$	$2q$	$GL_{2q}\times SO_{2n-4q}$	$\Lambda^2\mathbb{C}^{2q}$	$Sp_{2q}\times SO_{2n-4q}$
y_q	V_{2q-1}	$2q{+}2$	$GL_1\times GL_{2q-2}\times SO_{2n-4q+2}$	$\Lambda^2\mathbb{C}^{2q-2}\oplus\mathbb{C}^{2n-4q+2}$	$Sp_{2q-2}\times O_{2n-4q+1}$
x_m	$\Lambda^{\frac{n}{2}}_{\pm}V$	n	GL_n	$\Lambda^2\mathbb{C}^n$	Sp_n
z_m	$V_{n,\pm}$	$n{+}3$	$GL_2\times GL_{n-3}\times SO(2)$	$M_{2\times 2}\oplus\Lambda^2\mathbb{C}^{n-3}$	$SO(2)\times Sp_{n-3}$.

We observe that with one exception, we have $\lambda \prec h$, and M is contained in the derived group of L. In fact, except for $h = h_{z_m}$ with $n = 2m-1$, we have

$$\langle \lambda, \check{\alpha} \rangle = \alpha(h)$$

for all simple roots α.

Ignoring z_m for the moment, Proposition 1 in the introduction implies that V_λ appears once in $R^d(\mathcal{O}_{sd})$, where $2d = \lambda(h)$, and does not appear in $R(\mathcal{O})$ in any higher degree. The same conclusion holds for z_m, using corollary (2.3). Indeed, M sits in L as

$$M = \{(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, B, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) : \ a \in \mathbb{C}^\times, \ B \in Sp_{n-3}(\mathbb{C})\},$$

and a direct check shows that $V_{n,\pm}^U$ is two dimensional in each case, and contains a unique M-invariant, up to scalar.

We next show, for each small orbit $\mathcal{O}_{sd}$, that V_λ cannot appear in $R(\mathcal{O}_1)$ for another orbit $\mathcal{O}_1$, if $\mathcal{O}_1 \not\geq \mathcal{O}_{sd}$. All such $\mathcal{O}_1$ are among the x_q, y_q, z_q. By lemma (2.5), it suffices to see that $\lambda(h_1) < \lambda(h)$, where h_1 is attached to $\mathcal{O}_1$. For fixed λ, the values $\lambda(h_1)$ increase as we move up the partial order in one of the families x, y, z. It thus suffices to check only maximal $\mathcal{O}_1 \not\geq \mathcal{O}_{sd}$. We do this in the following table.

$\mathcal{O}_{sd}$	$\mathcal{O}_1$	$\lambda(h_1)$	$\lambda(h)$
x_q	z_{q-1}	$2q-2$	$2q$
y_q	z_{q-1}	$2q$	$2q+2$
y_q	x_{q+1}	$2q$	$2q+2$
z_m	z_{m-1}	$n+1$	$n+3$

Next we calculate, using the formulas in [C], the dimensions $d(u)$ of the fixed point varieties $\mathcal{B}_u$, for $u = u_q, \ v_q$. We find , remarkably, that in all cases we have

$$2d(u) = \lambda(h).$$

This finishes the proof of the main theorem for D_n.

If we exclude the non-normal orbits $x_m, \ z_m$, we can say more. Recall that $d(u)$ is also the birthday of V_λ in the harmonic polynomials R. The favorable conditions (1) and (2) of lemma (2.5) are satisfied, so for type D we have proved

Theorem. *Let $\mathcal{O}_{sd}$ be any small D_n orbit, except x_m, $(n = 2m)$ or z_m, $(n = 2m-1)$. Let V be the corresponding small D_n-module. Then*

