On some generic very cuspidal representations

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August 21, 2009

Abstract

Let G be a reductive p-adic group. Given a compact-mod-center maximal torus $S \subset G$ and sufficiently regular character χ of S, one can define, following Adler, Yu and others, a supercuspidal representation $\pi(S, \chi)$ of G. For S unramified, we determine when $\pi(S, \chi)$ is generic, and which generic characters it contains.

Keywords: Generic representation, very cuspidal, Whittaker model, Moy-Prasad depth

2000 Mathematics Subject Classification: 22E50; 20G25; 20G10

1 Introduction

Let k be a finite extension of the p-adic numbers $\mathbb{Q}_p$ for some prime p. A connected reductive k-group $\mathbf{G}$ is called **unramified** if it is quasi-split over k and split over an unramified extension of k. We let G denote the group of k-rational point of $\mathbf{G}$; this convention applies to all algebraic k-groups.

*Supported by NSF grant DMS-0500667
†Supported by NSF grant DMS-0801177

Let $\mathbf{G}$ be an unramified k-group with center $\mathbf{Z}$. Given an unramified maximal k-torus $\mathbf{S} \subset \mathbf{G}$ such that $\mathbf{S}/\mathbf{Z}$ is anisotropic, and a sufficiently regular character $\chi : S \to \mathbb{C}^\times$, one can construct (cf. [1], [3], [8], [10], [26]) an irreducible supercuspidal representation $\pi(S, \chi)$ of G; these are examples of **very cuspidal** representations and are the representations we shall consider in this paper. We have

$$\pi(S, \chi) = \mathrm{ind}_{K_x}^G \kappa(S, \chi),$$

(smooth compact induction) where $x = x(S)$ is the unique [25, §3.6.1] fixed-point of S in the reduced Bruhat-Tits building of G, K_x is an open subgroup of G that fixes x and has compact image in G/Z, and $\kappa(S, \chi)$ is a finite dimensional representation of K_x constructed from the pair (S, χ).

Let $\mathbf{B} \subset \mathbf{G}$ be a Borel subgroup defined over k. Fix a maximal k-torus $\mathbf{T} \subset \mathbf{B}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. A character $\psi : U \to \mathbb{C}^\times$ is called **generic** if the stabilizer of ψ in T is exactly the center Z. An irreducible admissible representation π of G is called **generic** if there exists a generic character ψ of U such that $\mathrm{Hom}_U(\pi, \psi)$ is nonzero (in which case, we say ψ occurs in $\pi(S, \chi)$). For any representation π, one may ask:

(i) Is π generic?

(ii) If π is generic, which generic characters occur in π?

The purpose of this paper is to answer both questions for the very cuspidal representation $\pi(S, \chi)$. The answer to question (i) is as follows.

Theorem 1.1 *The very cuspidal representation $\pi(S, \chi)$ is generic if and only if $x(S)$ is a hyperspecial vertex in the reduced Bruhat-Tits building of G.*

The second question is a bit more subtle. We now assume (as we may) that $x = x(S)$ is a hyperspecial vertex in the apartment of T in the reduced building of G. Let r (a positive integer) be the depth of χ, and let

$$G_{x,r}, \quad U_{x,r} := G_{x,r} \cap U, \quad T_r$$

be the Moy-Prasad filtration subgroups, with similar groups for r^+. We say that a character ψ of U has **generic depth** r at x if the restriction of ψ to U_{x,r^+} is trivial, giving a character ψ_r of $U_{x,r}/U_{x,r^+}$, and if the stabilizer in T_0 of ψ_r is contained in $Z \cdot T_{0+}$. Since x is hyperspecial, a character of generic depth r at x is indeed generic, as defined previously. One answer to question (ii) is as follows.

Theorem 1.2 *Let $\pi = \pi(S, \chi)$ as above have depth r (see section 2.5). Assume $x(S)$ is hyperspecial. Then $\mathrm{Hom}_U(\pi, \psi) \neq 0$ if and only if the T-orbit of ψ contains a character of generic depth r at $x(S)$.*

To give a quantitative answer to question (ii), let $H^1(k, \mathbf{L})$ denote the Galois cohomology of an algebraic k-group $\mathbf{L}$, and given an inclusion of k-groups $\mathbf{L} \subset \mathbf{M}$, let

$$\ker^1(\mathbf{L}, \mathbf{M}) := \ker[H^1(k, \mathbf{L}) \to H^1(k, \mathbf{M})]$$

denote the kernel of the map on cohomology induced by the inclusion. The group $\ker^1(\mathbf{Z}, \mathbf{G})$ acts simply-transitively on T-orbits of generic characters of U.

Let $\pi(S, \chi)$ be a very cuspidal representation of the type considered in this paper and assume $x(S)$ is hyperspecial.

Theorem 1.3 *The subgroup* $\ker^1(\mathbf{Z}, \mathbf{S})$ *of* $\ker^1(\mathbf{Z}, \mathbf{G})$ *acts simply-transitively on the* T*-orbits of generic characters which occur in* $\pi(S, \chi)$.

The final section of the paper relates this result to the L-packets of supercuspidal representations recently constructed in [7] and [20]. Roughly speaking, we show that Theorem 1.3 is compatible with the internal parametrization of the generic part of our L-packets. See section 7 below for more details.

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2 Some General Results

We begin with minimal hypotheses that will be strengthened as we proceed. Let k be a locally compact field, complete with respect to a discrete valuation $\mathrm{val} : k^\times \to \mathbb{Z}$. Denote by $\mathfrak{p}$ the prime ideal in the ring of integers $\mathfrak{o}$ of k.

Let $G = \mathbf{G}(k)$ be the group of k-rational points of a connected reductive k-group $\mathbf{G}$. Let $\mathcal{B}(G)$ denote the reduced Bruhat-Tits building of G. For $x \in \mathcal{B}(G)$ and $s \in \mathbb{R}_{\geq 0}$, let $G_{x,s}$ and G_{x,s^+} denote the Moy-Prasad filtration subgroups of G, as defined in [17].

2.1 A structure result

Given two points $x, y \in \mathcal{B}(G)$, let $[x, y]$ denote the geodesic in $\mathcal{B}(G)$ from x to y. There exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ containing both x and y; the geodesic $[x, y]$ is the straight line segment from x to y in the affine space $\mathcal{A}$.

Lemma 2.1 *Suppose $x, y \in \mathcal{B}(G)$ and $z \in [x, y]$. Then we have*

(1) $G_{z,s^+} = (G_{x,s^+} \cap G_{z,s^+}) \cdot (G_{z,s^+} \cap G_{y,s^+})$ *for all $s \geq 0$;*

(2) $G_{z,s} = (G_{x,s} \cap G_{z,s}) \cdot (G_{z,s} \cap G_{y,s})$ *for all $s > 0$.*

Proof: We prove statement (2). Statement (1) can be obtained by substituting "s^+" for "s" below.

Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing both x and y. Let $\mathbf{A}$ be the maximal k-split torus of $\mathbf{G}$ corresponding to $\mathcal{A}$. Then $\mathcal{A}$ is a homogeneous space for the vector group $V := X_*(\mathbf{A}) \otimes \mathbb{R}$, and there is $v \in V$ such that $y = x + v$.

Let $\mathbf{P}$ denote a minimal parabolic k-subgroup containing $\mathbf{A}$. Let Φ denote the set of roots of $\mathbf{A}$ in $\mathbf{G}$ and let $\Phi^+ \subset \Phi$ be the roots of $\mathbf{A}$ in $\mathbf{P}$. Without loss of generality, we may assume $\mathbf{P}$ is chosen so that

$$\langle \alpha, v \rangle \geq 0 \qquad \text{for all} \qquad \alpha \in \Phi^+.$$

Let $\mathbf{M}$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. Then we have the Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{U}$. Let $\bar{\mathbf{P}}$ denote the parabolic k-subgroup which is opposite to $\mathbf{P}$ with respect to $\mathbf{M}$ and let $\bar{\mathbf{U}}$ denote the unipotent radical of $\bar{\mathbf{P}}$.

Since $s > 0$ we have, for all $w \in \mathcal{A}$, the Iwahori decomposition

$$G_{w,s} = (G_{w,s} \cap \bar{U}) \cdot M_s \cdot (G_{w,s} \cap U),$$

where

$$M_s = \bigcap_{w' \in \mathcal{A}} G_{w',s}.$$

Since

$$M_s \subseteq G_{x,s} \cap G_{y,s} \cap G_{z,s},$$

it suffices to show

(a) $(G_{z,s} \cap \bar{U}) \subseteq (G_{x,s} \cap G_{z,s})$, and

(b) $(G_{z,s} \cap U) \subseteq (G_{z,s} \cap G_{y,s})$.

We prove (b). The proof of (a) is similar.

Let Ψ denote the set of affine roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and the valuation on k. If $\psi \in \Phi$, then let $\dot{\psi} \in \Phi$ denote the gradient of ψ. To prove (b) it suffices to show that if $\psi \in \Psi$ is such that $\psi(z) \geq s$ and $\langle \dot{\psi}, v \rangle \geq 0$, then $\psi(y) \geq s$. But we have $z = y - tv$ for some $t \geq 0$, so

$$\psi(y) = \psi(z + tv) = \psi(z) + t\langle \dot{\psi}, v \rangle \geq s + t\langle \dot{\psi}, v \rangle \geq s,$$

since $t\langle \dot{\psi}, v \rangle \geq 0$. ■

2.2 A result on fixed vectors

Fix a smooth representation (π, V) of G. For each compact open subgroup K of G, let

$$[K] : V \longrightarrow V^K$$

denote the projection operator, given by

$$[K]v = \int_K \pi(k)v \, dk,$$

where dk is the Haar measure on K for which $\int_K dk = 1$.

Lemma 2.2 *Suppose $x, y \in \mathcal{B}(G)$ and $s \in \mathbb{R}_{\geq 0}$. If $v \in V^{G_{y,s^+}}$ and $[G_{x,s^+}]v \neq 0$, then $V^{G_{z,s^+}} \neq \{0\}$ for all $z \in [x, y]$.*

Proof: Fix $z \in [x, y]$. We will actually show that $[G_{z,s^+}]v \neq 0$. From Lemma 2.1, we have

$$\begin{aligned}
[G_{x,s^+}][G_{z,s^+}]v &= [G_{x,s^+}][G_{x,s^+} \cap G_{z,s^+}][G_{z,s^+} \cap G_{y,s^+}]v \\
&= [G_{x,s^+}][G_{x,s^+} \cap G_{z,s^+}]v \\
&= [G_{x,s^+}]v \neq 0,
\end{aligned}$$

hence $[G_{z,s^+}]v \neq 0$. ■

2.3 Generalized s-facets

Let $\mathfrak{g}$ be the set of k-rational points of the Lie algebra of $\mathbf{G}$. We have analogous filtration subgroups $\mathfrak{g}_{x,s}$, $\mathfrak{g}_{x,s^+}$, for $x \in \mathcal{B}(G)$ and $s \in \mathbb{R}$. We recall here some basic facts about generalized s-facets from [5, section 3]. If we assume $s \geq 0$ then everything in this section remains valid when "$\mathfrak{g}$" is replaced by "G".

If $x, y \in \mathcal{B}(G)$, we say x **is related to** y if

$$\mathfrak{g}_{x,s} = \mathfrak{g}_{y,s} \qquad \text{and} \qquad \mathfrak{g}_{x,s^+} = \mathfrak{g}_{y,s^+}.$$

The equivalence classes in $\mathcal{B}(G)$ defined by this relation are called **generalized s-facets**. If F is a generalized s-facet and $x \in F$, we set

$$\mathfrak{g}_F := \mathfrak{g}_{x,s} \qquad \text{and} \qquad \mathfrak{g}_F^+ = \mathfrak{g}_{x,s^+}.$$

Suppose F is a generalized s-facet in $\mathcal{B}(G)$. If $\mathcal{A}$ is any apartment in $\mathcal{B}(G)$ meeting F, we let $\dim_{\mathcal{A}}(F)$ denote the dimension of the smallest affine subspace of $\mathcal{A}$ which contains $\mathcal{A} \cap F$. From [5, Cor. 3.2.14], if $\mathcal{A}'$ is another apartment in $\mathcal{B}(G)$ meeting F, then $\dim_{\mathcal{A}}(F) = \dim_{\mathcal{A}'}(F)$. Therefore it makes sense to define the **dimension** of F as:

$$\dim(F) = \dim_{\mathcal{A}}(F),$$

for any apartment $\mathcal{A}$ meeting F.

For a generalized s-facet F, we let $\bar{F}$ denote the closure of F in the natural (metric) topology on $\mathcal{B}(G)$. From [5, 3.2], the boundary

$$\partial F := \bar{F} - F$$

is a disjoint union of a finite number of generalized s-facets, each having dimension strictly less than that of F.

Lemma 2.3 *Let F_1, F_2 be two generalized s-facets. Then we have*

$$F_1 \subseteq \bar{F}_2 \quad \Leftrightarrow \quad \mathfrak{g}_{F_1}^+ \subseteq \mathfrak{g}_{F_2}^+ \subseteq \mathfrak{g}_{F_2} \subseteq \mathfrak{g}_{F_1}.$$

Proof: Implication "$\Rightarrow$" is [5, Cor. 3.2.19]. For the other implication, it is enough to show that for any two points x_1, x_2, with $x_i \in F_i$, we have the half-open segment $(x_1, x_2] := [x_1, x_2] - \{x_1\}$ contained in F_2.

