Epstein Surfaces, W-Volume, and the Osgood-Stowe Differential

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1 Introduction

In the paper [Eps], Epstein showed how a conformal metric on the conformal boundary $\partial_c M$ of a hyperbolic manifold $M$ gives rise to a surface in the interior of $M$. In particular these Epstein surfaces generalize the convex core of $M$, which arises as the Epstein surface of the projective metric on $\partial_c M$. In recent work motivated from physics, a natural functional called the W-volume has been introduced which is an analytic function on the space of conformal metrics and is defined in terms of the geometry of the associated Epstein surface. In work of Krasnov-Schlenker [KS] and Schlenker [Sch], they prove many fundamental properties of the W-volume and the related renormalized volume $V_R$, including a variational formula for $W$ and showing that $dV_R$ is the Schwarzian derivative of the uniformization of the conformal boundary $\partial_c M$.

In this paper we show that Epstein surfaces and W-volume can be described simply in terms of the Osgood-Stowe differential, a generalization of the Schwarzian derivative (see [OS]). Using this, we obtain short proofs of many of the properties of Epstein surfaces and W-volume derived by Epstein and Krasnov-Schlenker as well as new results describing the variation of the W-volume in terms of the Osgood-Stowe differential and a description of the W-volume in terms of curvature forms.

2 Results

2.1 Fundamental Pairs, Duals and Gauss-Codazzi

Given a smooth manifold $M$ a fundamental pair of real quadratic forms $(I, II)$ is a pair where $I$ is a Riemannian metric on $M$. For a fundamental pair $(I, II)$ we define the shape operator $B$ to be the linear map satisfying

$$II(X, Y) = I(BX, Y).$$
Then $B$ is self-adjoint with respect to $I$.

If $(I, II)$ is a fundamental pair on a surface $S$, we define the *dual fundamental pair at infinity* by $(I^*, II^*)$ where

\[
I^*(X,Y) = I((Id + B)X, (Id + B)Y) \quad \text{and} \quad II^*(X,Y) = I((Id + B)X, (Id - B)Y).
\]

We note that $(I^*, II^*)$ is a fundamental pair if and only if $Id + B$ is invertible. Thus $(I, II)$ is called a *good* fundamental pair if $Id + B$ is invertible. Also it follows that

\[
II^*(X,Y) = I(B^*X, Y) \quad \text{where} \quad B^* = (Id + B)^{-1}(Id - B).
\]

We note that $(Id + B)(Id + B^*) = 2Id$ and therefore $(I, II)$ is good if and only if $(I^*, II^*)$ is good. Thus the process is invertible and given a good fundamental pair $(I^*, II^*)$ at infinity, we have the good dual fundamental pair $(I, II)$ such that

\[
I(X,Y) = \frac{1}{4}I^*((Id + B^*)X, (Id + B^*)Y) \quad \text{and} \quad II(X,Y) = \frac{1}{4}I^*((Id + B^*)X, (Id - B^*)Y).
\]

**Gauss-Codazzi Equations in $\mathbb{H}^3$:**

If $S$ is an immersed surface in $\mathbb{H}^3$ with fundamental forms $(I, II)$ then they satisfy the Gauss-Codazzi equations

\[
d^\nabla B = 0 \quad \text{(Codazzi)} \quad (2.1)
\]
\[
det(B) = K + 1 \quad \text{(Gauss)} \quad (2.2)
\]

Here $\nabla$ is the connection for metric $I$ and $K$ is the Gaussian curvature of $I$.

In fact, a classical result says that the converse holds.

**Theorem 2.1 (Bonnet, [1])** Let $S$ be a simply connected surface with fundamental pair $(I, II)$. Then there is an isometric immersion of $S$ in $\mathbb{H}^3$ if and only if $(I, II)$ satisfy the Gauss-Codazzi equations.

Krasnov-Schlenker showed that if the fundamental forms $(I, II)$ they satisfy the Gauss-Codazzi equations the dual fundamental pair $(I^*, II^*)$ satisfy equations which called the *Gauss-Codazzi equations at infinity*. They are

\[
d^\nabla^* B^* = 0 \quad \text{(Codazzi)} \quad (2.3)
\]
\[
\text{Tr}(B^*) = -2K^* \quad \text{(Gauss)} \quad (2.4)
\]

Here $\nabla^*$ is the connection for metric $I^*$ and $K^*$ is the Gaussian curvature of $I$. In particular the Gauss equation at infinity says that the mean curvature at infinity $H^* = -K^*$.

We first prove the following lemma.

**Lemma 2.2** A good fundamental pair $(I, II)$ satisfies the Gauss-Codazzi equations if and only if the dual pair $(I^*, II^*)$ satisfies the Gauss-Codazzi equations at infinity.
Osgood-Stowe Differential: We now let $\rho$ be a conformal metric on a surface $\Sigma$ and let $\Sigma$ be projective structure on $\Sigma$. We define the Osgood-Stowe two-form $B_\Sigma(\rho)$ as follows; Let $\rho = e^{2\phi}|dz|^2$ in a projective chart $U$ in $\Sigma$. Then we define the Osgood-Stowe form of $\rho$ in $U$ by

$$B_\Sigma(\rho) = \frac{1}{2}(Q(\phi)dz^2 + Q(\bar{\phi})d\bar{z}^2)$$

where $Q(\phi) = 2(\phi_{zz} - (\phi_z)^2)$. Following work of Osgood-Stowe (see [OS]), this gives a well-defined two-form on $\Sigma$. We then let

$$II_\Sigma(\rho) = 2B_\Sigma(\rho) - K(\rho).$$

We show that

**Theorem 2.3** Let $\rho$ be a conformal metric on a Riemann surface $S$. Then a fundamental pair $(\rho, II)$ satisfies the Gauss-Codazzi equations at infinity if and only if $II = II_\Sigma(\rho)$ for some $\Sigma$ a projective structure on $S$.

Applying the above we get

**Theorem 2.4** Let $\rho$ be a conformal metric on a simply connected domain $\Omega \subseteq \hat{\mathbb{C}}$. Then for $t$ sufficiently large the dual fundamental form $(I_t, II_t)$ of $(e^{2t}\rho, II_\Sigma(\rho))$ is realizable as an immersed surface $S_t$ given by $f_t: \Omega \to \mathbb{H}^3$. Furthermore, the surfaces $S_t$ can be immersed in $\mathbb{H}^3$ such that they are obtained by normal flow and foliate a neighborhood of $\Omega$ in $\mathbb{H}^3$ with limiting metric $\rho$.

A simple calculation shows that the above describes the Epstein surface of $\rho$ and shows that Epstein surfaces are the simply the solutions to the Euler-Codazzi equations for the fundamental pair $(\rho, II_\Sigma(\rho))$.