(1) *V appears with multiplicity one in $R(\overline{\mathcal{O}}_{sd})$, in degree $d(u)$, for $u \in \mathcal{O}$.*
(2) *Let $\mathcal{O}_1$ be another nilpotent orbit. Then V appears in $R(\overline{\mathcal{O}}_1)$ if and only if $\mathcal{O} \leq \mathcal{O}_1$, and the occurrence can only be in degrees $d_1 \geq d(u)$. In particular, the functions in $V \subseteq R^{d(u)}(\overline{\mathcal{O}}_{sd})$ all vanish on the boundary of $\overline{\mathcal{O}}_{sd}$.*

6. Type E

In this section we prove the Main Theorem for type E_n. Number the diagrams

$$\begin{matrix} 1 & 2 & 3 & 4 & \cdots & (n-1) \\ & & n & & & \end{matrix}.$$

(6.1) We list the duals of the big orbits, their subduals, and the closure relations, pointing out the small orbits in bold print.

$G=E_6$		$G=E_7$			$G=E_8$	
$\mathbf{2A_2}$		A_2A_1			A_2A_1	
$\mathbf{A_2A_1}$	$\mathbf{4A_1}$					
				$\mathbf{4A_1}$		$\mathbf{A_2}$
A_2	$(\mathbf{3A_1})''$		A_2			
$\mathbf{3A_1}$			$(\mathbf{3A_1})'$		$\mathbf{3A_1}$	
$\mathbf{2A_1}$		$\mathbf{2A_1}$			$\mathbf{2A_1}$	
$\mathbf{A_1}$		$\mathbf{A_1}$			$\mathbf{A_1}$	
$\mathbf{0}$		$\mathbf{0}$			$\mathbf{0}$	

(6.2) For each nonzero small orbit $\mathcal{O}_{sd}$ we list its h, the type of M, the highest weight of the corresponding small G-module, written both in terms of fundamental weights and as a linear combination of roots, the value $\lambda(h)$, and the dimension $d(v) = \dim \mathcal{B}_v$, where $V_\lambda^T = H^{2d(v)}(\mathcal{B}_v)$ for $v \in \mathcal{O}$.

$\mathbf{E}_6$

$\mathcal{O}_{sd}$	h	M	λ	$\lambda(h)$	$d(v)$
$2A_2$	$\begin{smallmatrix}2&0&0&0&2\\&&0&&\end{smallmatrix}$	G_2	$3\omega_1,\ 3\omega_5 = \begin{smallmatrix}2&4&6&5&4\\&&3&&\end{smallmatrix}$	12	6
A_2A_1	$\begin{smallmatrix}1&0&0&0&1\\&&1&&\end{smallmatrix}$	$A_2 \times \mathbb{C}^\times$	$\omega_1+\omega_2,\ \omega_4+\omega_5 = \begin{smallmatrix}2&4&6&5&3\\&&3&&\end{smallmatrix}$	8	4
$3A_1$	$\begin{smallmatrix}0&0&1&0&0\\&&0&&\end{smallmatrix}$	A_2A_1	$\omega_3 = \begin{smallmatrix}2&4&6&4&2\\&&3&&\end{smallmatrix}$	6	3
$2A_1$	$\begin{smallmatrix}1&0&0&0&1\\&&0&&\end{smallmatrix}$	$B_3 \times \mathbb{C}^\times$	$\omega_1+\omega_5 = \begin{smallmatrix}2&3&4&3&2\\&&2&&\end{smallmatrix}$	4	2
A_1	$\begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix}$	A_5	$\omega_6 = \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}$	2	1