Choose an apartment $\mathcal{A}$ containing x_1 and x_2. Let $\mathbf{A}$ be the maximal k-split torus corresponding to $\mathcal{A}$ and let Ψ be the set of affine roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and the valuation on k. To prove that $(x_1, x_2] \subseteq F_2$, we must show that for any $\psi \in \Psi$, the affine function $\psi - s$ is always positive, always zero, or always negative on $(x_1, x_2]$.

If $\psi(x_1) > s$, then since $\mathfrak{g}_{F_1}^+ \subseteq \mathfrak{g}_{F_2}^+$, we have $\psi(x_2) > s$, so $\psi - s$ is positive on all of $[x_1, x_2]$. If $\psi(x_1) < s$, then since $\mathfrak{g}_{F_2} \subseteq \mathfrak{g}_{F_1}$, we have $\psi(x_2) < s$, so $\psi - s$ is negative on all of $[x_1, x_2]$. Finally, if $\psi(x_1) = s$, then either $\psi - s \equiv 0$ on $[x_1, x_2]$ or x_1 is the unique zero of $\psi - s$ on $[x_1, x_2]$, in which case $\psi - s$ is always positive or always negative on $(x_1, x_2]$. ■

2.4 Cuspidal representations

Let $\mathfrak{f}$ denote the residue field of k. Let $s \geq 0$ and fix a generalized s-facet F. Set

$$\mathsf{L}_F := G_F / G_F^+.$$

If $s = 0$ then L_F is the group of $\mathfrak{f}$-rational points of a connected reductive $\mathfrak{f}$-group. If $s > 0$ then L_F is a finite dimensional vector space over $\mathfrak{f}$.

Suppose H is a generalized s-facet containing F in its closure. From Lemma 2.3, we have

$$G_F^+ \subseteq G_H^+ \subseteq G_H \subseteq G_F.$$

Let L_F^H denote the image of G_H^+ in L_F. A finite dimensional complex representation (σ, W) of L_F is said to be **cuspidal** if for all generalized s-facets H for which $F \subseteq \partial H$, we have

$$W^{\mathsf{L}_F^H} = \{0\}.$$

Let $\mathcal{C}(\mathsf{L}_F)$ denote the set of equivalence classes of irreducible cuspidal representations of L_F.

If $s = 0$ then the above definition agrees with the usual definition of a cuspidal representation of a finite reductive group. If $s > 0$ then L_F is abelian and $\mathcal{C}(\mathsf{L}_F)$ consists of those characters of L_F which are non-trivial on L_F^H whenever $F \subseteq \partial H$.

2.5 A discreteness criterion

Suppose (π, V) is an irreducible admissible representation of G of depth s. This means there is some $x \in \mathcal{B}(G)$ for which $V^{G_{x,s^+}} \neq \{0\}$ and that $V^{G_{y,r^+}} = \{0\}$ for any $y \in \mathcal{B}(G)$ and $r < s$. The aim of this section is to give a criterion for the set

$$\mathcal{X}(\pi) := \{x \in \mathcal{B}(G) : \ V^{G_{x,s^+}} \neq \{0\}\}$$

to be discrete. Note first of all that $\mathcal{X}(\pi)$ is a disjoint union of generalized s-facets, preserved under the action of G on $\mathcal{B}(G)$, and $\mathcal{X}(\pi)$ is closed in $\mathcal{B}(G)$, by Lemma 2.3.

Lemma 2.4 *Suppose F is a generalized s-facet in $\mathcal{X}(\pi)$. The L_F-module $V^{G_F^+}$ is cuspidal if and only if F is maximal among the generalized s-facets in $\mathcal{X}(\pi)$.*

Proof: The generalized s-facet F is not maximal among the generalized s-facets in $\mathcal{X}(\pi)$ if and only if there is is a generalized s-facet H in $\mathcal{X}(\pi)$ for which $F \subset \partial H$; equivalently, $F \subset \partial H$ and $V^{G_H^+} \neq \{0\}$, or, from Lemma 2.3, $F \subset \partial H$ and

$$\left(V^{G_F^+}\right)^{\mathsf{L}_F^H} = V^{G_H^+} \neq \{0\}.$$

The lemma follows. ■

Corollary 2.5 *Suppose F is a generalized s-facet in $\mathcal{X}(\pi)$ and the L_F-module $V^{G_F^+}$ is cuspidal. If F is a minimal generalized s-facet in $\mathcal{B}(G)$, then $\mathcal{X}(\pi)$ is discrete; in fact, we have $\mathcal{X}(\pi) = \{gF : g \in G\}$.*

Proof: A minimal generalized s-facet is a point, so we have $F = \{x\}$ for some $x \in \mathcal{B}(G)$. From Lemma 2.4 it follows that x is isolated in $\mathcal{X}(\pi)$. We choose a nonzero vector $v \in V^{G_{x,s^+}}$.

Now suppose that y is another point in $\mathcal{X}(\pi)$. By definition we have $V^{G_{y,s^+}} \neq \{0\}$, so that $[G_{y,s^+}]V \neq \{0\}$. Since V is irreducible, there is $g \in G$ such that

$$[G_{y,s^+}]\pi(g)v \neq 0.$$

Applying $\pi(g)^{-1}$, this means that

$$[G_{g^{-1}y,s^+}]v \neq 0.$$

By Lemma 2.2, the geodesic $[x, g^{-1}y]$ is contained in $\mathcal{X}(\pi)$. But x is isolated in $\mathcal{X}(\pi)$, so $x = g^{-1}y$ and $y = gx$. Hence the generalized s-facet containing y is also minimal, so $\mathcal{X}(\pi)$ is discrete. ■

Remark: The first author and G. Prasad have shown (unpublished) that any two maximal generalized s-facets occurring in $\mathcal{X}(\pi)$ must be associate, in the sense of [5, def. 3.3.4]. Moreover, if $\mathcal{X}(\pi)$ is discrete, then π must be supercuspidal. There exist (nontrivial) examples of supercuspidal representations for which $\mathcal{X}(\pi)$ is not discrete.

2.6 Very cuspidal representations

We now impose the additional assumptions of [1, 2.1.1] on the residual characteristic p of k. Namely $p > 2$ and p does not divide the order of the center of the simply-connected cover of the derived group of $\mathbf{G}$ and moreover $p \neq 3$ if $\mathbf{G}$ has a simple factor of type G_2. If k has positive characteristic we also exclude $p = 3$ (resp. $p = 3, 5$) if $\mathbf{G}$ has a simple factor of type F_4 (resp. E_8).

Under these assumptions, there exists and we fix a non-degenerate symmetric $\mathrm{Ad}(G)$-invariant bilinear form $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g} \to k$ which restricts to a nondegenerate pairing $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}\times\mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)^+} \to \mathfrak{f}$ for all $r \in \mathbb{R}$. Fix also a character $\Lambda : k^+ \to \mathbb{C}^\times$ of the additive group of k, with $\ker\Lambda = \mathfrak{p}$.

Let $x \in \mathcal{B}(G)$ and $r > 0$. Identifying $G_{x,r}/G_{x,r^+} = \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$, as we may, any element $X \in \mathfrak{g}_{x,-r}$ determines a character

$$\chi_X : G_{x,r}/G_{x,r^+} \longrightarrow \mathbb{C}^\times,$$

by the formula

$$\chi_X(Y + \mathfrak{g}_{x,r^+}) = \Lambda\langle X, Y\rangle.$$

The assignment $X \mapsto \chi_X$ is a bijection

$$\mathfrak{g}_{x,-r}/\mathfrak{g}_{x,(-r)^+} \overset{\sim}{\longrightarrow} \mathrm{Irr}\left(G_{x,r}/G_{x,r^+}\right).$$

A semisimple element $X \in \mathfrak{g}$ has **depth** $-r < 0$ if $X \in \mathfrak{g}_{x,-r}$ for some $x \in \mathcal{B}(G)$ and $X \notin \mathfrak{g}_{y,(-r)^+}$ for any $y \in \mathcal{B}(G)$. As in [1, 2.2.3], a semisimple element $X \in \mathfrak{g}$ of depth $-r$ is called **good** if the centralizer $\mathbf{M} = C_{\mathbf{G}}(X)$ contains a maximal torus $\mathbf{S}$ splitting over a tame extension E/k, such that $\mathrm{val}_E(d\alpha(X)) = -r$ for every root of $\mathbf{S}$ in $\mathbf{G}$ outside of $\mathbf{M}$. Note that r is an integer if the extension E/k is unramified.

Suppose that X is good of depth $-r$ with centralizer $\mathbf{M} = C_{\mathbf{G}}(X)$. Let $\mathcal{B}(M)$ be the image of the building of M in $\mathcal{B}(G)$. By [11, 2.3.1] we have that

$$\mathcal{B}(M) = \{x \in \mathcal{B}(G) : \ X \in \mathfrak{g}_{x,-r} \setminus \mathfrak{g}_{x,(-r)^+}\}. \tag{1}$$

Assume further that $\mathbf{M}/\mathbf{Z}$ is anisotropic (we say $\mathbf{M}$ is **minisotropic**). Then $\mathcal{B}(M) = \{x\}$ consists of the unique point $x \in \mathcal{B}(G)$ such that $X \in \mathfrak{g}_{x,-r} \setminus \mathfrak{g}_{x,(-r)^+}$. For such an element X, Adler's construction in [1] produces many finite dimensional representations κ_X of the stabilizer K_x of x in G, with the property that the compactly induced representation

$$\mathrm{ind}_{K_x}^{G} \kappa_X \tag{2}$$

is irreducible supercuspidal of depth r and contains the character χ_X upon restriction to $G_{x,r}$. Let Π_X be the set of these representations (2). Each $\pi \in \Pi_X$ is an example of a very cuspidal representation.

Lemma 2.6 *Let the semisimple element $X \in \mathfrak{g}$ be good of depth $-r$, with minisotropic centralizer $\mathbf{M} = C_{\mathbf{G}}(X)$ and let $\pi \in \Pi_X$. Then $\mathcal{X}(\pi)$ is discrete.*

Proof: Note first that (1) implies that $\mathcal{B}(M) = \{x\}$ is a generalized $(-r)$-facet in $\mathcal{B}(G)$; we denote it by F. By [5, Lemma 3.2.5], F is also a generalized r-facet in $\mathcal{B}(G)$. Let V be the space of π. We will show that the L_F-module $V^{G_F^+}$ is cuspidal. The character χ_X appears in $V^{G_{x,r^+}}$. We first claim that any other character χ_Y of $G_{x,r}$ which appears in $V^{G_{x,r^+}}$ is K_x-conjugate to χ_X.

Since π is irreducible, there is a $g \in G$ so that

$$(X + \mathfrak{g}_{x,(-r)^+}) \cap \mathrm{Ad}(g)(Y + \mathfrak{g}_{x,(-r)^+})$$

is nonempty (see [17, 7.2]). This implies that there is $Z \in \mathfrak{g}_{x,(-r)^+}$ such that $X + Z \in \mathfrak{g}_{gx,-r}$. Let $\mathfrak{m}$ be the k-rational points in the Lie algebra of $\mathbf{M}$. From [1, 2.3.2], there is an $h \in G_{x,0^+}$ so that

$$\mathrm{Ad}(h)(X + Z) \in X + \mathfrak{m}_{x,(-r)^+}.$$

Moreover, the element $\mathrm{Ad}(h)(X + Z)$ is still good of depth $-r$. But also $\mathrm{Ad}(h)(X + Z)$ belongs to $\mathfrak{g}_{hgx,-r}$. From (1) we have $hgx = x$. Hence $g \in K_x$, and the claim is proved.

Hence it is enough to show that χ_X is cuspidal. If not, there exists a generalized r-facet H such that $F \subset \partial H$ and χ_X is trivial on L_F^H. This implies that $X \in \mathfrak{g}_{y,-r}$ for all $y \in H$. Using (1) again, we have $H \subset \{x\}$, a contradiction.

Since F is a minimal generalized s-facet and $V^{G_F^+}$ is cuspidal, it follows from Corollary 2.5 that $\mathcal{X}(\pi)$ is discrete. ■

3 Generic characters and representations

We now add the assumption that $\mathbf{G}$ is unramified. That is, $\mathbf{G}$ is quasi-split over k and $\mathbf{G}$ splits over an unramified extension of k. Let $\mathbf{U}$ denote the unipotent radical of a k-Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let $\mathbf{T}$ be a maximal k-torus in $\mathbf{B}$ and let $\mathbf{A}$ be the maximal k-split subtorus of $\mathbf{T}$. Let Φ (resp. Φ^+) be the set of roots of $\mathbf{A}$ in $\mathbf{G}$ (resp. $\mathbf{U}$) and let Π be the simple roots in Φ^+. For each $\alpha \in \Phi$ let $\mathbf{U}_\alpha$ be the corresponding root group; it is the product of $\mathbf{T}$-root groups for the roots of $\mathbf{T}$ which restrict to α. Then $\mathbf{U}_\alpha$ is defined over k and we let $U_\alpha = \mathbf{U}_\alpha(k)$.

Let $j : \mathbf{G} \to \bar{\mathbf{G}} := \mathbf{G}/\mathbf{Z}$ be the adjoint morphism. For any intermediate k-group $\mathbf{Z} \subset \mathbf{L} \subset \mathbf{G}$, we set

$$\bar{\mathbf{L}} = j(\mathbf{L}) \simeq \mathbf{L}/\mathbf{Z}$$

and let $\bar{L} = \bar{\mathbf{L}}(k)$ denote the group of k-rational points in $\bar{\mathbf{L}}$. For example, $\bar{\mathbf{B}}$ is a k-Borel subgroup of $\bar{\mathbf{G}}$ containing the maximal k-torus $\bar{\mathbf{T}}$ of $\bar{\mathbf{G}}$.