### 2.2 Variational Formula for $W$ volume and the Osgood-Stowe Differential

Given $N$ a convex submanifold of $M$, the $W$-volume of $N$ is defined by

$$W(N) = V(N) - \frac{1}{4}\int_{\partial N} HdA$$

where $H$ is the mean curvature given by $H = Tr(B)/2$. The $W$ volume has a number of nice analytic properties including the scaling property that if $N_t$ is obtained by normal flow from $N$ then a simple calculations (see [KS]) shows that

$$W(N_t) = W(N) + t\pi |\chi(\partial M)|.$$

Similarly, given a conformal metric $\rho$ on $\partial M$, then for $t$ large the metric $e^{2t}\rho$ has a convex Epstein surface $S_t$ bounding a convex submanifold $N_t$. The $W$ volume of $\rho$ is then defined by

$$W(\rho) = W(N_t) - t\pi |\chi(\partial M)|.$$
We let $\rho_M$ be the hyperbolic metric on $\partial_c M$ and then define the renormalized volume by

$$V_R(M) = W(\rho_M).$$

Let $M = \mathbb{H}^3/\Gamma$ with $\partial_c M = \Omega M/\Gamma$ where $\Gamma$ are Mobius transformations. Therefore the conformal boundary $\partial_c M$ is endowed with a natural projective structure $\Sigma_M$. This is uniquely defined by the quadratic differential $\phi_M$ on $\partial_c M$ given by the Schwarzian of the map uniformizing the components of $\Omega M$.

In their paper [KS], Krasnov-Schlenker proved the following:

**Theorem 2.5** (Krasnov-Schenker, [KS]) Let $CC(N)$ be the space of convex co-compact hyperbolic structures on $N$. Then for $M \in CC(N)$

$$dV_R(M) = Re(\phi_M)$$

We now generalize this to give a formula in terms of the Osgood-Stowe differential. We first denote by $\Sigma_M$ the natural projective structure on $\partial_c M$ and let $Q_M : Conf(\partial_c M) \rightarrow Q(\partial_c M)$ be the associated Osgood-Stowe differential map given by the projective structure $\Sigma_M$.

We define the space $CC_c(N) = \{(M, \rho) | M \in CC(N), \rho \in Conf(\partial_c M)\}$. We have the identification

$$T(M, \rho) = C^\infty(\partial_c M) \oplus T_M(Conf(\partial_c M))$$

given by letting $\rho_t = f_t \rho$ where $f_t$ is the hyperbolic metric and $f_t$ a smooth function. Then $\nu = \rho_0 = (\rho_0, \rho_0^2) = (f(\nu), \mu(\nu)) \in C^\infty(\partial_c M) \oplus T_M(Conf(\partial_c M))$. We note that $\mu(\nu)$ is a Beltrami differential on $\partial_c M$.

**Theorem 2.6** The variational formula for $W$ on $CC_c(N)$ is given by

$$dW(M, \rho)(\nu) = \frac{1}{4} \int_{\partial_c M} dK(\nu) d\rho - Re \int_{\partial_c M} Q_M(\rho) \mu(\nu).$$

We now generalize the scaling and monotonicity property of $W$ volume in a single theorem which also gives a new formulation for $W$ volume. If $\rho$ is a conformal metric, we let $\Omega_\rho = K(\rho)dA_\rho$ be the curvature form.

**Lemma 2.7** Let $\sigma = e^{2u} \rho$, then

$$W(\sigma) - W(\rho) = -\frac{1}{4} \left( \int u K_\rho dA_\rho + \int u K_\sigma dA_\sigma \right) = -\frac{1}{8} (\Omega_\rho + \Omega_\sigma)(\log(\sigma/\rho)).$$

It follows trivially that if $\rho \leq \sigma$ pointwise and both are non-positively curved then $W(\rho) \leq W(\sigma)$.
3 Osgood-Stowe Differential

We now describe the work of Osgood-Stowe in [OS] where they introduce the Osgood-Stowe differential and prove many of its properties. We let \((M,g)\) be a Riemannian n-manifold and \(\phi : M \to \mathbb{R}\) a smooth function. A natural symmetric two-tensor is given by

\[
Hess_g(\phi) - d\phi \otimes d\phi.
\]

Then the \textit{Osgood-Stowe differential} \(B_g(\phi)\) is defined to be the traceless part of this, i.e.

\[
B_g(\phi) = (Hess_g(\phi) - d\phi \otimes d\phi)_o = Hess_g(\phi) - d\phi \otimes d\phi - \frac{Trace(Hess_g(\phi) - d\phi \otimes d\phi)}{n} g.
\]

For \((U,g_e)\) an open subset \(U\) in \(\mathbb{R}^n\) with the Euclidean metric \(g_e\), we define

\[
B(\phi) = B_{g_e}(\phi).
\]

We note that if \(f : (M,g) \to (M',g')\) is a conformal map then \(f^*g' = e^{2\phi}g\) where \(\phi = \log |f'|\).

For \(f : \Omega \to \Omega'\) a conformal map on \(\Omega \subseteq \hat{\mathbb{C}}\), it can be easily checked that

\[
B(\log |f'|) = S(f) \text{ the Schwarzian derivative.}
\]

We have

\[
(Hess_{g_e}(\phi) - d\phi \otimes d\phi) = \begin{bmatrix}
\phi_{xx} - \phi_{yy} - \phi_x^2 + \phi_y^2 \\
\phi_{xy} - \phi_x \phi_y \\
\phi_{yy} - \phi_y^2 
\end{bmatrix}
\]

Taking the trace-free component

\[
B(\phi) = \begin{bmatrix}
\frac{1}{2}(\phi_{xx} - \phi_{yy} - \phi_x^2 + \phi_y^2) & \phi_{xy} - \phi_x \phi_y \\
\phi_{xy} - \phi_x \phi_y & -\frac{1}{2}(\phi_{xx} - \phi_{yy} - \phi_x^2 + \phi_y^2)
\end{bmatrix}
\]

We define

\[
Q(\phi) = \frac{1}{2} (\phi_{xx} - \phi_{yy} - \phi_x^2 + \phi_y^2) - i(\phi_{xy} - \phi_x \phi_y)
\]

and have

\[
B(\phi) = \begin{bmatrix}
Re(Q(\phi)) & -Im(Q(\phi)) \\
-Im(Q(\phi)) & -Re(Q(\phi))
\end{bmatrix}
\]

Writing as complex 2-forms we have \(Q\) is

\[
Q(\phi) = 2(\phi_{zz} - \phi_z^2)
\]

and

\[
B(\phi) = \frac{1}{2} (Q(\phi) dz^2 + \overline{Q(\phi)} d\overline{z}^2).
\]

Similarly for a metric \(g\),

\[
B_g(\phi) = \frac{1}{2} (Q_g(\phi) dz^2 + \overline{Q_g(\phi)} d\overline{z}^2)
\]

Osgood-Stowe showed (see [OS]) that \(B_g\) (and \(Q_g\)) satisfies an additive relation

\[
B_g(\phi + \psi) = B_g(\phi) + B_{e^{2\phi}}(\psi)
\]
3.1 Osgood-Stowe and conformal metrics

We now describe how the Osgood-Stowe differential can be used to define a cocycle on conformal metrics on a Riemann surface. Given a Riemann surface $S$ with two conformal metrics $\rho_1, \rho_2$, then $\rho_2 = e^{2\phi} \rho_1$ for some function $\phi$ and we define

$$Q(\rho_1, \rho_2) = Q_\rho(\phi) \quad B(\rho_1, \rho_2) = B_\rho(\phi).$$

In [Dum], Dumas showed that in local coordinates if $\rho_i = e^{2\phi_i} |dz|^2$ then

$$Q(\rho_1, \rho_2) = ((\phi_2 - \phi_1)_z - ((\phi_2)_z)^2 + ((\phi_1)_z)^2) dz^2.$$

It follows from the additivity property for $Q_\rho$ (and $B_\rho$) that we have the cocycle condition on $Q$ (and $B$)

$$Q(\rho_1, \rho_3) = Q(\rho_1, \rho_2) + Q(\rho_2, \rho_3) \quad Q(\rho_1, \rho_2) = -Q(\rho_2, \rho_1).$$

We have that $Q$ also satisfies the following functorial property that for $f$ conformal

$$Q(f^* \rho_1, f^* \rho_2) = f^* Q(\rho_1, \rho_2).$$

By direct calculation it can be shown that if $\rho$ is Möbius equivalent to the hyperbolic metric on the disk, the Euclidean metric or the spherical metric, and $\sigma = e^{2\phi} |dz|^2$ then

$$Q(\rho, \sigma) = Q(\phi)$$

It follows that for $f$ Möbius then,

$$Q(f^* \phi) = f^* Q(\phi).$$

The Möbius equation for Riemannian manifold $(M, g)$ is the equation

$$B_g(\phi) = H_0$$

where $H_0$ is a traceless symmetric bilinear form.