$\mathbf{E}_7$

$\mathcal{O}_{sd}$	h	M	λ	$\lambda(h)$	$d(v)$
$4A_1$	$\begin{smallmatrix}0&0&0&0&0&1\\&&1&&&\end{smallmatrix}$	C_3	$\omega_6+\omega_7 = \begin{smallmatrix}3&6&9&7&5&3\\&&5&&&\end{smallmatrix}$	8	4
$(3A_1)''$	$\begin{smallmatrix}0&0&0&0&0&2\\&&0&&&\end{smallmatrix}$	F_4	$2\omega_6 = \begin{smallmatrix}2&4&6&5&4&3\\&&3&&&\end{smallmatrix}$	6	3
$(3A_1)'$	$\begin{smallmatrix}0&1&0&0&0&0\\&&0&&&\end{smallmatrix}$	C_3A_1	$\omega_2 = \begin{smallmatrix}3&6&8&6&4&2\\&&4&&&\end{smallmatrix}$	6	3
$2A_1$	$\begin{smallmatrix}0&0&0&0&1&0\\&&0&&&\end{smallmatrix}$	B_4A_1	$\omega_5 = \begin{smallmatrix}2&4&6&5&4&2\\&&3&&&\end{smallmatrix}$	4	2
A_1	$\begin{smallmatrix}1&0&0&0&0&0\\&&0&&&\end{smallmatrix}$	D_6	$\omega_1 = \begin{smallmatrix}2&3&4&3&2&1\\&&2&&&\end{smallmatrix}$	2	1

$\mathcal{O}_{sd}$	h	M	λ	$\lambda(h)$	$d(v)$
$4A_1$	$0\ 0\ \underset{1}{0}\ 0\ 0\ 0\ 0$	C_4	$\omega_8 = 5\ 10\ \underset{8}{15}\ 12\ 9\ 6\ 3$	8	4
$3A_1$	$0\ 0\ \underset{0}{0}\ 0\ 0\ 1\ 0$	F_1A_1	$\omega_6 = 4\ 8\ \underset{6}{12}\ 10\ 8\ 6\ 3$	6	3
$2A_1$	$1\ 0\ \underset{0}{0}\ 0\ 0\ 0\ 0$	B_6	$\omega_1 = 4\ 7\ \underset{5}{10}\ 8\ 6\ 4\ 2$	4	2
A_1	$0\ 0\ \underset{0}{0}\ 0\ 0\ 0\ 1$	E_7	$\omega_7 = 1\ 2\ \underset{3}{4}\ 6\ 5\ 4\ 3\ 2$	2	1

These tables, combined with Proposition 1 of the introduction say that $\check{V}_\lambda$ appears in $R^d(\mathcal{O})$ with multiplicity one and in no higher degree. By Kempf's criterion [Ke], the classes A_1 and $2A_1$ are normal in each group, along with $(3A_1)''$ in E_7, since $\mathfrak{g}_2 = \mathfrak{g}_{\geq 2}$. The classes $2A_2$ and A_2A_1 in E_6 are not normal [Ri], and normality does not seem to be known for the other classes.

(6.3) As with D_n, we finish the proof of the Main Theorem by checking the conditions of (2.2) Corollary 2: For each small orbit $\mathcal{O}_{sd}$, we list the orbits $\mathcal{O}_1$ which do not contain $\mathcal{O}_{sd}$, and compute $\lambda(h_1)$ and $\lambda(h)$. One finds in all but one case that $\lambda(h) > \lambda(h_1)$, so V_λ does not appear in $R^{d(e)}(\mathcal{O}_1)$.

The exception is the orbit $\mathcal{O}_{sd} = 2A_2$ in E_6, with $\mathcal{O}_1 = A_3$. But A_3 is Richardson in the A_4-parabolic, and Alvis' tables [A] show that V_λ^T, which is 24_p in Frame's notation, does not have an invariant under W_{A_4}. Hence both V_λ's cannot appear in $R(\mathcal{O}_1)$ in any degree.

A sharper theorem analogous to that for D_n in §5 also holds for those small E_n-orbits with normal closures.

7. Characters

Our results so far are valid over the algebraic closure $\bar{\mathbb{Q}}_\ell$ of the ℓ-adic numbers, ℓ a prime. As in the introduction, let $\check{G}$ be the simply connected split group over $\bar{\mathbb{F}}_q$, where q is a power of a good prime for $\check{G}$, $\ell \nmid q$, such that the group G considered till now is the group over $\bar{\mathbb{Q}}_\ell$ with root datum dual to that of $\check{G}$. All cohomology is ℓ-adic. Let F be the Frobenius of $\check{G}$, let W be the Weyl group of G and $\check{G}$, and let $\check{\mathcal{B}}$ be the flag variety of $\check{G}$.