A character $\xi : U \to \mathbb{C}^\times$ is **generic** if its stabilizer in $\bar{T}$ is trivial. The group $\bar{T}$ acts simply-transitively on the set Ξ of generic characters of U. Hence the finite group $\bar{T}/j(T)$ acts simply-transitively on the set Ξ/T of T-orbits of generic characters.

3.1 Generic representations

We say that an irreducible admissible representation π of G is **generic** if the set

$$\Xi(\pi) := \{\xi \in \Xi : \ \mathrm{Hom}_U(\pi, \xi) \neq 0\}$$

is nonempty.

From now on our representations will have positive *integral* depth. Let (π, V) be an irreducible supercuspidal representation of G of depth $r \in \mathbf{Z}_{>0}$, of the form

$$\pi = \mathrm{ind}_{K_x}^G \kappa, \tag{3}$$

where x is a vertex in $\mathcal{B}(G)$, K_x is the stabilizer of x in $G(k)$, and κ is a finite dimensional representation of K_x which is trivial on G_{x,r^+}. In section 2.5 we studied the set

$$\mathcal{X}(\pi) = \{x \in \mathcal{B}(G) : \ V^{G_{x,r^+}} \neq \{0\}\}.$$

Since $\mathbf{G}$ is unramified, the building $\mathcal{B}(G)$ contains hyperspecial vertices.

Lemma 3.1 *Suppose π is generic of depth $r \in \mathbf{Z}_{>0}$ and $\mathcal{X}(\pi)$ is discrete. Then x is hyperspecial.*

Proof: This proof is very similar to that of Lemma 6.1.2 in [7]. Let Ψ be the set of affine roots of $\mathbf{A}$ in $\mathbf{G}$ with respect to the valuation on k. If $\psi \in \Psi$, let $\dot{\psi} \in \Phi$ denote its gradient.

Since $\mathbf{G}$ is unramified we may choose a hyperspecial vertex o in $\mathcal{A}$. Choose an alcove C in $\mathcal{A}$ so that $o \in \bar{C}$ and

$$\Phi^+ = \{\dot{\psi} \colon \psi(o) = 0 \text{ and } \psi|_C > 0\}.$$

Since we are free to conjugate x by elements of G, we may and shall assume that $x \in \bar{C}$.

For each $y \in \bar{C}$, set

$$\Psi_y := \{\psi \in \Psi : \psi(y) = 0\}, \qquad \Psi_y^+ := \{\psi \in \Psi_y : \psi|_C > 0\}.$$

Then Ψ_y is a spherical root system and Ψ_y^+ is a set of positive roots in Ψ_y. Let $\tilde{\Pi}_y$ be the unique base of Ψ_y contained in Ψ_y^+. Let $\Phi_y,\ \Phi_y^+,\ \Pi_y$ be the respective sets of gradients of the affine roots in $\Psi_y,\ \Psi_y^+,\ \tilde{\Pi}_y$. Note that $\Phi^+ = \Phi_o^+$. The roots in Π_y form a base of the reduced root system consisting of the non-divisible roots in Φ_y.

It follows from the affine Bruhat decomposition that $G = UNK_y$, where N is the normalizer of A in G and K_y is the stabilizer of y in G. We may choose a set $N(y) \subset N$ of representatives for the double cosets in $U\backslash G/K_y$, such that $n\Phi_y^+ \subset \Phi_o^+$ for each $n \in N(y)$.

Now let $\xi \in \Xi(\pi)$. Then from [21] we have

$$\mathbb{C} \simeq \mathrm{Hom}_G(\mathrm{ind}_{K_x}^G \kappa, \mathrm{Ind}_U^G \xi) \simeq \mathrm{Hom}_{K_x}(\kappa, \mathrm{Ind}_U^G \xi). \tag{4}$$

By Mackey theory, the restriction of $\mathrm{Ind}_U^G \xi$ to K_x is a direct sum

$$\left(\mathrm{Ind}_U^G \xi\right)\big|_{K_x} = \bigoplus_{n \in N(x)} \mathrm{Ind}_{U^n \cap K_x}^{K_x} \xi^n.$$

From (4) there is a unique $n \in N(x)$ such that ξ^n appears in the restriction of κ to $U^n \cap K_x$. Since κ is trivial on G_{x,r^+}, we have that ξ^n is trivial on $U^n \cap G_{x,r^+}$, so ξ is trivial on $U \cap G_{nx,r^+}$.

For $r > 0$, the Lie algebra L_x is abelian. However, since r is an integer, we can identify L_x and the Lie algebra of $\mathsf{G}_x(\mathfrak{f})$ as $\mathsf{T}(\mathfrak{f})$-modules. (Here, G_x is the connected reductive $\mathfrak{f}$-group associated to x and T denotes the $\mathfrak{f}$-torus in G_x corresponding to $\mathbf{T}$.) Consequently, we can speak of parabolic, Borel, Levi and nilradical subspaces of L_x, which are defined by the usual root-space decompositions.

Since $n\Psi_x^+ \subset \Psi_o^+$, it follows that the image of $U^n \cap G_{x,r}$ in L_x is the nilradical of a Borel subspace of L_x. Let $w = nA$ be the image of n in the Weyl group N/A. We claim that $w\Pi_x \subset \Pi_o$.

Since $n\Psi_x^+ \subset \Psi_o^+$, have $w\Pi_x \subset \Phi_o^+$. So suppose $\beta \in \Pi_x$ and $w\beta \in \Phi_o^+ - \Pi_o$. Then the root group $U_{w\beta}$ is contained in the commutator subgroup of U, so that ξ is trivial on $U_{w\beta}$. Hence $U_\beta \subset \ker \xi^n$. Since $\beta \in \Pi_x$, this implies ξ^n is trivial on the nilradical $\mathfrak{n}$ of the maximal parabolic subspace of L_x whose Levi subspace contains the β-root space. There is a facet $F \subset \mathcal{A}$ of positive dimension such that $x \in \bar{F}$ and $\mathfrak{n}$ is the image of $G(k)_{y,r^+}$ in L_x for any $y \in F$. Hence $V^{G_{y,r^+}} \neq \{0\}$ for all $y \in F$, contradicting the discreteness of $\mathcal{X}(\pi)$.

We have proved that $w\Pi_x \subset \Pi_o$. Since both x and o are vertices in $\mathcal{A}$, we have

$$|\Pi_x| = |\Pi_o| = \dim \mathcal{A},$$

implying that $w\Pi_x = \Pi_o$. Hence for any $\psi \in \tilde{\Pi}_o$ there is $k_\psi \in \mathbb{Z}$ such that $n^{-1}\psi(x) = k_\psi$.

Define $\lambda \in \bar{X}$ by the values $\langle \lambda, \beta \rangle = k_\psi$ for every absolute root β of $\mathbf{T}$ which restricts to $\dot{\psi}$. Then λ is Galois-fixed, so the translation t_λ preserves the apartment $\mathcal{A}$. For all $\psi \in \tilde{\Pi}_o$, we have

$$\psi(t_\lambda \cdot o) = \langle \lambda, \dot{\psi} \rangle = k_\psi = \psi(n \cdot x).$$

It follows that $n \cdot x = t_\lambda \cdot o$ is hyperspecial, so x is hyperspecial. ■

Corollary 3.2 *Suppose $\pi \in \Pi_X$ is very cuspidal, as in section 2.6, and generic. Then x is hyperspecial.*

Proof: This is immediate from Lemmas 3.1 and 2.6. ■

3.2 Depth of generic characters

Given $r \geq 0$ and a hyperspecial vertex $x \in \mathcal{A}$, we say a character ξ of U has **generic depth** r at x if ξ is trivial on $U \cap G_{x,r^+}$ and the restriction of ξ to $U \cap G_{x,r}$ has trivial stabilizer in $\bar{T}_0$. This makes sense because $\bar{T}_0$ fixes x and preserves the Moy-Prasad filtration subgroups at x.

Since x is hyperspecial, we have $U_\alpha \cap G_{x,r^+} \neq U_\alpha \cap G_{x,r}$ for all $\alpha \in \Phi$. It follows that ξ has generic depth r at x exactly when ξ is trivial on $U_\alpha \cap G_{x,r^+}$ and nontrivial on $U_\alpha \cap G_{x,r}$, for each $\alpha \in \Pi$. Moreover, characters of generic depth r are generic. Let $\Xi_{x,r} \subset \Xi$ denote the set of characters of U having generic depth r at x. It is clear that $\bar{T}_0$ preserves $\Xi_{x,r}$.

Lemma 3.3 *The group $\bar{T}_0$ acts simply-transitively on $\Xi_{x,r}$.*

Proof: We need only prove transitivity. Let $\xi, \xi' \in \Xi_{x,r}$. We have $\xi' = {}^t\xi$ for some (unique) $t \in \bar{T}$. We must show that $t \in \bar{T}_0$. We may assume that $t \in \bar{A}$ and it suffices to show that $|\alpha(t)| = 1$ for every $\alpha \in \Pi$. If $|\alpha(t)| > 1$ then

$$\mathrm{Ad}(t) \cdot (U_\alpha \cap G_{x,r}) \subset U_\alpha \cap G_{x,r^+} \subset \ker \xi' = \ker {}^t\xi,$$

so $U_\alpha \cap G_{x,r} \subset \ker \xi$, a contradiction. Interchanging ξ and ξ', we see that $|\alpha(t)| < 1$ is also impossible. Hence $|\alpha(t)| = 1$, as desired. ■

Lemma 3.4 *Suppose the representation π in (3) is generic and $\mathcal{X}(\pi)$ is discrete, so that x is hyperspecial, by Lemma 3.1. Then for any $\xi \in \Xi(\pi)$ there exists $t \in T$ such that $\xi^t \in \Xi_{x,r}$. Moreover, if t' is another element of T with the property that $\xi^{t'} \in \Xi_{x,r}$, then $t' \in UtK_x$.*

Proof: In fact, we'll see that we can choose $t \in A$. Recall that $N(x) \subset N$ is a set of representatives for $U\backslash G/K_x$. Since x is hyperspecial, the Iwasawa decomposition allows us to choose $N(x) \subset A$. If $\xi \in \Xi(\pi)$, then by Mackey theory again, we have

$$\mathrm{Hom}_U(\pi, \xi) \simeq \bigoplus_{t \in N(x)} \mathrm{Hom}_{U \cap K_x}(\kappa, \xi^t).$$

Hence there is a unique coset UtK_x such that

$$\mathrm{Hom}_{U\cap K_x}(\kappa, \xi^t) \neq 0.$$

It is immediate that ξ^t is trivial on $U \cap G_{x,r^+}$. The argument in the proof of Lemma 3.1 shows that ξ^t cannot be trivial on $U \cap G_{x,r}$. Hence ξ^t has generic depth r at x, as claimed. ■

Corollary 3.5 *Suppose the representation π in* (3) *is generic and $\mathcal{X}(\pi)$ is discrete, so that x is hyperspecial. Then every T-orbit in $\Xi(\pi)$ meets $\Xi_{x,r}$ in a single T_0-orbit. The group $\bar{T}_0/j(T_0)$ acts simply-transitively on $\Xi(\pi)/T$.*

Proof: The argument in the proof of Lemma 3.3, using instead $t \in T$, shows that if two characters in $\Xi_{x,r}$ are T-conjugate, then they are T_0-conjugate. Hence, we have an injection on orbit spaces:

$$\Xi_{x,r}/T_0 \hookrightarrow \Xi/T. \tag{5}$$

Lemma 3.4 shows every T-orbit in $\Xi(\pi)$ meets $\Xi_{x,r}$. Hence every T-orbit in $\Xi(\pi)$ meets $\Xi_{x,r}$ in a single T_0-orbit. The last assertion follows from Lemma 3.3 itself. ■

4 Local expansions

In Lemma 3.1 we proved one direction of Theorem 1.1; in this section we prove the other direction. We now assume that k has characteristic zero. Until Corollary 4.8 below, we require only that $\mathbf{G}$ be quasi-split over k. We will use some results on Galois cohomology, whose proofs are deferred to section 5.

4.1 Regular nilpotent elements

Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$. An element $Y \in \mathfrak{g}$ is **regular** if its centralizer $C_{\mathbf{G}}(Y)$ has smallest possible dimension, namely $\dim C_{\mathbf{G}}(Y) = \dim \mathbf{T}$. The regular nilpotent elements in $\mathfrak{g}$ form a single $\mathbf{G}$-orbit and the centralizer $C_{\mathbf{G}}(F)$ of a regular nilpotent element $F \in \mathfrak{g}$ is the product of its unipotent radical and the center $\mathbf{Z}$ of $\mathbf{G}$.

A reductive group is quasi-split over k exactly when its Lie algebra contains regular nilpotent elements rational over k. Since $\mathbf{G}$ is quasi-split by assumption, the set $\mathcal{N}_{\mathsf{reg}}$ of k-rational regular nilpotent elements in $\mathfrak{g}$ is non-empty.