The space of solutions to the homogeneous Möbius equation $B_g(\phi) = 0$ is defined to be $\mathcal{U}(M)$. The constant functions give the trivial solutions and define a subset homeomorphic to $\mathbb{R}$ in $\mathcal{U}(M)$ corresponding to conformal metrics homothetic to $g$.

In [OS], Osgood and Stowe classified which Riemannian manifolds $(M, g)$ have non-trivial solutions to the homogeneous Möbius equation. For surfaces it follows that

**Theorem 3.1** (Osgood-Stowe, [OS, Theorem 5.4]) Let $(M, g)$ be a complete Riemannian surface. Then $\mathcal{U}(M) \neq \mathbb{R}$ if and only if $(M, g)$ is simply connected and has constant curvature.
3.2 Osgood-Stowe and Projective Structures

Given $\Sigma \in P(X)$ then the developing map of the projective structure gives a locally univalent map $f_\Sigma : \Delta \to \hat{\mathbb{C}}$ which is well defined up to Möbius transformation. The Schwarzian derivative of $f_\Sigma$ defines a quadratic differential $\tilde{\phi}_\Sigma$ on $\Delta$ which descends to a quadratic differential $\phi_\Sigma \in Q(X)$. The map $P(X) \to Q(X)$ given by $\Sigma \to \phi_\Sigma$ is an analytic homeomorphism.

Given $\Sigma \in P(X)$ we now define a natural map $Q_\Sigma : Conf(X) \to Q(X)$. The map is given by mapping a conformal metric to its Osgood-Stowe differential with respect to the projective structure $\Sigma$. To see this is well-defined, we write a conformal metric $\rho \in Conf(X)$ in $\Sigma$ projective coordinates on a chart $U$ in the form $e^{2\phi(z)}|dz|^2$ and define $Q_\Sigma(\rho) = Q(\phi)$ on $U$. We note if we choose another projective coordinate system on a set $V$ with $\rho = e^{2\phi'(w)}|dw|^2$, then $w = f(z)$ on $U \cap V$ is Möbius and
\[
f^*(e^{2\phi'(z)}|dz|^2) = e^{2\phi'(w)}|dw|^2
\]
giving $f^*\phi = \phi'$ as conformal metrics. As $f$ is Möbius then by the above invariance we have $Q(\phi') = f'Q(\phi')$. Similarly we define $B_\Sigma$.

**Lemma 3.2** Let $\Sigma \in P(X)$ be a projective structure. Then $Q_\Sigma(\rho) = Q_\Sigma(\sigma)$ if and only if $\rho, \sigma$ are homothetic metrics. Furthermore if $\rho_X$ is the hyperbolic metric on $X$ then
\[
Q_\Sigma(\rho) = Q(\rho_X, \rho) - \phi_\Sigma.
\]

Let $f_\Sigma : \Delta \to \Omega$ be the associated locally univalent map of $\Sigma$. The conformal metric $\rho$ on $X$ lifts to give a conformal metric on $\Delta$ of the form $\tilde{\rho}$. Then as $f = f_\Sigma$ is a projective chart, $Q_\Sigma(\rho) = f'Q(|dz|^2, f, \tilde{\rho})$. We let $\rho_h$ be the hyperbolic metric on $\Delta$ and $\rho_f = f^*|dz|^2 = |f'(z)|^2|dz|^2$. Then using the cocycle property we have
\[
Q(|dz|^2, f, \tilde{\rho}) = Q(|dz|^2, f, \rho_h) + Q(f, \rho_h, f, \tilde{\rho})
\]
Therefore
\[
Q_\Sigma(\rho) = f^*Q(|dz|^2, f, \rho_h) + Q(f, \rho_h, f, \tilde{\rho}) = Q(\rho_f, \rho_h) + Q(\rho_h, \tilde{\rho})
\]
As $Q(\rho_f, \rho_h) = -Q(\rho_h, \rho_f) = -S(f)$ we obtain
\[
Q_\Sigma(\rho) = Q(\rho_h, \tilde{\rho}) - S(f_\Sigma).
\]
The result then follows by descending to the quotient $X$.

By the above, if $Q_\Sigma(\rho_1) = Q_\Sigma(\rho_2)$ then $Q(\rho_X, \rho_1) = Q(\rho_X, \rho_2)$. Therefore $B(\rho_X, \rho_1) = B(\rho_X, \rho_2)$. By the cocycle relation we have
\[
B(\rho_X, \rho_2) = B(\rho_X, \rho_1) + B(\rho_1, \rho_2)
\]
giving $B(\rho_1, \rho_2) = 0$. Letting $\rho_1 = e^{2\phi} \rho_X$, then $B(\rho_1, \rho_2) = B_{\rho_1}(\phi_2 - \phi_1) = 0$. As $X$ is not simply connected, by Theorem 3.1 we have $\phi_2 - \phi_1 = c$ a constant. Thus $\rho_1, \rho_2$ are homothetic. □
4 Convex submanifolds, Normal flow

Given an immersion $f: S \to \mathbb{H}^3$ there is a canonical lift $F: S \to T^1\mathbb{H}^3$. If $g_t: T^1\mathbb{H}^3 \to T^1\mathbb{H}^3$ is the geodesic flow then we define $F = g_t: F$ and $f_t = \pi \circ F$ where $\pi: T^1\mathbb{H}^3 \to \mathbb{H}^3$ is the projection.

We let $S$ be an immersed $\pi$, injective hyperbolic surface. We let $I$ be the intrinsic metric on $S$ and $B: TS \to TS$ the associated shape operator given by $B(v) = -\nabla_v n$.

The metric $I$ is also called the first fundamental form of $S$. Similarly we define the second and third fundamental forms by $II(v,w) = I(v,Bw)$ and $III(v,w) = III(v,Bw)$.

We let $S_t$ be the surface obtained by taking normal flow for time $t$ on $S$. These surfaces foliate the ends of $M$. It can be easily seen (see [KS]) that if $I_t$ is the pullback of the first fundamental form of $S_t$ then

$$I_t(v,w) = I((\cosh(t)Id + \sinh(t)B)(v),(\cosh(t)Id + \sinh(t)B)(w)).$$

We obtain a conformal metric $I^*$ on $\partial M$ by taking the limit of the metrics $I_t$. We define

$$I^*(v,w) = \frac{1}{\cosh^2(t)}I_t(v,w).$$

Taking the limit we obtain

$$I^*(v,w) = I((Id+B)(v),(Id+B)(w))$$

If we expand $I_t$ we can define the fundamental forms at infinity by

$$I_t = \frac{1}{4}(e^{2t}I^* + 2II^* + e^{-2t}III^*).$$

A simple calculation also shows that

$$II^*(v,w) = I((Id+B)(v),(Id-B)(w))$$

Thus it is natural to define the shape operator at infinity by $B^* = (Id+B)^{-1}(Id-B)$ and have $II^*(v,w) = I^*(v,B^*w)$ and $III^*(v,w) = II(v,B^*w)$.