(7.1) Let Π be the principal series representation $\check{G}^F$, induced from trivial on a Borel subgroup of $\check{G}^F$. The Hecke algebra $\mathbf{H} = \mathrm{End}_G(\Pi)$ is semisimple, and its virtual modules are in canonical bijection with those of W, so we will not distinguish between them. For any irreducible W-module E we define

$$\rho(E) = \mathrm{Hom}_{\mathbf{H}}(E, \Pi),$$

and extend to virtual modules by linearity. For example, if $E = \mathfrak{t}$ is the reflection representation of W, the representation $\rho(\mathfrak{t})$, called the "reflection representation" of $\check{G}^F$, has dimension $\sum q^{m_i}$, where the m_i are the exponents of W.

On the other hand, to E also corresponds a virtual representation R_E of $\check{G}^F$, defined as follows [L]. For any $w \in W$, let R_w be the Lefschetz character on the cohomology of the variety of Borel subgroups in $\mathcal{B}$ in relative position w with respect to F, and define the virtual $\check{G}^F$-module

$$R_E = \frac{1}{|W|} \sum_{w \in W} \mathrm{tr}(w, E) R_w.$$

In general, $\rho(E) \neq R_E$. However, we have

Proposition. *If V is a small G-module, then*

$$\rho(V^T) = R_{V^T}.$$

This will follow from more general considerations. The irreducible unipotent representations of $\check{G}^F$, among which are the constituents of Π, can be partitioned into families, one family $\mathcal{F}(v)$ for each special unipotent orbit of $v \in G$. Let $A(v)$ be the component group of $\check{G}_v$. For each $\mathcal{F}(v)$ we have a finite group $\Gamma = \Gamma(v)$, which is a quotient of $A(v)$, such that the irreducible representations in $\mathcal{F}(v)$ are in bijection with Γ-conjugacy classes of pairs (x, σ), where $x \in \Gamma$ and σ is an irreducible character of the centralizer Γ_x of x in Γ. Let $\chi_{x,\sigma}$ be the irreducible $\check{G}^F$ representation in $\mathcal{F}(v)$ corresponding to (x, σ). If $\chi_{x,\sigma} \subset \Pi$, then there is also a corresponding W-module $E_{x,\sigma}$ such that $\chi_{x,\sigma} = \rho(E_{x,\sigma})$, as described above.

The virtual character $R_{E_{x,\sigma}}$ is orthogonal to all irreducible $\check{G}^F$ representations outside $\mathcal{F}(v)$, and if $\chi_{y,\tau} \in \mathcal{F}(v)$, then we have

$$\langle R_{E_{x,\sigma}}, \chi_{y,\tau} \rangle = \frac{1}{|\Gamma_x||\Gamma_y|} \sum_{\substack{z \in \Gamma \\ zxz^{-1} \in \Gamma_y}} \tau(zxz^{-1}) \overline{\sigma(z^{-1}yz)}.$$

Lemma. *Fix $x \in \Gamma$, and assume that $\chi_{x,\sigma} \subset \Pi$ for all irreducible characters σ of Γ_x, so we can define the virtual W-module*

$$E_x = \sum_{\sigma \in \hat{\Gamma}_x} \sigma(x) E_{x,\sigma}.$$

Then

$$R_{E_x} = \rho(E_x).$$

proof.