Any two elements of $\mathcal{N}_{\mathsf{reg}}$ are $\mathbf{G}$-conjugate, but they need not be G-conjugate. The G-orbits in $\mathcal{N}_{\mathsf{reg}}$ are parametrized by the first Galois cohomology set $H^1(k, C_{\mathbf{G}}(F))$, for any $F \in \mathcal{N}_{\mathsf{reg}}$. By Hilbert's Theorem 90 and a simple exact sequence argument, the first Galois cohomology set of a unipotent group is trivial. It follows that if $F \in \mathcal{N}_{\mathsf{reg}}$, then $H^1(k, C_{\mathbf{G}}(F)) \simeq H^1(k, \mathbf{Z})$. This means that any two elements in $\mathcal{N}_{\mathsf{reg}}$ are conjugate by an element $g \in \mathbf{G}$ for which $\gamma(g)^{-1}g \in \mathbf{Z}$

for all $\gamma \in \mathrm{Gal}(\bar{k}/k)$. It follows that the group $\bar{G}$ acts transitively on the elements of $\mathcal{N}_{\mathsf{reg}}$, and the finite group $\bar{G}/j(G)$ acts simply-transitively on the set of G-orbits in $\mathcal{N}_{\mathsf{reg}}$.

Let $\mathbf{\mathfrak{v}}$ be the span of the negative simple root spaces for $\mathbf{T}$ in $\mathbf{\mathfrak{g}}$ and let $\mathfrak{v} = \mathbf{\mathfrak{v}}(k)$. An element $F \in \mathfrak{v}$ belongs to $\mathcal{N}_{\mathsf{reg}}$ precisely when the coefficient of every root vector in F is nonzero. Since $X_*(\bar{\mathbf{T}})$ has a basis dual to the roots in $\mathbf{\mathfrak{v}}$, it follows that $\bar{T}$ acts transitively on $\mathfrak{v} \cap \mathcal{N}_{\mathsf{reg}}$.

Lemma 4.1 *Every G-orbit in $\mathcal{N}_{\mathsf{reg}}$ meets $\mathfrak{v}$ in a single T-orbit. This gives a bijection between the set of G-orbits in $\mathcal{N}_{\mathsf{reg}}$ and the set of T-orbits in $\mathfrak{v} \cap \mathcal{N}_{\mathsf{reg}}$.*

Proof: Given $F \in \mathcal{N}_{\mathsf{reg}}$, choose $g \in \bar{G}$ such that $\mathrm{Ad}(g)F \in \mathfrak{v}$. By Lemma 5.1 below, we can write $g = t \cdot j(h)$, with $t \in \bar{T}$ and $h \in G$. Then

$$\mathrm{Ad}(h)F = \mathrm{Ad}(t^{-1}g)F \in \mathfrak{v} \cap \mathcal{N}_{\mathsf{reg}}.$$

Now suppose F, F' belong to $\mathfrak{v} \cap \mathcal{N}_{\mathsf{reg}}$ and that there is $h \in G$ such that $\mathrm{Ad}(h)F = F'$. Choose $t \in \bar{T}$ such that $\mathrm{Ad}(t)F' = F$. Then $t \cdot j(h)$ is a k-rational point in the centralizer $C_{\bar{\mathbf{G}}}(F)$. Since j maps the unipotent radical of $C_{\mathbf{G}}(F)$ bijectively onto $C_{\bar{\mathbf{G}}}(F)$, there is a k-rational element $\ell \in C_{\mathbf{G}}(F)$ such that $t \cdot j(h) = j(\ell)$. Hence $t \in \bar{T} \cap j(G) = j(T)$, so that F and F' are T-conjugate. ∎

4.2 Regular semisimple orbital integrals

Let $\mathcal{O}'$ be an arbitrary nilpotent G-orbit in $\mathfrak{g}$. By [19], the $G(k)$-invariant measure on $\mathcal{O}'$ (which is uniquely determined by our choices of the pairing $\langle\ ,\ \rangle$ and additive character Λ) may be uniquely extended to a distribution $\mu_{\mathcal{O}'}$ on $\mathfrak{g}$ which vanishes on elements of $C_c^\infty(\mathfrak{g})$ whose support does not meet $\mathcal{O}'$.

Let X be a regular semisimple element in $\mathfrak{g}$. By [9, Thm 5.11], there exists a lattice $L = L(X) \subset \mathfrak{g}$ and complex constants $c_{\mathcal{O}'}(X)$, indexed by the nilpotent G-orbits $\mathcal{O}' \subset \mathfrak{g}$, such that the orbital integral μ_X over the G-orbit of X has the expansion

$$\mu_X(f) = \sum_{\mathcal{O}'} c_{\mathcal{O}'}(X)\mu_{\mathcal{O}'}(f), \tag{6}$$

for all $f \in C_c(\mathfrak{g}/L)$, where the sum runs over all nilpotent G-orbits in $\mathfrak{g}$.

A result of Shelstad [23] gives necessary and sufficient conditions for the nonvanishing of $c_{\mathcal{O}}(X)$, when $\mathcal{O}$ is a regular nilpotent G-orbit. (In fact, Shelstad computes an exact formula for $c_{\mathcal{O}}(X)$, but we do not need this.) Kottwitz [14] has recast Shelstad's nonvanishing criterion in terms of Kostant sections. In the next section we review Kostant sections, and then give Kottwitz' formulation of Shelstad's nonvanishing result.

4.3 Kostant sections

Let $\mathcal{O}$ be a regular nilpotent G-orbit in $\mathfrak{g}$. Choose $F \in \mathcal{O}$. A **Kostant section** for $\mathcal{O}$ (at F) is an affine subspace $\mathbf{V} \subset \mathfrak{g}$ obtained as follows. Choose, as we may, elements $H, E \in \mathfrak{g}$, satisfying the $\mathfrak{sl}_2$-relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H,$$

and let

$$\mathbf{V} = F + C_{\mathfrak{g}}(E).$$

(This is not the most general Kostant section, but it will suffice for our purposes.) Kostant showed that every regular $\mathbf{G}$-orbit $\mathcal{O}_0 \subset \mathfrak{g}$ meets $\mathbf{V}$ in exactly one point. Since E and F are k-rational, the Kostant section $\mathbf{V}$ is defined over k. Hence, if the regular $\mathbf{G}$-orbit $\mathcal{O}_0$ is also defined over k, the unique point in $\mathcal{O}_0 \cap \mathbf{V}$ must be k-rational. Thus $\mathbf{V}$ determines a k-rational point in every regular $\mathbf{G}$-orbit which is defined over k.

Any two triples (F, H, E) and (F, H', E'), with the same F, are conjugate by the unipotent radical of $C_{\mathbf{G}}(F)$ [4, 5.5.10]. Since unipotent groups have trivial Galois cohomology in degree one, it follows that any two Kostant sections for $\mathcal{O}$ are G-conjugate.

4.4 A result of Kottwitz and Shelstad

Proposition 4.2 *Let $\mathcal{O}$ be a regular nilpotent G-orbit in $\mathfrak{g}$, let $\mathbf{V}$ be a Kostant section for $\mathcal{O}$ and let $X \in \mathfrak{g}$ be regular semisimple. Then the constant $c_{\mathcal{O}}(X)$ is nonzero exactly when the G-orbit of X meets $\mathbf{V}$.*

In [14] Kottwitz provides a direct proof of 4.2, based on the fact that the map $\mathbf{G} \times \mathbf{V} \to \mathfrak{g}$, arising from the adjoint action, is a submersion. We offer a slightly different proof, still based on the "submersion principle." We begin by establishing some notation.

Choose $F \in \mathcal{O}$ with E, H as in section 4.3, such that $\mathbf{V} = F + C_{\mathfrak{g}}(E)$. There are unique Borel subgroups $\mathbf{B}, \bar{\mathbf{B}}$ in $\mathbf{G}$, with Lie algebras $\mathfrak{b}, \bar{\mathfrak{b}}$, such that $E \in \mathfrak{b}$, $F \in \bar{\mathfrak{b}}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$.

Lemma 4.3 *The map*

$$\bar{\mathbf{B}} \times \mathbf{U} \times \mathbf{V} \longrightarrow \mathfrak{g}, \qquad (\bar{b}, u, F + A) \mapsto \mathrm{Ad}(\bar{b}u)(F + A) \tag{7}$$

is a submersion.

Proof: Kostant showed (see [13, 2.4]) that the adjoint action gives a k-isomorphism

$$\mathbf{U} \times \mathbf{V} \xrightarrow{\sim} F + \mathfrak{b}.$$

Hence it is enough to show that the map

$$\varrho : \bar{\mathbf{B}} \times (F + \mathfrak{b}) \longrightarrow \mathfrak{g}, \qquad (\bar{b}, F + A) \mapsto \mathrm{Ad}(\bar{b})(F + A) \tag{8}$$

is a submersion.

After conjugating, it is enough to show that the differential $d\varrho$ is surjective at $(1, F + A)$, for all $A \in \mathfrak{b}$, which is to say that

$$\mathfrak{b} + [F + A, \bar{\mathfrak{b}}] = \mathfrak{g}, \tag{9}$$

for all $A \in \mathfrak{b}$.

For integers i, let $\mathfrak{g}_i = \{Z \in \mathfrak{g} : \ [Z, H] = iZ\}$. Since $\mathcal{O}$ is regular, we have $\mathfrak{g}_i = 0$ for odd i. We have

$$\mathfrak{b} = \bigoplus_{i \le 0} \mathfrak{g}_i, \qquad \bar{\mathfrak{b}} = \bigoplus_{i \ge 0} \mathfrak{g}_i, \qquad \bar{\mathfrak{u}} := \bigoplus_{i > 0} \mathfrak{g}_i$$

and

$$[F, \mathfrak{g}_i] = \mathfrak{g}_{i+2}, \qquad \text{if} \quad i \ge 0.$$

To prove (9), it suffices to show that

$$\bigoplus_{i \le m} \mathfrak{g}_i \subseteq \mathfrak{b} + [F + A, \bar{\mathfrak{b}}]$$

for all $m \ge 0$. We shall prove this by induction on m. This is obvious for $m \in \{0, 1\}$. Let $m \ge 2$ and let

$$Y = \sum_{i \le m} Y_i, \qquad Y_i \in \mathfrak{g}_i.$$

Choose

$$Z \in \mathfrak{g}_{m-2} \subset \bar{\mathfrak{b}}$$

such that $[F, Z] = Y_m$. Note that since $A \in \mathfrak{b}$, we have

$$[A, Z] \in \bigoplus_{i < m} \mathfrak{g}_i.$$

Hence we have

$$Y - [F + A, Z] \in \bigoplus_{i < m} \mathfrak{g}_i.$$

By induction, we may assume that

$$Y - [F + A, Z] \in \mathfrak{b} + [F + A, \bar{\mathfrak{b}}].$$

It follows that

$$Y \in \mathfrak{b} + [F + A, \bar{\mathfrak{b}}],$$

as desired. ■

Corollary 4.4 *The map* $\mathbf{G} \times \mathbf{V} \to \mathfrak{g}$ *given by sending* (g, v) *to* $\mathrm{Ad}(g)v$ *is a submersion.*

Corollary 4.5 *The set*

$$\mathcal{U} := \bigcup_{\substack{\bar{b}\in\bar{B}\\ u\in U}} \mathrm{Ad}(\bar{b}u)\mathbf{V}(k)$$

is open in $\mathfrak{g}$.

Lemma 4.6 *For all* $t \in k^{\times}$*, we have* $t^2\mathcal{U} = \mathcal{U}$.

Proof: Let γ be the one parameter k-subgroup of $\mathbf{G}$ such that $d\gamma(1) = -H$. Then for all $t \in \bar{k}^{\times}$ we have

$$\mathrm{Ad}(\gamma(t))F = t^{-2}F \qquad \text{and} \qquad \mathrm{Ad}(\gamma(t))E = t^2E.$$

Thus, for all $t \in \bar{k}^{\times}$, we have $\mathrm{Ad}(\gamma(t))$ normalizes $C_{\mathfrak{g}}(E)$ and

$$\mathrm{Ad}(\gamma(t))\mathbf{V} = t^2F + C_{\mathfrak{g}}(E) = t^2(F + C_{\mathfrak{g}}(E)) = t^2\mathbf{V}.$$

Since γ is defined over k, the lemma now follows from the definition of $\mathcal{U}$ in Corollary 4.5. ■

Lemma 4.7 *Suppose* $X \in \mathbf{V}(k)$ *and* $L_E \subset C_{\mathfrak{g}}(E)$ *is a lattice so that* $X \in F + L_E$*. Then for all* $n \in \mathbb{Z}_{\geq 0}$ *the G-orbit of* $\varpi^{2n}X$ *meets* $F + L_E$*.*

Proof: Let γ be the one parameter k-subgroup of $\mathbf{G}$ such that $d\gamma(1) = -H$. Then

$$\mathrm{Ad}(\gamma(\varpi^n))F = \varpi^{-2n}F \qquad \text{and} \qquad \mathrm{Ad}(\gamma(\varpi^n))L_E \subset L_E.$$

Consequently, $\mathrm{Ad}(\gamma(\varpi^n))(\varpi^{2n}X) \in F + L_E$. ■

We are now ready to prove Proposition 4.2.
Proof: Fix a regular semisimple element $X \in \mathfrak{g}$. Choose a lattice $L = L(X)$ as at the start of section 4.4.

Suppose $(\mathrm{Ad}(G)X)$ meets $\mathbf{V}$. Without loss of generality, we assume $X \in \mathbf{V}(k)$. Choose a lattice $L_E \subset (C_{\mathfrak{g}}(E))(k)$ so that $X \in F + L_E$. Let K be any compact open subgroup of G. The set

$$\mathcal{F} := \{\mathrm{Ad}(k)(F + W) \,|\, k \in K \text{ and } W \in L_E\}$$

is compact and, from Corollary 4.4, open in $\mathfrak{g}$. Thus, there exists an $N \in \mathbb{Z}$ such that

$$\mathcal{F} + \varpi^{2N}L = \mathcal{F}.$$

Consequently, $[\varpi^{-2N}\mathcal{F}] \in C_c(\mathfrak{g}/L)$. From Lemma 4.7 we conclude that

$$0 \neq \mu_{\varpi^{2n}X}([\mathcal{F}]) = \mu_X([\varpi^{-2N}\mathcal{F}]).$$

Thus, in the notation of (6)

$$0 \neq \mu_X([\varpi^{-2N}\mathcal{F}]) = \sum_{\mathcal{O}'} c_{\mathcal{O}'}(X)\mu_{\mathcal{O}'}([\varpi^{-2N}\mathcal{F}]).$$

Since $\mathcal{O}$ is the only nilpotent orbit that meets $\mathrm{Ad}(G)(\mathbf{V}(k))$, we conclude that $c_{\mathcal{O}}(X) \neq 0$.