It follows from above that we have the linear relations

$$I^* = I + 2II + III$$
$$II^* = I - III$$
$$III^* = I - 2II + III$$

$$I = \frac{1}{4}(I^* + 2II^* + III^*)$$
$$II = \frac{1}{4}(I^* - III^*)$$
$$III = \frac{1}{4}(I^* - 2II^* + III^*).$$

Furthermore, inverting the formula for $B^*$ we obtain $B = (Id+B^*)^{-1}(Id-B^*)$.

**Lemma 4.1** Let $S$ be an immersed surface in $\mathbb{H}^3$ with $\rho = I^*$ on $\Omega \subseteq \mathbb{C}$. Then the second fundamental form at infinity is

$$II^* = 2B_{\Sigma}(\rho) - K(\rho)dA(\rho)$$

{epstein2}
I then σ be the Möbius map given by normal projection to the upper hemisphere and assume p is the origin in ℍ³ and P = TpS the xy-plane with S. Let f : P → C be the Möbius map given by normal projection to the upper hemisphere and let σ = f°ρ = e²ψ|dz|². We note that ψ(0) = 0, ψx(0) = ψy(0) = 0. We have at z = 0 that both I, I* = Id and the curvature at z = 0 is K(σ) = −Δψ. Therefore at z = 0

\[ 2B(σ) - K(σ)dA(σ) = 2 \left[ \frac{1}{2} (ψ_{xx} - ψ_{yy}) \right] + Δψ.Id = \begin{bmatrix} 2ψ_{xx} & 2ψ_{xy} \\ 2ψ_{xy} & 2ψ_{yy} \end{bmatrix} = 2ψ''(0). \]

Thus we need only show B* = 2ψ''(0).

We note that for S, the boundary of the r-neighborhood of a hyperbolic plane, then I' = e²ρh, where ρh is the hyperbolic metric, B* = e⁻²Id and B*(e²ρh) = 0. As K(e²ρh) = −e⁻² the equation holds for Sr.

To finish, we assume that the x and y axes are lines of curvature of I. Then it follows that ψxy(0) = 0 (Obvious?). Thus we only need show the equality for tangent lines to the x, y axes at z = 0. This follows from the case Sr above. □

4.1 Epstein surfaces

In [Eps], Epstein described how, given a conformal metric on a domain Ω ⊆ C, one can construct an immersed surface in ℍ³. We consider a conformal metric ρ on Ω. Given p ∈ ℍ³, we let ψ be the visual metric on C from x, i.e. the spherical measure on C given by radial projection from p. Given ξ ∈ Ω we define the set

\[ H_ρ(ξ) = \{ p ∈ Hs | ρ(ξ) ≤ ρp(ξ) \}. \]

This set is a horoball in ℍ³ with basepoint ξ. Epstein considered the envelope of all these.

\[ EP_ρ = \partial \left( \bigcup_{ξ ∈ Ω} H_ρ(ξ) \right). \]

Epstein showed that for large t, EPρ will be a convex embedded surface.

We make the following observation that Epstein’s construction corresponds to inverting Krasnov-Schlenker’s construction.

**Lemma 4.2** Let S be an immersed submanifold of M such that the induced metric at infinity by normal flow is ρ. Then ρ is a conformal metric on ∂M and S is the Epstein surface EPρ.

**Proof:** We let S be immersed such that I* = ρ. We let F : S → ∂M be the map induced by the normal flow. Then we have for v, w ∈ TpS

\[ ρ_ρ(F_s(v), F_s(w)) = (Id + B)(v), (Id + B)(w)). \]

For each p ∈ S let H_ρ be the hyperbolic plane tangent to S at p and f_p : H_ρ → ∂M be the map obtained by normal flow on H_ρ. A simple calculation gives that (f_p)'(F_s(v)) = (Id + B)(v). Therefore

\[ ρ_ρ(F_s(v), F_s(w)) = (f_p(F_s(v)), f_p(F_s(w)) = ((f_p), (f_p)(F_s(v), F_s(w))). \]
It follows that \( \rho_{F(p)} = (f_p)_p \) and therefore \( \rho \) is conformal as \( f_p \) is conformal. Taking \( p \) at the origin of the Poincaré model, then the visual measure \( v_p \) is regular spherical measure and \( \rho_\xi = 1 \). Thus \( p \in \partial H_\rho(\xi) \) giving \( p \in E_p \). □

5 The Dual of a Fundamental Pair

Given a smooth manifold \( M \) a fundamental pair of real quadratic forms \((I, II)\) is a pair where \( I \) is a Riemannian metric on \( M \). Given a fundamental pair \((I, II)\) we define the shape operator \( B \) to be the operator satisfying

\[
II(X, Y) = I(BX, Y).
\]

Then \( B \) is self-adjoint with respect to \( I \). We call fundamental pair \((I, II)\) good if \((Id + B)\) is invertible. Equivalently, \( B \) has no eigenvalues equal to \(-1\).

If \((I, II)\) is a fundamental pair on a surface \( S \), following the prior section, it is natural to define the dual fundamental pair by \((I^*, II^*)\) where

\[
I^*(X, Y) = I((Id + B)X, (Id + B)Y) \quad II^*(X, Y) = I((Id + B)X, (Id - B)Y).
\]

We note that \((I^*, II^*)\) is a fundamental pair if and only if \((I, II)\) is a good fundamental pair. Also it follows that

\[
II^*(X, Y) = I(B^*X, Y) \quad \text{where} \quad B^* = (Id + B)^{-1}(Id - B).
\]

As \((Id + B)(Id + B^*) = 2Id\) we have trivially that \((I, II)\) is a good fundamental pair if and only if \((I^*, II^*)\) is a good fundamental pair.

5.1 Gauss-Codazzi Equations

If \( S \) is an immersed surface in \( \mathbb{H}^3 \) with fundamental forms \((I, II)\) then they satisfy the Gauss-Codazzi equations

\[
d^\nabla B = 0 \quad \text{(Codazzi)} \quad \det(B) = K + 1 \quad \text{(Gauss)}
\]

Here \( \nabla \) is the connection for metric \( I \), \( K \) is the Gaussian curvature of \( I \) and for \( X, Y \) vector fields

\[
(d^\nabla B)(X, Y) = \nabla_X(B(Y)) - \nabla_Y(B(X)) - B([X, Y])
\]

In fact, we have the converse.

**Theorem 5.1** Let \( S \) be a simply connected surface with a good fundamental pair \((I, II)\). Then there is an isometric immersion of \( S \) in \( \mathbb{H}^3 \) if and only if \((I, II)\) satisfy the Gauss-Codazzi equations.
Krasnov-Schlenker showed that if the fundamental forms \((I, II)\) they satisfy the Gauss-Codazzi equations the dual fundamental pair \((I^*, II^*)\) satisfy equations which they called the Gauss-Codazzi equations at infinity. They are

\[
d^\nabla^* B^* = 0 \quad \text{(Codazzi)}
\]
\[
Tr(B^*) = -2K^* \quad \text{(Gauss)}
\]

Here \(\nabla^*\) is the connection for metric \(\star I\) and \(K^*\) is the Gaussian curvature of \(I\). In particular the Gauss equation at infinity says that the mean curvature at infinity \(H^* = -K^*\).