$$\begin{aligned}
\langle R_{E_x}, \chi_{y,\tau} \rangle &= \sum_{\sigma \in \hat{\Gamma}_x} \sigma(x) \langle R_{E_{x,\sigma}}, \chi_{y,\tau} \rangle \\
&= \frac{1}{|\Gamma_x||\Gamma_y|} \sum_{\sigma} \sum_{\substack{z \in \Gamma \\ zxz^{-1} \in \Gamma_y}} \sigma(x) \tau(zxz^{-1}) \overline{\sigma(z^{-1}yz)} \\
&= \frac{1}{|\Gamma_y|} \sum_{\substack{z \in \Gamma \\ zxz^{-1} \in \Gamma_y}} \tau(zxz^{-1}) \frac{1}{|\Gamma_x|} \sum_{\sigma} \sigma(x) \overline{\sigma(z^{-1}yz)}.
\end{aligned}$$

Since x is in the center of Γ_x, the inner sum is $|\Gamma_x|$ if $z^{-1}yz = x$, zero otherwise. Thus,

$$\langle R_{E_x}, \chi_{y,\tau} \rangle = \frac{1}{|\Gamma_y|} \sum_{\substack{z \in \Gamma \\ z^{-1}yz = x}} \tau(zxz^{-1}) = \tau(y),$$

if y is conjugate to x in Γ, zero otherwise.

By definition, this also holds with R_{E_x} replaced by $\rho(E_x)$. Since $\rho(E_x)$ and R_{E_x} are orthogonal to all characters outside $\mathcal{F}(u)$, the lemma is proved. □

Now let V be a small G-module, with zero weight space $V^T \simeq H^{2d(v)}(\mathcal{B}_v)$. Inspection of cases [R] shows that group $A(v)$ has order one or two, is always isomorphic to $\Gamma(v)$, and all representations of $A(v)$ appear in $H^{2d(v)}(\mathcal{B}_v)$. Thus, we have $V^T = E_1$, for the family $\mathcal{F}(v)$, so the proposition follows from the lemma.

(7.2) For any $g \in \check{G}^F$, Lusztig has generalized the Springer construction to an action of W on $H(\check{\mathcal{B}}_g)$, and we have the following formula, known as the "Springer hypothesis".

Theorem ([Kaz],[L],[Sp]). *For any virtual W-module E, and $g \in \check{G}^F$, we have*

$$R_E(g) = tr(F, [H(\check{\mathcal{B}}_g) \otimes E]^W).$$

The usual statement of (7.2) is $R_w(u) = \sum_i (-1)^i \mathrm{tr}(Fw, H^i(\check{\mathcal{B}}_u))$, for u unipotent in $\check{G}^F$. This formula extends to any g by applying it to the semisimple part of g (c.f. 8.2), and $H^i(\check{\mathcal{B}}_g) = 0$ for odd i by [CLP], whence the formulation given above. From this and proposition (7.1) we have

Proposition. *If V is a small G-module and $g \in \check{G}^F$, then*

$$tr(g, \rho(V^T)) = tr(F, [H(\check{\mathcal{B}}_g) \otimes E]^W).$$

When $g = 1$, the only eigenvalue of F on $H^{2d}(\check{\mathcal{B}})$ is q^d, and $\mathcal{H}^d = H^{2d}(\check{\mathcal{B}})$ as W-modules, hence

Corollary. $\dim \rho(V_T) = P(V^T, \mathcal{H}, q)$.

8. Motives

We can give a motivic version of the formula for the character of $\rho(V_T)$. Our discussion here is more general than our present needs, but may be useful elsewhere.

(8.1) In this section we do not assume $\check{G}$ to be split. Being defined over a finite field, it is automatically quasi-split. Let Γ be the Galois group of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$. For present purposes, a *motive* is just a graded Γ-module.