Suppose $(\mathrm{Ad}(G)X)$ does not meet $\mathbf{V}$. Consider the function $[\varpi^{-2N}F + L]$ which, from the above paragraph and Lemma 4.6, belongs to

$$D := C_c^\infty(\mathrm{Ad}(G)\mathbf{V}(k)) \cap C_c(\mathfrak{g}/L).$$

For all $f \in D$, we have

$$0 = \mu_X(f) = c_{\mathcal{O}}(X)\mu_{\mathcal{O}}(f).$$

Since $\mu_{\mathcal{O}}([\varpi^{-2N}F + L]) \neq 0$, we conclude that $c_{\mathcal{O}}(X) = 0$. ■

Let Δ be an index set for the regular nilpotent G-orbits, so that

$$\mathcal{N}_{\mathsf{reg}} = \coprod_{\delta\in\Delta} \mathcal{O}_\delta$$

is the partition of $\mathcal{N}_{\mathsf{reg}}$ into G-orbits. By Lemma 5.1, the group $\bar{T}/j(T)$ acts simply-transitively on $\{\mathcal{O}_\delta :\ \delta \in \Delta\}$. If $X \in \mathfrak{g}$ is regular semisimple, we abbreviate $c_\delta(X) = c_{\mathcal{O}_\delta}(X)$ for the coefficient of $\mu_{\mathcal{O}_\delta}$ in (6).

Corollary 4.8 *Suppose the centralizer of X in $\mathbf{G}$ is an unramified anisotropic torus $\mathbf{S}$ whose unique fixed-point in $\mathcal{B}(G)$ is a hyperspecial point. Then $\bar{T}_0$ preserves the set $\{\mathcal{O}_\delta :\ c_\delta(X) \neq 0\}$.*

Proof: If $c_\delta(X) \neq 0$, then by Proposition 4.2 there is a Kostant section $\mathbf{V}_\delta$ for $\mathcal{O}_\delta$ such that $X \in \mathbf{V}_\delta$. In Proposition 5.2 below, (which requires $\mathbf{S}$ to be unramified), it follows that $\mathrm{Ad}(t)X \in \mathrm{Ad}(G)X$, for all $t \in \bar{T}_0$. Hence $\mathrm{Ad}(t)\mathbf{V}_\delta$ is a Kostant section for $\mathrm{Ad}(t)\mathcal{O}_\delta$ which meets $\mathrm{Ad}(G)X$. The result follows from another application of Proposition 4.2. ■

4.5 Local character expansions

Given a generic character ξ of U, there is a regular nilpotent element $F_\xi \in \mathfrak{v}$ defined by the condition

$$\Lambda(\langle X, F_\xi\rangle) = \xi(\exp X) \tag{10}$$

for all $X \in \mathfrak{u}$. The assignment $\xi \mapsto F_\xi$ is a $\bar{T}$-equivariant bijection between the set of generic characters of U and the set of regular nilpotent elements in $\mathfrak{v}$.

For any irreducible admissible representation π of G, let Θ_π be the character of π, viewed as a function on the set G^{rss} of regular semisimple elements in G. There is a neighborhood $\mathcal{V}$ of the identity in G such that on $\mathcal{V} \cap G^{\mathrm{rss}}$ we have the identity

$$\Theta_\pi(\gamma) = \sum_{\mathcal{O}'} c_{\mathcal{O}'}(\pi)\hat{\mu}_{\mathcal{O}'}(\log \gamma), \tag{11}$$

where, as in (6), $\mathcal{O}'$ runs over the set of nilpotent G-orbits in $\mathfrak{g}$, $\hat{\mu}_{\mathcal{O}'}$ is the Fourier transform of the orbital integral over $\mathcal{O}'$, and the complex numbers $c_{\mathcal{O}'}(\pi)$ are uniquely determined, given our choices of Λ and $\langle\ ,\ \rangle$. The following is a special case of the main result in [16].

Proposition 4.9 *We have $\xi \in \Xi(\pi)$ if and only if $c_{\mathcal{O}}(\pi) \neq 0$, where $\mathcal{O}$ is the G-orbit of F_ξ.*

4.6 Regular very cuspidal characters

Let $\pi \in \Pi_X$ be a very cuspidal representation, as in section 2.6. Assume now that X is regular, so that $C_{\mathbf{G}}(X)$ is a torus, which we now denote by $\mathbf{S}$. From [2, 6.3.1] we have the Murnaghan-Kirillov formula, valid for regular semisimple γ in an explicit neighborhood of the identity of G:

$$\Theta_\pi(\gamma) = \deg(\pi)\hat{\mu}_X(\log \gamma). \tag{12}$$

Inserting (6) into (12), comparing with (11) and invoking the uniqueness of the coefficients, we find that

$$c_{\mathcal{O}'}(\pi) = \deg(\pi)c_{\mathcal{O}'}(X) \tag{13}$$

for each nilpotent G-orbit $\mathcal{O}'$ in $\mathfrak{g}$.

Since π is generic, we may further assume that x is hyperspecial (see Corollary 3.2). Let ξ be a generic character of U, let $\mathcal{O}_\xi$ be the G-orbit of F_ξ, and choose a Kostant section $\mathbf{V}_\xi$ for $\mathcal{O}_\xi$. Combining Proposition 4.9, equation (13) and Proposition 4.2, we have

Proposition 4.10 *Assume that $\pi \in \Pi_X$ is very cuspidal, where $C_{\mathbf{G}}(X) = \mathbf{S}$ is a torus such that $\mathcal{B}(S)$ is a hyperspecial point in $\mathcal{B}(G)$. Then for any generic character ξ of U, we have $\mathrm{Hom}_U(\pi, \xi) \neq 0$ if and only if the G-orbit of X meets $\mathbf{V}_\xi$.*

4.7 Completion of the proof of Theorem 1.1

We must show that if $\pi \in \Pi_X$ is very cuspidal and the centralizer $\mathbf{S} = C_{\mathbf{G}}(X)$ is an unramified minisotropic torus whose unique fixed point x in $\mathcal{B}(G)$ is hyperspecial, then π is generic.

Let $\xi \in \Xi$ be any generic character with corresponding regular nilpotent element $F_\xi \in \mathfrak{v}$ and choose a Kostant section $\mathbf{V}_\xi$ for the G-orbit of F_ξ. Since X is regular, there is $g \in \mathbf{G}$ such that

$$\mathrm{Ad}(g)X \in \mathbf{V}_\xi. \tag{14}$$

Since the $\mathbf{G}$-orbit of X is defined over k, the point $\mathrm{Ad}(g)X$ is k-rational, so its centralizer ${}^g\mathbf{S}$ is defined over k. Since $\mathbf{S}$ is abelian, it follows that the map $\mathrm{Ad}(g) : \mathbf{S} \to {}^g\mathbf{S}$ is a k-isomorphism. Since X and $\mathrm{Ad}(g)X$ have the same set of root values, the element $\mathrm{Ad}(g)X$ is also good, in the sense of section 2.6. Moreover, ${}^g\mathbf{S}$ is also minisotropic and unramified. Hence we have another very cuspidal representation

$$ {}^g\pi := \pi(\mathrm{Ad}(g)X) $$

of G. For regular semisimple γ near the identity in G we have the expansion

$$ \Theta_{{}^g\pi}(\gamma) = \deg({}^g\pi) \cdot \hat{\mu}_{\mathrm{Ad}(g)X}(\log \gamma). $$

By (14) and Proposition 4.10, we have

$$ \mathrm{Hom}_U({}^g\pi, \xi) \neq 0. \tag{15} $$

By the other direction of Theorem 1.1, which was already proved in Lemma 3.1, the unique fixed point y of ${}^gS$ in $\mathcal{B}(G)$ is hyperspecial. By Lemma 5.5 below, we may adjust g in its coset $g\mathbf{S}$ so that $j(g) \in \bar{G}$. By uniqueness of the fixed point, we have $y = j(g) \cdot x$.

Moreover, $\mathrm{Ad}(g)^{-1}F_\xi$ is k-rational, so by Lemma 4.1 we may choose $h \in G$ such that the regular nilpotent element

$$ F' := \mathrm{Ad}(hg^{-1})F_\xi $$

lies in $\mathfrak{v}$. Let $\xi' \in \Xi$ be the generic character such that $F' = F_{\xi'}$, as in (10). Then $\mathrm{Ad}(h)X$ is contained in the Kostant section $\mathbf{V}' = \mathrm{Ad}(hg^{-1})\mathbf{V}_\xi$ for the G-orbit of F'. From Proposition 4.10, we now have

$$ \mathrm{Hom}_U(\pi, \xi') \neq 0, $$

showing that π is generic. This completes the proof of Theorem 1.1.

5 Some Galois cohomology

In this chapter we prove those results used above whose proofs were postponed. Though not phrased as such, these results concern the Galois cohomology of the center $\mathbf{Z}$ of $\mathbf{G}$ and the map $H^1(k, \mathbf{Z}) \to H^1(k, \mathbf{L})$ for various subgroups of $\mathbf{L} \subset \mathbf{G}$ containing $\mathbf{Z}$.

Fix an algebraic closure $\bar{k}$ of k and let K be the maximal unramified extension of k in $\bar{k}$. Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group of k and let $\mathcal{I} = \mathrm{Gal}(\bar{k}/K)$ be the inertia subgroup of Γ. If $\mathbf{L}$ is an algebraic k-group (identified with its set of $\bar{k}$-rational points) and $\gamma \in \Gamma$, then $\gamma_{\mathbf{L}}$ denotes the automorphism of $\mathbf{L}$ arising from the given k-structure. Given a containment $\mathbf{L} \subseteq \mathbf{M}$ of k-groups, we let

$$ \iota(\mathbf{L}, \mathbf{M}) : H^1(k, \mathbf{L}) \to H^1(k, \mathbf{M}) $$

denote the map induced on (non-abelian) Galois cohomology sets by the inclusion $\mathbf{L} \hookrightarrow \mathbf{M}$, and we let $\ker^1(\mathbf{L}, \mathbf{M})$ denote the kernel of $\iota(\mathbf{L}, \mathbf{M})$.

5.1 The arithmetic of the adjoint morphism for unramified groups

Recall that our connected reductive k-group $\mathbf{G}$ is quasi-split over k and split over K. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $\mathbf{B}$ is defined over k and let $\mathbf{T}$ be a maximal k-torus in $\mathbf{B}$.

The adjoint morphism j (introduced in section 3) is generally not surjective on rational points. Given $\mathbf{Z} \subset \mathbf{L} \subset \mathbf{G}$ as above, the group

$$\Delta(L) := \bar{L}/j(L)$$

fits into the exact sequence of pointed sets:

$$1 \longrightarrow \Delta(L) \xrightarrow{\delta_{\mathbf{L}}} H^1(k, \mathbf{Z}) \xrightarrow{\iota(\mathbf{Z},\mathbf{L})} H^1(k, \mathbf{L}) \xrightarrow{j_{\mathbf{L}}} H^1(k, \bar{\mathbf{L}}),$$

where $\delta_{\mathbf{L}}$ is the coboundary map and $j_{\mathbf{L}}$ is induced by the adjoint morphism $j : \mathbf{L} \to \bar{\mathbf{L}}$. Note that the inclusion $\bar{L} \hookrightarrow \bar{G}$ induces an injection $\Delta(L) \hookrightarrow \Delta(G)$.

Lemma 5.1 *We have $\bar{G} = \bar{T} \cdot j(G)$. Hence the inclusion $\bar{T} \hookrightarrow \bar{G}$ induces an isomorphism*

$$\Delta(T) \simeq \Delta(G).$$

Proof: If we replace $\mathbf{G}$ by the simply-connected cover of its derived subgroup, then both $\bar{T}$ and $\bar{G}$ are unchanged while $j(G)$ can only become smaller. Hence we may as well assume that $\mathbf{G}$ is semisimple and simply-connected. Then $H^1(k, \mathbf{G}) = 1$ by Steinberg's Theorem [24], so

$$\Delta(G) \simeq H^1(k, \mathbf{Z}).$$

But we also have $H^1(k, \mathbf{T}) = 1$, as is well-known (cf. [18, Lemma 2.0]), so that

$$\Delta(T) \simeq H^1(k, \mathbf{Z}),$$

which proves the Lemma. ■

The group $\Delta(G) \simeq \Delta(T)$ factors into a geometric part and an arithmetic part, as follows. If we fix a uniformizer $\varpi \in k$ then since $\mathbf{T}$ splits over K, we can identify X with a subgroup of $\mathbf{T}(K)$, via evaluation at ϖ, and we have

$$\mathbf{T}(K) = X \times \mathbf{T}(K)_0, \tag{16}$$

where

$$\mathbf{T}(K)_0 = \{t \in \mathbf{T}(K) : \ \mathrm{val}_K(\chi(t)) = 0 \quad \text{for all} \quad \chi \in X^*(\mathbf{T})\}$$

and val_K is the extension of the valuation val to K. The two factors in (16) are stable under the Frobenius F and $T_0 = \mathbf{T}(K)_0^{\mathrm{F}}$, so we have

$$T = X^{\vartheta} \times T_0. \tag{17}$$

Let $\bar{X} = X_*(\bar{\mathbf{T}})$. Then we have a similar decomposition

$$\bar{T} = \bar{X}^\vartheta \times \bar{T}_0. \tag{18}$$

It follows that

$$\Delta(T) = \Delta(X) \times \Delta(T_0), \tag{19}$$

where $\Delta(X) = \bar{X}^\vartheta/j(X^\vartheta)$ is the geometric part and $\Delta(T_0) = \bar{T}_0/j(T_0)$ is the arithmetic part. The following result was used in the proof of Corollary 4.8.