We have the following simple lemma.

**Lemma 5.2** A good fundamental pair \((I, II)\) satisfies the Gauss-Codazzi equations if and only if the dual pair \((I^*, II^*)\) satisfy the Gauss-Codazzi equations at infinity.

**Proof:** The proof that if \((I, II)\) satisfy the Gauss-Codazzi equations then the pair \((I^*, II^*)\) satisfy the Gauss-Codazzi equations at infinity appears in [KS]. The argument reverses easily as follows.

We now assume that \((I^*, II^*)\) satisfy the Gauss-Codazzi equations at infinity. Krasnov-Schelnker [KS, Lemma 5.2] observe that the pair \((I^*, II^*)\) satisfy

\[
\nabla_x^* Y = (\text{Id} + B)^{-1} \nabla_x ((\text{Id} + B)(Y)) \quad K^* = \frac{K}{\det(\text{Id} + B)}.
\]

It follows (see [KS, Remark 5.5]) that

\[
d^\nabla^* B^* = (\text{Id} + B)^{-1} d^\nabla (\text{Id} - B).
\]

We let \(\lambda_1, \lambda_2\) be the eigenvalues of \(B\) and \(\lambda_1^*, \lambda_2^*\) the eigenvalues of \(B^*\). As \(B^* = (\text{Id} + B)^{-1} (\text{Id} - B)\)

\[
Tr(B^*) = \lambda_1^* + \lambda_2^* = \frac{1 - \lambda_1}{1 + \lambda_1} + \frac{1 - \lambda_2}{1 + \lambda_2} = \frac{(1 - \lambda_1)(1 + \lambda_2) + (1 - \lambda_2)(1 + \lambda_1)}{(1 + \lambda_1)(1 + \lambda_2)} = \frac{2 - 2\lambda_1\lambda_2}{(1 + \lambda_1)(1 + \lambda_2)}
\]

\[
= \frac{2 - 2\det(B)}{\det(\text{Id} + B)} = \frac{2(1 - \det(B))}{\det(\text{Id} + B)}.
\]

By the Gauss equation at infinity \(Tr(B^*) = -2K^*\). Therefore

\[
-2K^* = \frac{2K}{\det(\text{Id} + B)} = \frac{2(1 - \det(B))}{\det(\text{Id} + B)}
\]

giving \(\det(B) - 1 = K\) the Gauss equation for \((I, II)\).

To show that \(I, II\) satisfy the Codazzi equation, by inverting the above, we have

\[
d^\nabla B = (\text{Id} + B^*)^{-1} d^\nabla^* (\text{Id} - B^*) = (\text{Id} + B^*)^{-1} (d^\nabla^* (\text{Id}) - d^\nabla^* (B^*)) = 0
\]
as \( d^{\nabla^*} B^r = 0 \) by the Codazzi equation at infinity and \( d^{\nabla^*} (Id) = 0 \) as \( \nabla^* \) is torsion-free. □

We now let \( \rho \) be a conformal metric on a projective structure \( \Sigma \). We let

\[
II^*(\rho) = 2B_{\Sigma}(\rho) - K(\rho).\rho
\]

where \( B_{\Sigma}(\rho) \) is the Osgood-Stowe differential for \( \rho \) and \( K(\rho) \) is the curvature of \( \rho \).

Will will need the following elementary fact concerning conformal metrics.

**Lemma 5.3** Let \( \rho = e^{2\phi}|dz|^2 \) be a conformal metric on \( \Omega \subseteq \mathbb{C} \). Then the co-variant derivative \( \nabla \) of \( \rho \) satisfies

\[
\nabla \partial_z = 2\phi \partial_z \quad \nabla \partial_{\bar{z}} = 0 = \partial_z \partial_{\bar{z}} = 2\phi \partial_{\bar{z}}.
\]

**Proof:** If \( \rho = e^{2\phi} \sigma \) and \( \nabla \) is the covariant derivative of \( \sigma \) then a standard formula gives

\[
\nabla_X Y = \hat{\nabla}_X Y + X(\phi).Y + Y(\phi).X - <X,Y> \cdot grad(\phi)
\]

where \(<,...,>\) is the inner product and \( grad \) the gradient of \( \sigma \). Letting \( \sigma \) be the Euclidean metric and noting that

\[
<\partial_z, \partial_z> = <\partial_{\bar{z}}, \partial_{\bar{z}}> = 0 <\partial_z, \partial_{\bar{z}}> = \frac{1}{2}
\]

the result follows. □

**Theorem 5.4** The fundamental pair \((\rho, II^*)\) on Riemann surface \( S \) satisfies the Gauss-Codazzi equations at infinity if and only if

\[
II^* = II_{\Sigma}^*(\rho)
\]

for some projective structure \( \Sigma \) on \( S \).

**Proof:** We take \( \rho = e^{2\phi}|dz|^2 \) in projective coordinates corresponding to projective structure \( \Sigma \). We let \((\rho, II^*)\) be a fundamental pair. Then as \( II^* \) is real symmetric, we have

\[
II^* = R(dz^2) + R(d\bar{z}^2) - Se^{2\phi}|dz|^2
\]

for some functions \( R,S \). The traceless part of \( II^* \) is

\[
II_0^* = R(dz)^2 + R(d\bar{z})^2.
\]

It follows that

\[
Tr(B^*) = Tr((I^*)^{-1} II^*) = Tr(-S.id) = 2S.
\]

Therefore \((\rho, II^*)\) satisfy the Gauss equation at infinity if and only if \( S = K(\rho) \).
We consider the covariant derivative $\nabla^*$ of $\rho$ in complex coordinates. We let

$$\partial_z = \frac{\partial}{\partial z}, \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}.$$ 

In these coordinates we have

$$B^* = \begin{bmatrix} -\frac{S}{2} & Re^{-2\phi} \\ Re^{-2\phi} & -\frac{S}{2} \end{bmatrix}.$$ 

Therefore

$$B^* \partial_z = -\frac{S}{2} \partial_z + Re^{-2\phi} \partial_{\bar{z}}, \quad B^* \partial_{\bar{z}} = Re^{-2\phi} \partial_z - \frac{S}{2} \partial_{\bar{z}}.$$ 

As $dV^* B^*$ is a skew-symmetric tensor, we need only check for $X = \partial_z, Y = \partial_{\bar{z}}$ to see if it is zero. Expanding we have

$$(dV^* B^*)(\partial_z, \partial_{\bar{z}}) = \nabla_{\partial_z} B^* (\partial_{\bar{z}}) = \nabla_{\partial_{\bar{z}}} B^* (\partial_z) - B^* ([\partial_z, \partial_{\bar{z}}])$$

$$= \nabla_{\partial_z} (Re^{-2\phi} \partial_z - \frac{S}{2} \partial_{\bar{z}}) - \nabla_{\partial_{\bar{z}}} (Re^{-2\phi} \partial_z - \frac{S}{2} \partial_{\bar{z}}).$$