Gross has defined "the motive of $\check{G}$" as follows [G]. Fix a Γ-stable maximal torus and Borel subgroup $\check{T} \subset \check{B}$. Recall that $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g} = Lie(G)$. We may identify $\mathfrak{t} = X^*(\check{T}) \otimes \bar{\mathbb{Q}}_\ell$, where $X^*(\ )$ denotes rational character group. Let $\check{\mathfrak{t}}$ be the dual space of $\mathfrak{t}$. Let I_+ be the W-invariant polynomials of positive degree on $\check{\mathfrak{t}}$. The cotangent space I_+/I_+^2 has dimension equal to that of $\mathfrak{t}$. It is naturally a Γ-module, via the action on the root datum of $\check{G}$, with grading $I_+/I_+^2 \simeq \oplus V_d$ inherited from that of I_+. The motive $M(\check{G})$ is by definition the graded Γ-module $\bar{\mathbb{Q}}_\ell(-1) := H^2(\mathbb{P}^1)$. The Galois action on the latter is induced by the natural action on $\mathbb{P}^1$.

In fact, this motive belongs to a family of motives $M(\check{G}, E, g)$, parametrized by pairs (E, g) of W-representations E and rational points $g \in \check{G}^F$. To define these, we first need

Lemma. *Let Γ_E be the stabilizer of E in Γ. Then the W action on E extends to the semidirect product $W \rtimes \Gamma_E$.*

proof. Without loss E and $\mathfrak{t}$ are irreducible W-modules. Consider the birthday occurrence of E in $R(\check{\mathfrak{t}})$. If this occurrence has multiplicity one, then the extension is clear. Otherwise, the multiplicity can be at most two, and either $G = E_{7,8}$ (but then Γ acts trivially on $\mathfrak{t}$ so there is nothing to prove) or $G = D_n$ and E is the restriction of two distinct representations of $W(B_n)$. But in that case the action of Γ passes through $W(B_n)$ as well, so the extension is automatic. $\square$

Thus, from E we get the $W\Gamma$-module $\mathrm{Ind}_{W\Gamma_E}^{W\Gamma} E$. On the other hand, given a rational point $g \in \check{G}^F$, we have a natural action of Γ on $\check{\mathcal{B}}_g$, as well as the Lusztig action of W on cohomology, whence a $W\Gamma$-module $H(\check{\mathcal{B}}_g)$. This cohomology is only nonzero in even degrees, as recalled in (8.2). We now define the motive $M(\check{G}, E, g) = \oplus M^d(\check{G}, E, g)$ by

$$M^d(\check{G}, E, g) = [H^{2d}(\mathcal{B}_g) \otimes \mathrm{Ind}_{W\Gamma_E}^{W\Gamma} E]^W.$$

For example, when $\check{G}$ is split, the Γ-action on $\mathfrak{t}$ and W is trivial, hence it is also trivial on E, so

$$M^d(\check{G}, E, g) = [H^{2d}(\mathcal{B}_g) \otimes E]^W \qquad (\check{G} \text{ split}).$$

Proposition. *We have $M(\check{G}) \simeq M(\check{G}, \mathfrak{t}, 1)$, as graded Γ-modules.*

proof. We may assume W acts irreducibly on $\mathfrak{t}$. Recall that J_W is the ideal in $R(\mathfrak{t})$ generated by I_+. The differential

$$R(\check{\mathfrak{t}}) \longrightarrow R(\check{\mathfrak{t}}) \otimes \mathfrak{t}$$

is $W\Gamma$- equivariant, and induces a map

$$\delta : I_+ \longrightarrow [R(\check{\mathfrak{t}})/J_W \otimes \mathfrak{t}]^W,$$

which has I_+^2 in its kernel. By Solomon's theorem [S], the right side has a basis δF_i, where the F_i's are homogeneous generators of I_+. It follows that

$$I_+/I_+^2 \simeq [R(\check{\mathfrak{t}})/J_W \otimes \mathfrak{t}]^W$$

as $W\Gamma$-modules.