Proposition 5.2 *Let* $\mathbf{S}$ *be a minisotropic unramified maximal* k*-torus in* $\mathbf{G}$*. Then* $\Delta(S) = \Delta(T_0)$*.*

Proof: The following proof was suggested by the referee; it is much shorter than our original proof. Extend the valuation val to $\bar{k}^\times$. For any diagonalizable k-group $\mathbf{D}$, define

$$\mathbf{D}_0 = \{d \in \mathbf{D}(\bar{k}) :\ \mathrm{val}(\chi(d)) = 0 \quad \text{for all} \quad \chi \in X^*(\mathbf{D})\},$$

and set $D_0 = \mathbf{D}_0 \cap D(k)$. Let $\mathbf{S}$ be any maximal k-torus in $\mathbf{G}$ with image $\bar{\mathbf{S}}$ in $\bar{\mathbf{G}}$ under the adjoint morphism $j : \mathbf{G} \to \bar{\mathbf{G}}$. We first claim that j restricts to a surjection $\mathbf{S}_0 \to \bar{\mathbf{S}}_0$. For this we may, upon replacing $\mathbf{G}$ by its derived subgroup, assume that $\mathbf{G}$ is semisimple. If $\bar{s} \in \bar{\mathbf{S}}_0$ has lift $s \in \mathbf{S}$, then the map $X^*(\mathbf{S}) \to \mathbf{Q}^+$ given by $\lambda \mapsto \mathrm{val}(\lambda(s))$ vanishes on the subgroup $j^*X^*(\bar{\mathbf{S}})$ of finite index in $X^*(\mathbf{S})$. Since $\mathbf{Q}^+$ has no finite subgroups, the claim follows. We therefore have an exact sequence

$$1 \longrightarrow \mathbf{Z}_0 \longrightarrow \mathbf{S}_0 \longrightarrow \bar{\mathbf{S}}_0 \longrightarrow 1.$$

Since $H^1(k, \mathbf{S}_0) = 1$ by the profinite version of Lang's theorem, the coboundary $\delta_{\mathbf{S}_0} : \bar{S}_0 \to H^1(k, \mathbf{Z}_0)$ is surjective. Hence, the image of the composition $\bar{S}_0 \to \bar{S}/j(S) \to H^1(k, \mathbf{Z})$ coincides with the image of $H^1(k, \mathbf{Z}_0)$ in $H^1(k, \mathbf{Z})$. It follows that $\bar{S}_0/j(S_0)$ is independent of $\mathbf{S}$. We therefore have $\Delta(S_0) = \Delta(T_0)$ for any maximal k-torus $\mathbf{S}$ in $\mathbf{G}$. Now, if S is minisotropic, we have $\Delta(S_0) = \Delta(S)$, so the result is proved. ■

5.2 Unramified cohomology

This section contains a technical calculation in Galois cohomology that will be used in section 7. More background can be found in [7, Chap.2]. If $\mathbf{L}$ is any connected k-group, then the natural map

$$H^1(K/k, \mathbf{L}(K)) \longrightarrow H^1(k, \mathbf{L})$$

is a bijection. The action of $\mathrm{Gal}(K/k)$ on $\mathbf{L}(K)$ is completely determined by the endomorphism

$$\mathrm{F} = \mathrm{Frob}_{\mathbf{L}}$$

of $\mathbf{L}$. Likewise, a cocycle $c : \mathrm{Gal}(K/k) \to \mathbf{L}(K)$ is determined by the element

$$u_c = c(\mathrm{Frob}),$$

which belongs to the set

$$Z^1(\mathrm{F}, \mathbf{L}(K)) := \{u \in \mathbf{L}(K) : \ u \cdot \mathrm{F}(u) \cdots \mathrm{F}^{n-1}(u) = 1, \quad \text{for some} \quad n \geq 1\}.$$

Thus, an unramified cocycle is identified with an element in $Z^1(\mathrm{F}, \mathbf{L}(K))$, and we identify $H^1(K/k, \mathbf{L}(K))$ with the set $H^1(\mathrm{F}, \mathbf{L}(K))$ of $\mathbf{L}(K)$-orbits in $Z^1(\mathrm{F}, \mathbf{L}(K))$ under the action: $\ell * u = \ell u \, \mathrm{F}(\ell)^{-1}$. Let $[u]_L \in H^1(\mathrm{F}, \mathbf{L}(K))$ denote the class of an element $u \in Z^1(\mathrm{F}, \mathbf{L}(K))$.

Given any unramified maximal torus $\mathbf{S}$ in $\mathbf{G}$, we will use unramified cohomology to study the diagram:

$$\begin{array}{ccccccc} 1 \longrightarrow \Delta(G) & \xrightarrow{\delta_{\mathbf{G}}} & H^1(k, \mathbf{Z}) & \longrightarrow & H^1(k, \mathbf{G}) & \xrightarrow{j_{\mathbf{G}}} & H^1(k, \bar{\mathbf{G}}) \\ & & \| & & \uparrow & & \\ 1 \longrightarrow \Delta(S) & \xrightarrow{\delta_{\mathbf{S}}} & H^1(k, \mathbf{Z}) & \longrightarrow & H^1(k, \mathbf{S}) & \xrightarrow{j_{\mathbf{S}}} & H^1(k, \bar{\mathbf{S}}), \end{array} \tag{20}$$

where the unlabeled maps are induced by inclusion.

Let $X = X_*(\mathbf{T})$ be the lattice of algebraic one-parameter subgroups of $\mathbf{T}$ and let $\vartheta \in \mathrm{Aut}(X)$ be the automorphism of X induced by the Frobenius endomorphism F. Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$. For $w \in \mathbf{N}/\mathbf{T}$, let $\mathbf{T}_w$ be the unramified twist of $\mathbf{T}$. Denoting the twisted action of $\gamma \in \Gamma$ by $\gamma_{\mathbf{T}_w}$, we have

$$\gamma_{\mathbf{T}_w} = \begin{cases} \gamma_{\mathbf{T}} & \text{if} \quad \gamma \in \mathcal{I} \\ \mathrm{F}_w := \mathrm{Ad}(w) \circ \mathrm{F} & \text{if} \quad \gamma = \mathrm{Frob}\,. \end{cases}$$

Note that F_w acts on X via $w\vartheta$. We have the explicit isomorphism

$$[X/(1 - w\vartheta)X]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}_w, \mathbf{T}(K)) \simeq H^1(k, \mathbf{T}_w), \tag{21}$$

which sends the class of $\lambda \in X$ to the unramified class $[\lambda(\varpi)]_{\mathbf{T}_w} \in H^1(\mathrm{F}_w, \mathbf{T}(K))$.

Let $p_0 \in \mathbf{G}(K)$ be an element such that $p_0^{-1}\,\mathrm{F}(p_0) \in \mathbf{N}(K)$ is a representative of w. Then $\mathbf{S} := p_0 \mathbf{T} p_0^{-1}$ is an unramified maximal k-torus in $\mathbf{G}$, and the map $\mathrm{Ad}(p_0) : \mathbf{T}_w \to \mathbf{S}$ is a k-isomorphism. Hence we have the explicit isomorphism

$$[X/(1 - w\vartheta)X]_{\mathrm{tor}} \longrightarrow H^1(\mathrm{F}, \mathbf{S}(K)) \simeq H^1(k, \mathbf{S}), \tag{22}$$

which sends the class of $\lambda \in X$ to the unramified class $[p_0 \lambda(\varpi) p_0^{-1}]_{\mathbf{S}} \in H^1(\mathrm{F}, \mathbf{S}(K))$.

Let $\bar{X} = X_*(\bar{\mathbf{T}})$ and again write ϑ for the automorphism of $\bar{X}$ induced by the Frobenius F. We have similar isomorphisms

$$\left[\bar{X}/(1 - w\vartheta)\bar{X}\right]_{\mathrm{tor}} \xrightarrow{\sim} H^1(k, \bar{\mathbf{T}}_w) \xrightarrow{\sim} H^1(k, \bar{\mathbf{S}}). \tag{23}$$

Our basic diagram (20) becomes

$$\begin{array}{ccccccc} 1 \longrightarrow \Delta(T) & \xrightarrow{\delta_T} & H^1(k,\mathbf{Z}) & \xrightarrow{\iota(\mathbf{Z},\mathbf{G})} & H^1(k,\mathbf{G}) & \xrightarrow{j_{\mathbf{G}}} & H^1(k,\bar{\mathbf{G}}) \\ & & \| & & r_w \uparrow & & \\ 1 \longrightarrow \Delta(T_w) & \xrightarrow{\delta_{\mathbf{T}_w}} & H^1(k,\mathbf{Z}) & \xrightarrow{\iota(\mathbf{Z},\mathbf{T}_w)} & H^1(k,\mathbf{T}_w) & \xrightarrow{j_w} & H^1(k,\bar{\mathbf{T}}_w), \end{array} \tag{24}$$

where r_w is the composition

$$r_w : H^1(k,\mathbf{T}_w) \xrightarrow{\mathrm{Ad}(p_0)} H^1(k,\mathbf{S}) \xrightarrow{\iota(\mathbf{S},\mathbf{G})} H^1(k,\mathbf{G})$$

and $j_w = j_{\mathbf{T}_w}$ is induced by the restriction of j to $\mathbf{T}$.

The group $\Delta(T)$ parametrizes generic characters, and the group $H^1(k,\mathbf{T}_w)$ parametrizes representations in an L-packet of very cuspidal representations (see [20]). In section 7 we will show that the map

$$\iota_w : \Delta(T) \to H^1(k,\mathbf{T}_w),$$

given by the composition

$$\iota_w : \Delta(T) \xrightarrow{\delta_{\mathbf{T}}} H^1(k,\mathbf{Z}) \xrightarrow{\iota(\mathbf{Z},\mathbf{T}_w)} H^1(k,\mathbf{T}_w) \tag{25}$$

determines which generic characters appear in which representation in the L-packet. Our goal here is to calculate the map ι_w explicitly. Diagram (24) shows that

$$\iota_w(\Delta(T)) = \iota(\mathbf{Z},\mathbf{T}_w)\left(\ker^1(\mathbf{Z},\mathbf{G})\right) = \ker r_w \cap \ker j_w.$$

Lemma 5.3 *Let $t \in \mathbf{T}$ be such that $j(t)$ is k-rational. Then $t \cdot w(t)^{-1} \in Z^1(\mathrm{F}_w, \mathbf{T}(K))$ and*

$$\iota_w([t]_\Delta) = [t \cdot w(t)^{-1}]_{\mathbf{T}_w} \in H^1(\mathrm{F}_w, \mathbf{T}(K)).$$

Proof: Recall that $[t]_\Delta$ denotes the class of $j(t)$ in $\Delta(T)$ and $\delta_{\mathbf{T}}([t]_\Delta) \in H^1(k,\mathbf{Z})$ is the class of the cocycle $\gamma \mapsto z_\gamma = t^{-1} \cdot \gamma_{\mathbf{T}}(t) \in \mathbf{Z}$, for $\gamma \in \Gamma$. In $\mathbf{T}_w$, the cocycle z_γ is cohomologous to the cocycle

$$z'_\gamma := t \cdot z_\gamma \cdot \gamma_{\mathbf{T}_w}(t)^{-1} = \gamma_{\mathbf{T}}(t) \cdot \gamma_{\mathbf{T}_w}(t)^{-1}.$$

Note that $z'_\sigma = 1$ for $\sigma \in \mathcal{I}$ and

$$z'_{\mathrm{Frob}} = \mathrm{F}(t) \cdot \mathrm{F}_w(t)^{-1}.$$

Since $t^{-1} \cdot \mathrm{F}(t) \in \mathbf{Z}$ which is centralized by w, we have

$$t^{-1} \cdot \mathrm{F}(t) = w(t^{-1} \cdot \mathrm{F}(t)) = w(t)^{-1} \cdot \mathrm{F}_w(t).$$

It follows that

$$z'_{\mathrm{Frob}} = t \cdot w(t)^{-1}.$$

Since z' is trivial on $\mathcal{I}$, it is an unramified cocycle, so the Lemma is proved. ■

We now obtain a more explicit formula for ι_w using the factorization $\Delta(T) = \Delta(X) \times \Delta(T_0)$ from (19). Let $X^\circ \subset X$ be the lattice of co-roots.

Lemma 5.4 *Assume that $\mathbf{T}_w$ is minisotropic. Then* $\ker \iota_w = \Delta(T_0)$ *and on* $\Delta(X)$ *we have the formula*

$$\iota_w[\mu(\varpi)]_\Delta = [\lambda(\varpi)]_{\mathbf{T}_w} \in H^1(\mathrm{F}_w, \mathbf{T}(K)),$$

where $\mu \in \bar{X}^\vartheta$ and λ is the unique element of X° such that $j\lambda = (1-w)\mu$.