By the formulae for the covariant derivative above

$$(dV^* B^*)(\partial_z, \partial_{\bar{z}}) = \left( (Re^{-2\phi})_z \partial_z + Re^{-2\phi} \nabla_{\partial_z} \partial_{\bar{z}} - \frac{S}{2} \partial_{\bar{z}} \right) - \left( -\frac{S}{2} \partial_z + (Re^{-2\phi})_{\bar{z}} \partial_z + Re^{-2\phi} \nabla_{\partial_{\bar{z}}} \partial_z \right)$$

$$= \left( Re^{-2\phi} - 2\phi \cdot Re^{-2\phi} \right) \partial_z + Re^{-2\phi} 2\phi \cdot \partial_{\bar{z}} - \frac{S}{2} \partial_{\bar{z}} - \left( -\frac{S}{2} \partial_z + (Re^{-2\phi})_{\bar{z}} \partial_z + Re^{-2\phi} 2\phi \partial_{\bar{z}} \right)$$

$$= \left( Re^{-2\phi} + \frac{S}{2} \right) \partial_z - \left( \frac{S}{2} + Re^{-2\phi} \right) \partial_{\bar{z}} = \left( \frac{S}{2} + Re^{-2\phi} \right) \partial_z - \left( \frac{S}{2} + Re^{-2\phi} \right) \partial_{\bar{z}}$$

Therefore the Codazzi equation $dV^* B^* = 0$ corresponds is

$$S = -2Re^{-2\phi}.$$ 

Therefore $(\rho, II^*)$ satisfies the Gauss-Codazzi equations at infinity if and only if

$$S = K \quad S_{\bar{z}} = -2Re^{-2\phi}.$$ 

We have $K = -e^{-2\phi} \Delta \phi = -4e^{-2\phi} \phi_{zz}$ and $Q = 2(\phi_{zz} - (\phi_z)^2)$.

$$K_z = -4e^{-2\phi} (\phi_{zz} - 2\phi \phi_z), \quad Q_z = 2(\phi_{zz} - 2\phi \phi_z).$$

Therefore $(\rho, II^*(\rho))$ satisfies the Gauss-Codazzi equations at infinity.

If $(I, II^*)$ satisfies the Gauss-Codazzi equations at infinity then $S = K$ and $S_{\bar{z}} = -2Re^{-2\phi}$. Therefore

$$-2Re^{-2\phi} = S = K = -2Qe^{-2\phi}.$$ 

Therefore $R_{\bar{z}} = Q_{\bar{z}}$ giving $R - Q = f(z)$ for some holomorphic function $f$. We now pullback to the disk $D$ uniformizing surface $S$ and let $\rho_h$ be the hyperbolic
structure on $D$. We let $\hat{\mathcal{R}}, \hat{\mathcal{Q}}$ be the pullbacks of the quadratic differentials $\hat{R}dz^2, \hat{Q}d\bar{z}^2$ and $\hat{\rho}$ the pullback of the conformal metric $\rho$. Then

$$\hat{\mathcal{R}} - \hat{\mathcal{Q}} = \psi$$

where $\psi$ is a holomorphic quadratic differential on $D$. We also have that

$$\hat{\mathcal{Q}} = Q(\rho, \hat{\rho}) - \phi_\Sigma$$

where $\phi_\Sigma$ is the Schwarzian of the locally univalent map given by projective structure $\Sigma$. Then

$$\hat{\mathcal{R}} = Q(\rho, \hat{\rho}) - \psi + \phi_\Sigma = Q(\rho, \hat{\rho}) - \phi_\Sigma,$$

as the space of projective structures on $S$ is homeomorphic to quadratic differentials. Thus

$$\hat{\mathcal{R}} = Q_\Sigma(\rho).$$

\[\square\]

**Corollary 5.5** Let $\rho$ be a conformal metric on a simply connected domain $\Omega \subseteq \mathbb{C}$. Then the dual fundamental form $(I, II)$ to $(\rho, II^\rho_{\Omega}(\rho))$ is realizable as an immersed surface in $\mathbb{H}^3$. Furthermore, the surfaces dual to the fundamental pair $(e^2\rho, II^\rho_{\Omega}(\rho))$ can be immersed such that they are obtained by normal flow and foliate a neighborhood of $\Omega$ in $\mathbb{H}^3$ with limiting metric $\rho$. In particular, the immersed surface dual to $(\rho, II^\rho_{\Omega}(\rho))$ is the Epstein surface of $\rho$.

**Proof:** We have $II^\rho_{\Omega}(e^2\rho) = II^\rho_{\Omega}(\rho)$, therefore by the prior lemma, $(e^2\rho, II^\rho_{\Omega}(\rho))$ satisfies the Euler-Codazzi equations at infinity. We let $B_7^r$ be the shape operator of $(e^2\rho, II^\rho_{\Omega}(\rho))$. Then $B_7^r = e^{-2t}B^r$ where $B^r$ is the shape operator at $t = 0$. Therefore for $t$ sufficiently large $(Id + B^r_t)$ is invertible and $(e^2\rho, II^\rho_{\Omega}(\rho))$ is a good fundamental pair. Therefore the dual fundamental pair $(I, II)$ is defined. By the prior lemma, $(I, II)$ satisfies the Euler-Codazzi equations and therefore there is a map $f_\Omega : \Omega \rightarrow \mathbb{H}^3$, unique up to composition by a Mobius transformation, such that the intrinsic metric of the image has fundamental forms $(f_\Omega) \ast (I, II)$. Then we have by definition of $I_\Omega$ that

$$I_\Omega = \frac{1}{4}(e^{2\rho}((Id + B^r_\Omega),(Id + B^r_\Omega)) = \frac{1}{4}\rho(e',Id + e^{-t}B^*, e'.Id + e^{-t}B^*).$$

We assume (by rescaling) that $(\rho, II^\rho(\rho))$ is a good fundamental pair and let $(I, II)$ be the fundamental forms for the dual with shape operator $B$. Then

$$I = \frac{1}{4}\rho((Id + B^r),(Id + B^r)).$$

Therefore the metric on the surface given by normal flow on the image of $f_0$ is

$$I(cosh(t).Id + \sinh(t)B, cosh(t).Id + \sinh(t)B) = \frac{1}{4}\rho(J(t), J(t))$$
where $J(t) = (\cosh(t).Id + \sinh(t)B)(Id + B^*)$.

We note that $B^* = (Id + B)^{-1}(Id - B)$ giving $(Id + B)(Id + B^*) = 2.Id$. Therefore

$$J(t) = (\cosh(t).Id + \sinh(t)B).(Id + B^*) = \frac{1}{2}(e^t(\cosh(t) + e^t(\cosh(t) + e^{-t}(Id - B))(Id + B^*)$$

$$= \frac{1}{2}(e^tId + e^{-t}B^*)(Id + B)(Id + B^*) = (e^tId + e^{-t}B^*).$$

Thus

$$I_t = I(\cosh(t).Id + \sinh(t)B, \cosh(t).Id + \sinh(t)B).$$

Therefore the surface given by normal flow on the image of $f_0$ has fundamental pair given by $(I_t, I_t)$ and therefore by uniqueness, $f_0$ can be taken to be normal flow on the image of $f_0$. We obtain boundary conformal metric $\sigma = e^{2\psi}|dz|^2$ on $\Omega'$ with second fundamental form $\mathcal{II}^*$. By Lemma 4.2 we have that the surfaces are the Epstein surfaces of $\sigma$ and therefore by Lemma 4.1 $\mathcal{II}^* = \mathcal{II}^*(\sigma)$.