As graded W-algebras, we have $R^\bullet(\check{\mathfrak{t}})/J_W \simeq H^{2\bullet}(\check{\mathcal{B}})$ induced via the map $\mathfrak{t} \longrightarrow H^2(\check{\mathcal{B}})$ sending $\chi \in X^*(\check{T})$ to the first Chern class of the line bundle on $\check{\mathcal{B}}$ determined by χ. Since all of Γ stabilizes $\mathfrak{t}$, we have

$$[H^{2d}(\check{\mathcal{B}}) \otimes \check{\mathfrak{t}}]^W = M^d(\check{G}, \mathfrak{t}, 1).$$

It remains to check that

$$[R^{d-1}(\check{\mathfrak{t}})/J_W](1-d) \simeq H^{2d-2}(\check{\mathcal{B}})$$

as Γ-modules. Both sides are generated by terms with $d = 2$, so it suffices to see that $\mathfrak{t}(-1) \simeq H^2(\check{\mathcal{B}})$.

Let $\check{\mathcal{B}}_1, \dots, \check{\mathcal{B}}_r$ be the one dimensional Schubert varieties in $\check{\mathcal{B}}$, corresponding to the conjugacy classes of minimal parabolic subgroups in $\check{G}$. The restriction map

$$H^2(\check{\mathcal{B}}) \longrightarrow \bigoplus_{i=1}^{r} H^2(\check{\mathcal{B}}_i)$$

is a Γ-equivariant isomorphism. I claim the right side is $\mathfrak{t}(-1)$, as Γ-modules. We have $\mathfrak{t} = \oplus \mathfrak{t}_i$, where each $\mathfrak{t}_i$ is the line through the simple root α_i corresponding to $\check{\mathcal{B}}_i$, and Γ permutes the $\mathfrak{t}_i$'s just as it permutes the $\check{\mathcal{B}}_i$'s. Considering orbits, we may assume the action is transitive. Let Γ_1 be the stabilizer of α_1. The corresponding Levi subgroup L_{α_1} of $\check{G}$ is split over the fixed field of Γ_1, and $H^2(\check{\mathcal{B}}_1) \simeq \mathfrak{t}_1(-1)$ as Γ_1 modules, by definition of the twist. The claim, and proposition, follow by inducing this isomorphism from Γ_1 to Γ. $\square$

(8.2) The motive can be reduced to g unipotent as follows. Let $g = su$ be the Jordan decomposition of $g \in \check{G}^F$. Let $\mathcal{B}^s$ be the flag variety for the centralizer $\check{G}_s$. Then it is known (cf [Ka, 3.2]) that

$$H^\bullet(\check{\mathcal{B}}_g) = \mathrm{Ind}_{W_s}^{W} H^\bullet(\check{\mathcal{B}}_u^s).$$

Clearly

$$[\mathrm{Ind}_{W\Gamma_E}^{W\Gamma} E]\big|_{W_s\Gamma} = \mathrm{Ind}_{W_s\Gamma_E}^{W_s\Gamma} E,$$

so

$$\begin{aligned} M(\check{G}, E, g) &= [H(\mathcal{B}_g) \otimes \mathrm{Ind}_{W\Gamma_E}^{W\Gamma} E]^W \\ &= [H(\mathcal{B}_u^s) \otimes \mathrm{Ind}_{W_s\Gamma_E}^{W_s\Gamma} E]^{W_s} \\ &= M(\check{G}_s, E, u). \end{aligned}$$

If $\check{G}$ is split, then the Springer hypothesis (7.2) says

$$R_E(g) = \mathrm{tr}(F, M(\check{G}, E, g)) = \mathrm{tr}(F, M(\check{G}_s, E, u)).$$

Taking $E = \check{\mathfrak{t}}$, $u = 1$, and using Proposition (8.1), we recover Gan's formula [Ga] for the character of the reflection representation on a semisimple element.

More generally, taking $E = V^T$ for a small G-module V, we can express the harmonic multiplicity polynomial for V in terms of the motive $M(\check{G}, V^T, 1)$ by the identity

$$P(V, R(\mathcal{N}), q) = \mathrm{tr}(F, M(\check{G}, V^T, 1)).$$

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