Proof: Note that the formula makes sense because $(1-w)\bar{X} \subset jX^\circ$ and j is injective on X°. Since jX has finite index in $\bar{X}$, there is an integer $m \geq 1$ and $\eta \in X$ such that

$$m\mu = j\eta. \tag{26}$$

Let $\varpi^{1/m} \in \bar{k}$ be a root of $x^m = \varpi$. Set

$$t := \eta(\varpi^{1/m}).$$

Then $j(t) = \mu(\varpi)$, and

$$t \cdot w(t)^{-1} = (1-w)\eta(\varpi^{1/m}).$$

Now

$$j(1-w)\eta = (1-w)j\eta = m(1-w)\mu = mj\lambda.$$

Since j is injective on X°, we have

$$(1-w)\eta = m\lambda, \tag{27}$$

so that

$$t \cdot w(t)^{-1} = \lambda(\varpi).$$

The formula for ι_w on $\Delta(X)$ now follows from Lemma 5.3. This formula implies that ι_w is injective on $\Delta(X)$. Indeed, if $[\lambda(\varpi)]_{\mathbf{T}_w} = 1$ then there is $\nu \in X$ such that

$$\lambda = (1-w\vartheta)\nu. \tag{28}$$

Applying j to both sides and remembering that $\vartheta\mu = \mu$, we get

$$(1-w\vartheta)\mu = (1-w)\mu = j\lambda = (1-w\vartheta)j\nu.$$

Since $\mathbf{T}_w$ is minisotropic, the map $1 - w\vartheta$ is injective on $\bar{X}$, so we have

$$\mu = j\nu.$$

In equation (26) we can then take $\eta = \nu$ and $m = 1$, so that equation (27) reads as

$$(1-w)\nu = \lambda. \tag{29}$$

Comparing equations (28) and (29), we see that $\nu \in X^\vartheta$, so that $\mu \in j(X^\vartheta)$, proving the injectivity of ι_w on $\Delta(X)$.

Finally, it follows from Prop. 5.2 that ι_w vanishes on $\Delta(T_0)$. Hence $\ker \iota_w = \Delta(T_0)$ and the proof of Lemma 5.4 is complete. ■

5.3 Hyperspecial points and stable conjugacy

In this section we prove Lemma 5.5, which was used in section 4.7

Let $X \in \mathfrak{g}$ be regular semisimple, let $\mathcal{O} = \mathrm{Ad}(\mathbf{G})X$, and let $\mathbf{S} = C_{\mathbf{G}}(X)$. Any k-rational point $Y \in \mathcal{O}(k)$ is of the form $Y = \mathrm{Ad}(g)X$ for some $g \in \mathbf{G}$ such that $s_\gamma := g^{-1}\gamma(g) \in \mathbf{S}$ for all $\gamma \in \Gamma = \mathrm{Gal}(\bar{k}/k)$. The mapping $\gamma \mapsto s_\gamma$ is a Galois cocycle whose class

$$\mathrm{inv}(X, Y) := [s_\gamma] \in H^1(k, \mathbf{S})$$

is independent of the choice of g. It is clear that $[s_\gamma]$ lies in the kernel $\ker^1(\mathbf{S}, \mathbf{G})$ of the map $H^1(k, \mathbf{S}) \to H^1(k, \mathbf{G})$ induced by the inclusion $\mathbf{S} \hookrightarrow \mathbf{G}$. Two rational points $Y, Y' \in \mathcal{O}(k)$ are G-conjugate if and only if $\mathrm{inv}(X, Y) = \mathrm{inv}(X, Y')$, and we have

$$\ker^1(\mathbf{S}, \mathbf{G}) = \{\mathrm{inv}(X, Y) : \ Y \in \mathcal{O}(k)\}.$$

Let $Y = \mathrm{Ad}(g)X \in \mathcal{O}(k)$ as above, and let $\mathbf{S}_1 = \mathrm{Ad}(g)\mathbf{S} = C_{\mathbf{G}}(Y)$. Since $\mathbf{S}$ is abelian, the isomorphism $\mathrm{Ad}(g) : \mathbf{S} \to \mathbf{S}_1$ is defined over k.

Assume now that $\mathbf{S}$ is minisotropic and unramified over k and therefore has a unique fixed-point x in $\mathcal{B}(G)$. Then $\mathbf{S}_1$ is also minisotropic unramified over k and has a unique fixed-point $y \in \mathcal{B}(G)$.

Lemma 5.5 *In the situation above, assume that x is hyperspecial in $\mathcal{B}(G)$. Then the point y is also hyperspecial in $\mathcal{B}(G)$ if and only if* $\mathrm{inv}(X, Y)$ *belongs to the image of the map $H^1(k, \mathbf{Z}) \to H^1(k, \mathbf{S})$ induced by the inclusion $\mathbf{Z} \hookrightarrow \mathbf{S}$.*

Proof: We have $\mathrm{inv}(X, Y) \in \mathrm{im}[H^1(k, \mathbf{Z}) \to H^1(k, \mathbf{S})]$ exactly when there exists $g' \in \mathbf{G}$ such that $Y = \mathrm{Ad}(g')X$ and $j(g')$ is k-rational. In this case we have $y = j(g') \cdot x$, by uniqueness of fixed-points. It follows that y is hyperspecial.

Assume now that y is hyperspecial in $\mathcal{B}(G)$. Since $\bar{G}$ acts transitively on hyperspecial points in $\mathcal{B}(G)$, we have $y = j(h) \cdot x$ for some $h \in \mathbf{G}$ with the property that $j(h) \in \bar{G}$. Set $z_\gamma := h^{-1}\gamma(h) \in \mathbf{Z}$, for $\gamma \in \Gamma$. The tori $\mathbf{S}_1 = {}^g\mathbf{S}$ and $\mathbf{S}_2 = {}^h\mathbf{S}$ both fix the vertex y and are isomorphic over k, via the map $\mathrm{Ad}(hg^{-1}) : \mathbf{S}_1 \to \mathbf{S}_2$.

Set $\mathsf{G}_y := \mathbf{G}(K)_y / \mathbf{G}(K)_{y,0^+}$ and $\bar{\mathsf{G}}_y := \bar{\mathbf{G}}(K)_y / \bar{\mathbf{G}}(K)_{y,0^+}$. These are connected reductive groups over the residue field $\mathfrak{f}$. Since y is hyperspecial, we may identify

$$(\bar{\mathsf{G}})_y = \mathsf{G}_y / Z(\mathsf{G}_y),$$

where $Z(\mathsf{G}_y)$ is the center of G_y. Indeed, $\mathbf{G}(K)_y$ projects naturally onto both groups and the kernel of both projections is $\mathbf{G}(K)_y^+ \cdot (\mathbf{Z} \cap \mathbf{G}(K)_y)$.

For $i = 1, 2$, the intersections $\mathbf{S}_i \cap \mathbf{G}(K)_y$ project to maximal tori S_i in G_y. In turn, each S_i projects to a maximal torus $\bar{\mathsf{S}}_i$ in $\mathsf{G}_y / Z(\mathsf{G}_y)$.

The hyperspecial condition on y also implies that $\bar{\mathbf{G}}(K)_y$ is the full stabilizer of y in $\bar{\mathbf{G}}(K)$. The element $\mathrm{Ad}(hg^{-1}) \in \bar{\mathbf{G}}(K)_y$ thus projects to an element $d \in \mathsf{G}_y / Z(\mathsf{G}_y)$, and we have ${}^d\bar{\mathsf{S}}_1 = \bar{\mathsf{S}}_2$.

By Lemma 5.6 below, the tori S_1 and S_2 are $\mathsf{G}_x(\mathfrak{f})$-conjugate. From [6] we conclude that $\mathbf{S}_1$ and $\mathbf{S}_2$ are G_x-conjugate. Choose $\ell \in G_x$ such that ${}^{\ell}\mathbf{S}_1 = \mathbf{S}_2$.

The element $n := h^{-1}\ell g$ belongs to the normalizer $\mathbf{N}$ of $\mathbf{S}$ in $\mathbf{G}$. For any $\gamma \in \Gamma$, we have

$$h^{-1}\gamma(h) = z_\gamma \in \mathbf{Z}, \qquad g^{-1}\gamma(g) = s_\gamma \in \mathbf{S}, \qquad \gamma(\ell) = \ell.$$

It follows that

$$\gamma(n) = n \cdot z_\gamma^{-1} \cdot s_\gamma.$$

Since $z_\gamma^{-1} \cdot s_\gamma \in \mathbf{S}$, the image of n in $\mathbf{N}/\mathbf{S}$ is k-rational.

Since $\mathbf{S}$ is unramified, we have $H^1(\bar{k}/K, \mathbf{S}) = 1$, which implies that

$$[\mathbf{N}/\mathbf{S}](K) = \mathbf{N}(K)/\mathbf{S}(K).$$

Since x is hyperspecial, we can apply Lemma 6.2.3 of [7], to conclude that

$$[\mathbf{N}/\mathbf{S}](k) = \mathbf{N}(k)/\mathbf{S}(k).$$

Hence there exists $s \in \mathbf{S}$ such that ns is k-rational. For all $\gamma \in \Gamma$, we then have

$$ns = \gamma(ns) = \gamma(n)\gamma(s) = n \cdot z_\gamma^{-1} \cdot s_\gamma \cdot \gamma(s)$$

so that $s^{-1}s_\gamma\gamma(s) = z_\gamma \in \mathbf{Z}$. This means that $[s_\gamma] = [z_\gamma] \in \mathrm{im}[H^1(k, \mathbf{Z}) \to H^1(k, \mathbf{S})]$, as claimed. ■

In the proof above, we used the following result.

Lemma 5.6 *Let* G *be a connected reductive group over the finite field* $\mathfrak{f}$*, with center* Z *and adjoint group* $\bar{\mathsf{G}} = \mathsf{G}/\mathsf{Z}$*. Let* F *be the Frobenius endomorphism of* G *and* $\bar{\mathsf{G}}$*. Suppose we have two* F*-stable maximal tori* $\mathsf{S}_1, \mathsf{S}_2$ *in* G*, projecting to maximal tori* $\bar{\mathsf{S}}_1, \bar{\mathsf{S}}_2$ *in* $\bar{\mathsf{G}}$*. Suppose also that there is* $d \in \bar{\mathsf{G}}$ *satisfying*

$$\text{(i) } {}^{d}\bar{\mathsf{S}}_1 = \bar{\mathsf{S}}_2 \qquad \text{and} \qquad \text{(ii) } \mathrm{Ad}(d) \circ F = F \circ \mathrm{Ad}(d) \quad \text{on} \quad \bar{\mathsf{S}}_1.$$

Then S_1 *and* S_2 *are* $\mathsf{G}(\mathfrak{f})$*-conjugate.*

Condition (ii) means that $d^{-1}F(d) \in \bar{\mathsf{S}}_1$. By Lang's theorem applied to $\bar{\mathsf{S}}_1$, there is $s_1 \in \bar{\mathsf{S}}$ such that $\bar{\mathsf{S}}_1$ and $\bar{\mathsf{S}}_2$ are conjugate by the element $d_1 := ds_1 \in \bar{\mathsf{G}}(\mathfrak{f})$. Let d_2 be a lift of d_1 in G and let $z := d_2^{-1}F(d_2) \in \mathsf{Z}$. Since $\mathsf{Z} \subset \mathsf{S}$, there is $t \in \mathsf{S}_1$ such that $z = t \cdot F(t)^{-1}$. Then the element $d_3 := d_2 t$ belongs to $\mathsf{G}(\mathfrak{f})$ and conjugates S_1 to S_2, proving the lemma. ■

6 The generic characters in a very cuspidal representation

We next consider question (ii) in the introduction, concerning which generic characters are afforded by our generic very cuspidal representations $\pi \in \Pi_X$.

Given two regular nilpotent elements F, F' in $\mathfrak{g}$ and $g \in \mathbf{G}$ such that $\mathrm{Ad}(g)F = F'$, we have a cocycle $z_\gamma = g^{-1}\gamma(g) \in \mathbf{Z}$ whose class

$$\mathrm{inv}(F, F') := [z_\gamma] \in \ker^1(\mathbf{Z}, \mathbf{G})$$

vanishes if and only if F and F' are G-conjugate. By Lemma 4.1, we know that if F, F' belong to $\mathfrak{v}$ then we may take $g \in \mathbf{T}$.

Lemma 6.1 *Let $\pi \in \Pi_X$ be a generic very cuspidal representation such that $C_{\mathbf{G}}(X)$ is a min-isotropic torus $\mathbf{S}$, and let $\xi \in \Xi(\pi)$. Then if $\xi' \in \Xi$ is another generic character of U, we have $\xi' \in \Xi(\pi)$ if and only if $\mathrm{inv}(F_\xi, F_{\xi'}) \in \ker^1(\mathbf{Z}, \mathbf{S})$.*

Proof: We have $F_{\xi'} = \mathrm{Ad}(t)F_\xi$ for some $t \in \mathbf{T}$, with cocycle $z_\gamma = t^{-1}\gamma(t) \in \mathbf{Z}$. Let $\mathbf{V}_\xi$ be a Kostant section at F_ξ for the G-orbit of F_ξ. Then $\mathbf{V}_{\xi'} := \mathrm{Ad}(t)\mathbf{V}_\xi$ is a Kostant section at $F_{\xi'}$ for the G-orbit of $F_{\xi'}$. By Proposition 4.10, we may assume that $X \in \mathbf{V}_\xi$. Moreover, we have $\mathrm{Ad}(h)X \in V_{\xi'}$ for some $h \in G$. The elements X and $\mathrm{Ad}(t^{-1}h)X$ both belong to $\mathbf{V}_\xi$, hence must coincide. It follows that $h = ts$ for some $s \in \mathbf{S}$. We have

$$s^{-1}z_\gamma\gamma(s) = s^{-1} \cdot t^{-1}\gamma(t) \cdot \gamma(s) = h^{-1} \cdot \gamma(h) = 1.$$

This shows that $[z_\gamma]$ becomes trivial in $H^1(k, \mathbf{S})$, as claimed.