We let $F : \Omega \to \Omega'$ be the composition of $f_0$ with normal flow to infinity. Then $F_*\rho = \sigma$ and thus $F$ is conformal. Also $F_*(\mathcal{II}^*(\rho)) = \mathcal{II}^* = \mathcal{II}^*(\sigma)$ and in particular $F_*(Q(\rho)) = Q(\sigma)$. We have

$$Q(\sigma) = F_*(Q(\rho)) = F_*(Q(|dz|^2, \rho)) = Q(F_*(|dz|^2, F_*\rho) = Q(F_*(|dz|^2, \sigma) = Q(\sigma) - S(F)$$

where $S(F)$ is the Schwarzian derivative. Therefore $S(F) = 0$ giving that $F$ is Möbius. Composing by $F^{-1}$ we obtain the Epstein surface $Ep_\rho$. □

### 5.2 Determinant, principal curvatures, convexity

Given $\rho$ a conformal metric on $\Omega \subseteq \hat{\mathbb{C}}$, it follows that $B^*$ satisfies

$$det(B^*) = 4||Q_{\Sigma_\rho}(\rho)||^2 - K(\rho)^2$$

where

$$||Q_{\Sigma_\rho}(\rho)|| = \frac{|Q_{\Sigma_\rho}(\rho)|}{\rho}.$$ 

We have that $B^*$ has eigenvalues

$$\lambda^* = -K(\rho) \pm 2||Q_{\Sigma_\rho}(\rho)||.$$ 

Thus the dual $(I, \mathcal{II})$ is well-defined provided

$$||Q_{\Sigma_\rho}(\rho)|| \neq \frac{|K(\rho) - 1|}{2}.$$ 

As $B = (Id + B^*)^{-1}(Id - B^*)$, the principal curvatures of the associated Epstein surface are

$$\lambda = \frac{1 - (-K(\rho) \pm 2||Q_{\Sigma_\rho}(\rho)||)}{1 + (-K(\rho) \pm 2||Q_{\Sigma_\rho}(\rho)||)} = \frac{||Q_{\Sigma_\rho}(\rho)|| \pm (1 + K(\rho))/2}{||Q_{\Sigma_\rho}(\rho)|| \pm (1 - K(\rho))/2}.$$
Note for the hyperbolic metric (i.e. \( K(\rho) = -1 \)) we obtain the curvature formulae of Epstein.

\[
\lambda = -\frac{||S(f)||}{||S(f)|| + 1}.
\]

We note that \( e^{2t} \rho \) has shape operator \( e^{-2t} B^* \). Thus \( S_t \) has principal curvatures

\[
\lambda = \frac{1 - e^{-2t}( -K(\rho) \pm 2||Q_{\Sigma_M}(\rho)||)}{1 + e^{-2t}( -K(\rho) \pm 2||Q_{\Sigma_M}(\rho)||)}.
\]

In particular \( S_t \) is locally convex if

\[
e^{2t} > 2||Q_{\Sigma_M}(\rho)|| \pm K(\rho).
\]

It follows that if \( \rho \) is non-positively curved and \( \rho_\Sigma \) is the projective metric on \( \Sigma \)
then

\[
\rho_\Sigma \leq \rho ||2||Q_{\Sigma_M}(\rho)|| - K(\rho)||_\infty \leq \rho (2||Q_{\Sigma_M}(\rho)||_\infty + ||K(\rho)||_\infty)
\]

6 Variational Formula for W volume in terms of Osood-Stowe Differential

Given \( N \) a convex submanifold of \( M \), the W-volume of \( N \) is defined by

\[
W(N) = V(N) - \frac{1}{4} \int_{\partial N} H dA
\]

where \( H \) is the mean curvature given by \( H = Tr(B)/2 \). The W volume has a number of nice analytic properties including the scaling property that if \( N_t \) is obtained by normal flow from \( N \) then a simple calculations (see [KS]) shows that

\[
W(N_t) = W(N) + t\pi|\chi(\partial M)|.
\]

Similarly, given a conformal metric \( \rho \) on \( \partial_c M \), then for \( t \) large the metric \( e^{2t} \rho \) has a convex Epstein surface \( S_t \) bounding a convex submanifold \( N_t \). The W volume of \( \rho \) is then defined by

\[
W(\rho) = W(N_t) - t\pi|\chi(\partial M)|.
\]

We let \( \rho_M \) be the hyperbolic metric on \( \partial_c M \) and then define the renormalized volume by

\[
V_{\rho}(M) = W(\rho_M).
\]

Let \( M = \mathbb{H}^3/\Gamma \) with \( \partial_c M = \Omega_M/\Gamma \) where \( \Gamma \) are Mobius transformations. Therefore the conformal boundary \( \partial_c M \) is endowed with a natural projective structure \( \Sigma_M \). This is uniquely defined by the quadratic differential \( \phi_M \) on \( \partial_c M \) given by the Schwarzian of the map uniformizing the components of \( \Omega_M \).

In their paper [KS], Krasnov-Schlenker proved the following:
**Theorem 6.1** (Krasnov-Schenker, [KS]) Let $CC(N)$ be the space of convex co-compact hyperbolic structures on $N$. Then for $M \in CC(N)$

$$dV_R(M) = Re(\phi_M)$$

We now generalize this to give a formula in terms of the Osgood-Stowe differential. We first denote by $\Sigma_M$ the natural projective structure on $\partial_c M$ and let $Q_M : Conf(\partial_c M) \rightarrow Q(\partial_c M)$ be the associated Osgood-Stowe differential map given by the projective structure $\Sigma_M$.

We define the space

$$CC_c(N) = \{(M, \rho) \mid M \in CC(N), \rho \in Conf(\partial_c M)\}.$$ 

We note that if $\alpha : (-1, 1) \rightarrow CC_c(N)$ is a smooth curve given by $\alpha(t) = (M_t, \rho_t)$ then $\rho^t = f_t \cdot \rho^0_h$ where $\rho^0_h$ is the uniformizing metric on $\partial_c M_t$ and $f_t : \partial_c M_t \rightarrow \mathbb{R}_+$ is a smooth function. Therefore the tangent vector $v = \hat{\rho}_0$ can be identified with

$$v = \rho_0 = \hat{f}_0 \rho^0_h + f_0 \rho^0_\hat{\rho} = (f(v), \mu(v)) \in C^\infty(\partial_c M) \oplus T_M(CN)$$

giving the identification $T_{\Sigma M, \rho}CC_c(N) = T_M(CN) \oplus C^\infty(\partial_c M)$.