The argument is reversible: if there exists $s \in \mathbf{S}$ such that $z_\gamma = s \cdot \gamma(s)^{-1}$, then the element $h = ts$ belongs to G and $\mathrm{Ad}(h)X = \mathrm{Ad}(t)X \in \mathbf{V}_{\xi'}$, so that $\xi' \in \Xi(\pi)$, by Proposition 4.10. ■

6.1 Example: SL_2

Assume $p \neq 2$. Let $\mathbf{G} = \mathbf{SL}_2$, with $\bar{\mathbf{G}} = \mathbf{PGL}_2$. Here $\mathbf{Z} = \{\pm 1\}$, so

$$H^1(k, \mathbf{Z}) = k^\times / k^{\times 2}.$$

Let k_2 be the unramified quadratic extension of k with norm mapping $N : k_2^\times \to k^\times$. Set $k_2^1 = \ker N$. Take an unramified torus $\mathbf{S} \subset \mathbf{G}$ with $S \simeq k_2^1$. We have $\bar{S} = k_2^\times / k^\times$ and

$$j(S) = k_2^1 / \pm 1 \simeq k_2^1 \cdot k^\times / k^\times \subset \bar{S}.$$

Hence

$$\Delta(S) \simeq k_2^\times / k_2^1 \cdot k^\times,$$

which is isomorphic, via N, to the group $N(k_2^\times)/k^{\times 2}$. It follows that $\Delta(S)$ is isomorphic to an index two subgroup of $k^\times / k^{\times 2}$, so that $|\Delta(S)| = 2$.

Let T, U be the diagonal and upper triangular matrices (with ones on the diagonal) in G and let α be the root of T in $\mathfrak{u}$. We may identify $T = k^\times = \bar{T}$, such that $j : T \to \bar{T}$ is the squaring map. Hence $\Delta(X) = \mathbb{Z}/2\mathbb{Z}$ and

$$|\Delta(T_0)| = \tfrac{1}{2}[k^\times : k^{\times 2}] = |\Delta(S)|.$$

Let $\mathcal{A}$ be the apartment of T and let $o, y \in \mathcal{A}$ be the hyperspecial vertices whose stabilizers are

$$G_o = SL_2(\mathfrak{o}), \qquad G_y = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} SL_2(\mathfrak{o}) \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}^{-1}.$$

Taking o as the origin, we view α as a linear functional on $\mathcal{A}$ with

$$\alpha(o) = 0, \qquad \alpha(y) = 1.$$

For $F = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}$ with $f \in k^\times$, the corresponding generic character $\xi_f : U \to \mathbb{C}^\times$ is given by

$$\psi_f \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right) = \Lambda(ft).$$

A Kostant section at F for the G-orbit of F is given by

$$\mathbf{V}_f = \left\{ \begin{bmatrix} 0 & t \\ f & 0 \end{bmatrix} : \ t \in \bar{k} \right\}.$$

The T-orbits of such F are represented by

$$f \in \mathcal{F} := \{1, \epsilon, \varpi, \epsilon\varpi\},$$

where ϵ is a fixed non-square unit in $\mathfrak{o}$.

For $x \in \{o, y\}$ and integers $r \geq 0$, we have

$$U \cap G_{x,r} = \begin{bmatrix} 1 & \mathfrak{p}^{r-\alpha(x)} \\ 0 & 1 \end{bmatrix}.$$

Since Λ has conductor $\mathfrak{p}$, it follows that ξ_f has generic depth r at x iff

$$\operatorname{val}(f) + r = \alpha(x). \tag{30}$$

Each $x \in \{o, y\}$ is the fixed-point set in $\mathcal{B}(G)$ of the group S_x of k-rational points in an unramified anisotropic torus $\mathbf{S}_x$. These groups are given by

$$S_o = \begin{bmatrix} a & b \\ b\epsilon & a \end{bmatrix}, \qquad S_y = \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} S_o \begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}^{-1}.$$

Very cuspidal representations $\pi_x = \pi(X_x)$ of G arise from elements $X_x \in \mathfrak{g}$ of the form

$$X_o = u\varpi^{-r} \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \qquad X_y = u\varpi^{-r} \begin{bmatrix} 0 & \varpi^{-1} \\ \epsilon\varpi & 0 \end{bmatrix} = \operatorname{Ad}\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix} \right) X_o,$$

where u is a unit in $\mathfrak{o}$. One can check that there exists $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ such that $\mathrm{Ad}(g)X_x \in \mathbf{V}_f$ exactly when the equations

$$\begin{cases} d^2\epsilon - c^2 = u^{-1} f \varpi^r & \text{for } x = o \\ (d\varpi)^2 - c^2\epsilon = u^{-1} f \varpi^{r+1} & \text{for } x = y \end{cases}$$

have a solution (c, d) in $k \times k$. The left side of these equations is a norm from the unramified quadratic extension of k. It follows that

$$\mathrm{Ad}\,(G)\, X_x \cap \mathbf{V}_f \neq \varnothing \quad \Leftrightarrow \quad r + \mathrm{val}(f) \equiv \alpha(x) \mod 2. \tag{31}$$

Hence from Proposition 4.10 we have

$$\Xi(\pi_x) = \{\xi_f :\ r + \mathrm{val}(f) \equiv \alpha(x) \mod 2\}. \tag{32}$$

From (30) we see that $\Xi(\pi_x)$ is the union of the two T-orbits of characters having generic depth r at x and these two orbits are interchanged by the projective matrix

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \in \bar{T}_0.$$

7 Generic representations in L-packets

Let $\mathcal{W}$ be the Weil group of k and let $\hat{G}$ be the dual group of $\mathbf{G}$. Let $\hat{\vartheta} \in \mathrm{Aut}(\hat{G})$ be the automorphism of $\hat{G}$ arising from the action of the Frobenius F on the root datum of $\mathbf{G}$. In [7] and [20] we considered homomorphisms

$$\varphi : \mathcal{W} \longrightarrow {}^L G = \langle \hat{\vartheta} \rangle \ltimes \hat{G}$$

with the properties

- The map φ is trivial on $\mathcal{I}^{(r+1)}$ and nontrivial on $\mathcal{I}^{(r)}$, for some integer $r \geq 0$, where $\mathcal{I}^{(r)}$ is the upper-numbering filtration on $\mathcal{I}$ (see [22, chap.4]).
- The centralizer of $\varphi(\mathcal{I}^{(r)})$ in $\hat{G}$ is a maximal torus $\hat{T}$ of $\hat{G}$.
- $\varphi(\mathrm{Frob}) \in \hat{\vartheta} \ltimes \hat{G}$, and the centralizer of $\varphi(\mathcal{W})$ in $\hat{G}$ is finite, modulo the center $Z(\hat{G})$ of $\hat{G}$.

Let C_φ be the component group of the centralizer of the image of φ in $\hat{G}$. The element $\varphi(\mathrm{Frob})$ is of the form

$$\varphi(\mathrm{Frob}) = \hat{\vartheta} \ltimes \hat{w},$$

where $\hat{w} \in \hat{G}$ normalizes $\hat{T}$ and corresponds via duality to an element $w \in \mathbf{N}/\mathbf{T}$. We have

$$C_\varphi = \hat{T}^{\hat{\vartheta}\hat{w}} / Z(\hat{G})^\circ.$$

Hence C_φ is abelian, and the set of irreducible characters of C_φ may be identified with the torsion subgroup of $X/(1 - w\vartheta)X$. For each class $\rho \in [X/(1 - w\vartheta)X]_{\mathrm{tor}}$ we have an an explicitly constructed isomorphism class of supercuspidal representations $\pi(\varphi, \rho)$ which have depth-zero [7] or are very cuspidal [20].

7.1 Generic representations in L-packets

For a general $\rho \in \mathrm{Irr}(C_\varphi)$, the class $\pi(\varphi,\rho)$ consists of representations of the group of k-points of a certain pure inner form of $\mathbf{G}$. This pure inner form is k-isomorphic to $\mathbf{G}$ itself exactly when $r_w(\rho) = 1 \in H^1(k,\mathbf{G})$, where r_w is the map in (24). Let

$$\Pi(\varphi,1) = \{\pi(\varphi,\rho) : \ \rho \in \ker r_w\}.$$

For $\rho \in \ker r_w$, the class $\pi(\varphi,\rho)$ contains a representation induced from a hyperspecial parahoric subgroup of G if and only if $\rho \in \ker j_w$ (see [7, 6.2.1], which applies also to the positive-depth case). From Theorem 1.1 it follows that $\pi(\varphi,\rho)$ is generic if and only if $\rho \in \ker r_w \cap \ker j_w$. Lemma 5.4 shows that the map ι_w restricts to an isomorphism

$$\iota_w : \Delta(X) \xrightarrow{\sim} \ker r_w \cap \ker j_w.$$

In particular, we have

Corollary 7.1 *The set* $\Pi(\varphi,1)$ *contains exactly* $|\Delta(X)| = [\bar{X}^\vartheta : j(X^\vartheta)]$ *generic representations.*

This was proved in the depth-zero case in [7].

7.2 Generic characters and the parametrization of L-packets

We now consider the generic characters appearing in a representation belonging to the class $\pi(\varphi,\rho)$, for $\rho \in \ker r_w \cap \ker j_w$.

Proposition 7.2 *Let* $\mu \in \bar{X}^\vartheta$ *have image* $\rho = \iota_w\left([\mu(\varpi)]_\Delta\right) \in \ker r_w \cap \ker j_w$. *Then under the conjugation action of* $\bar{G}$ *on isomorphism classes of representations of* G*, we have*

$$\mathrm{Ad}(\mu(\varpi)) \cdot \pi(\varphi,1) = \pi(\varphi,\rho).$$

On the level of generic characters, this gives the immediate

Corollary 7.3

$$\mathrm{Ad}(\mu(\varpi)) \cdot \Xi\left(\pi(\varphi,1)\right) = \Xi\left(\pi(\varphi,\rho)\right).$$

This can be stated more cohomologically, as follows. Via the coboundary $\delta_{\mathbf{T}}$ and Lemma 5.1, we identify $\ker^1(\mathbf{Z},\mathbf{G})$ with the union of the cosets of $j(T)$ in $\bar{T}$. Let $\ker^1_\rho(\mathbf{Z},\mathbf{G})$ be the fiber over ρ under the map $H^1(k,\mathbf{Z}) \to H^1(k,\mathbf{T}_w)$ induced by inclusion. Corollary 7.3 asserts that if ξ is any generic character in $\Xi(\pi(\varphi,1))$, then in the free $\ker^1(\mathbf{Z},\mathbf{G})$-orbit through ξ we have

$$\ker^1_\rho(\mathbf{Z},\mathbf{G}) \cdot \xi = \Xi\left(\pi(\varphi,\rho)\right).$$

To prove Proposition 7.2, we assume, as we may, that $\bar{\mathbf{G}}$ is simple. Let o be a hyperspecial vertex in $\mathcal{A}$ such that some representation in the class $\pi(\varphi,1)$ is induced from K_o. Let C be a

ϑ-stable alcove in $\mathcal{A}(K)$ containing o in its closure. Let $\mathcal{M}$ (for "minuscule weight") be the set of $\nu \in \bar{X}$ such that $t_\nu \cdot o \in \bar{C}$. Then $\vartheta\mathcal{M} = \mathcal{M}$ and $\mathcal{M}^\vartheta$ contains a set of representatives for $\bar{X}^\vartheta/j(X^\vartheta)$. We may assume that $\mu \in \mathcal{M}^\vartheta$. Then $t_\mu \cdot o$ is a hyperspecial vertex in $\bar{C}^\vartheta \subset \mathcal{A}$.

Recall from Lemma 5.4 that ρ is the class of $\lambda \in X^\circ$, where $j\lambda = (1-w)\mu$. Then the unique fixed-point of $t_\lambda w\vartheta$ in $\mathcal{A}(K)$ is $x_\lambda = t_\mu \cdot o \in \bar{C}$. According to the recipe of [7, chap.4] (using notation therein) we have

$$w_\lambda = t_\lambda w, \quad y_\lambda = 1, \quad \mathrm{F}_\lambda = \mathrm{F}\,.$$

The element

$$p_\lambda := \mathrm{Ad}(\mu(\varpi))p_0 \in \mathbf{G}(K)_{x_\lambda}$$

has the property that $p_\lambda^{-1}F(p_\lambda)$ normalizes $\mathbf{T}(K)$ and is a lift of $t_\lambda w$. Let χ_φ be the character of T_w given by the abelian Langlands correspondence [15], [7], [20]. As in section 2.6, the character χ_φ is determined by an element X_φ in the Lie algebra of T_w.

If we set

$$X_0 = \mathrm{Ad}(p_0)X_\varphi, \qquad X_\lambda = \mathrm{Ad}(p_\lambda)X_\varphi,$$

then the class $\pi(\varphi, 1)$ contains $\pi(X_0)$ and the class $\pi(\varphi, \rho)$ contains $\pi(X_\lambda)$. Since $\mathrm{Ad}(\mu(\varpi))X_0 = X_\lambda$, this implies Proposition 7.2. ■

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