**Theorem 6.2** The variational formula for $W$ on $CC_c(N)$ is given by

$$dW_{(M, \rho)}(v) = \frac{1}{4} \int_{\partial_c M} dK(v) d\rho - Re \int_{\partial_c M} Q_M(\rho) \mu(v).$$

**Proof:** By Krasnov-Schenker (see [KS])

$$\delta W = -\frac{1}{8} \int (\delta Tr(B^*) + <\delta I^*, II^*_0>) d\rho$$

We note that $H^* = -K(\rho)$ and by prior calculations $II^*_0 = 2B_M(\rho) = 2B(\phi)$ where $\rho = e^{2\phi} |dz|^2$ in projective coordinates. We have that

$$\delta W = \frac{1}{4} \int K(v) d\rho - \frac{1}{4} \int <I^*, B(\phi)> d\rho$$

Let $\mu(z)$ be a Beltrami differential giving a change in conformal structure and let $f^*$ be the solution to the Beltrami equation

$$\mu(f^*) = \frac{(f^*)_z}{(f^*)_\bar{z}} = t \mu.$$

Then the new metric $I^*_0$ is conformal with respect to the conformal structure induced by $f^*$. Therefore the pullback $(f^*)^* I^*_0$ is conformal with respect to $I^*$ giving

$$(f^*)^* I^*_0 = u I^*.$$ 

In terms of Jacobians we have

$$I^*_0 = (f^*)_* u I^* = u (Jf^*)^2 I^*(Jf^*).$$
As \( I^* = e^{2\phi}Id \) then

\[
\dot{I}^* = \dot{u}_0 I^* + (J\dot{J}^0)^T I^* + I^*(J\dot{J}^0) = (\dot{u}_0 I^* + (J\dot{J}^0)^T + J\dot{J}^0)e^{2\phi}
\]

We have \(<X,Y> = Tr(X(I^*)^{-1}Y(I^*)^{-1}) = Tr(XY)e^{-4\phi} \). Therefore

\[
< I^*, B(\phi) > = \dot{u}_0 e^{-2\phi} Tr(B(\phi)) + e^{-2\phi} Tr((J\dot{J}^0)^T + J\dot{J}^0)B(\phi)) = 2e^{-2\phi} Tr(J\dot{J}^0)B(\phi)
\]
as \( B(\phi) \) is traceless and the Jacobian is symmetric. Therefore

\[
\int < I^*, B(\phi) > dA^* = 2 \int Tr((J\dot{J}^0)B(\phi))d\mathcal{A} =
\]

Letting \( f^t(z) = u^t(z) + iv^t(z) \)

\[
Tr(Jf^t)B(\phi) = Tr \begin{pmatrix} (u^t)'_x & (v^t)'_y \ & -Im(Q(\phi)) & -Re(Q(\phi)) \ & -Im(Q(\phi)) & -Re(Q(\phi)) \ 
\end{pmatrix} =
\]

\[
= ((u^t)'_x - (v^t)'_y) Re(Q(\phi)) - ((u^t)'_y + (v^t)'_x) Im(Q(\phi)) = 2Re(Q(\phi) \partial_z f^t)
\]

Therefore differentiating

\[
Tr(J\dot{J}^0)B(\phi) = \frac{d}{dt} \bigg|_{t=0} Tr(Jf^t)B(\phi) = 2Re(Q(\phi) \frac{d}{dt} \bigg|_{t=0} (\partial_z f^t))
\]

As \( f^t(z) = z + tf^1(z) + o(t^2) \) we have

\[
\mu(f^t) = \frac{t \partial_z f^1 + o(t^2)}{1 + t \partial_z f^1 + o(t^2)}
\]

and therefore

\[
\mu(v) = \lim_{t \to 0} \frac{\mu(f^t)}{t} = \partial_z f^1 = \frac{d}{dt} \bigg|_{t=0} (\partial_z f^t)
\]

Therefore combining we get

\[
\frac{1}{4} \int < I^*, B(\phi) > d\rho = Re \int Q(\phi) \mu(v)
\]
giving

\[
W = \frac{1}{4} \int \dot{K}(v)d\rho - Re \int Q_M(\rho) \mu(v).
\]

\( \square \)

It follows trivially from above that for conformal changes in metric we have

\[
W = \frac{1}{4} \int \dot{K}(v)d\rho.
\]

**Renormalized Volume Variational Formula**

We now check that we obtain the same variation as Krasnov-Schlenker in the case of renormalized volume. By Lemma 3.2 and the fact that \( Q(\rho, \rho) = 0 \) for all \( \rho \)

\[
dV_R = -ReQ_M(\rho_M) = -Re(Q(\rho_M, \rho_M) - \phi_M) = Re(\phi_M).
\]

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7 Properties of W volume

7.1 Monotonicity

We use the above to give an elementary proof of Krasnov-Schenker’s Monotonicity for W volume.

Theorem 7.1 (Krasnov-Schenker, [KS]) Let $\rho, \sigma$ be non-positively curved smooth conformal metrics on $\partial M$ with $\rho(z) \leq \sigma(z)$ pointwise. Then $W(\rho) \leq W(\sigma)$.

Proof: We have $\sigma = e^{2u}\rho$ where $u : X \to \mathbb{R}$ is a non-negative function. We interpolate by $\rho_t = e^{2tu}\rho$ as before. Then we have

\[
\int K_t dA_t = 2\pi \chi(\partial M)
\]

Therefore

\[
\frac{d}{dt} W(\rho_t) = \frac{1}{4} \int K_t dA_t = -\frac{1}{4} \int K_t dA_t
\]

We let $dA$ be the area element for $\rho$ and $K$ its curvature. As $dA_t = e^{2tu}dA$ then $dA_t = 2udA_t$ and

\[
\frac{d}{dt} W(\rho_t) = -\frac{1}{2} \int K_t udA_t
\]

We have the curvature $K_t = e^{-2u}(K - t\Delta u)$ and as $K_0, K_1 \leq 0$ then by linearity $K_t \leq 0$. Therefore as $K_t \leq 0$ and $u \geq 0$ then

\[
\frac{d}{dt} W(\rho_t) \geq 0.
\]

\[\square\]

7.2 W volume in terms of scaling function

We now generalize the scaling and monotonicity property of W volume in a single theorem which gives a new formulation for W volume. Given two conformally equivalent metrics $\rho, \sigma$ then there is a non-zero function $u$ such that $\sigma = e^{2u}\rho$. Equivalently, we can define $u = \log(\sigma/\rho)/2$. Further for a conformal metric $\rho$ we define $K_\rho$ to be its curvature and $\Omega_\rho = K_\rho dA_\rho$ its curvature 2-form.

Lemma 7.2 Let $\sigma = e^{2u}\rho$, then

\[
W(\sigma) - W(\rho) = -\frac{1}{4} \left( \int uK_\rho dA_\rho + \int uK_\sigma dA_\sigma \right) = -\frac{1}{8} (\Omega_\rho + \Omega_\sigma)(\log(\sigma/\rho)).
\]

Proof We let consider the path from $\rho$ to $\sigma$ given by $\rho_t = e^{2tu}\rho$, then we have

\[
\int K_t dA_t = 2\pi \chi(\partial M)
\]

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Therefore
\[
\frac{d}{dt}(W(\rho_t)) = \frac{1}{4} \int K_t dA_t = -\frac{1}{4} \int K_t dA_t
\]
We let \(dA\) be the area element for \(\rho\) and \(K\) its curvature. As \(dA_t = e^{2u}dA\) then \(dA_t = 2udA_t\) and
\[
\frac{d}{dt}(W(\rho_t)) = -\frac{1}{2} \int K_t dA_t
\]
We have \(K_t = e^{-2u}(K - t\Delta u)\), where \(\Delta\) is the Laplacian of \(\rho\). Therefore
\[
\frac{d}{dt}(W(\rho_t)) = -\frac{1}{2} \int u(K + t\Delta u)dA = -\frac{1}{2} \int uKdA + \frac{1}{2} t \int u\Delta u dA
\]
Therefore integrating we have
\[
W(\sigma) = W(\rho) - \frac{1}{4} \int uKdA - \frac{1}{4} \int u\Delta(u)dA.
\]
If \(K_{\sigma}\) is the curvature of \(\sigma\) then \(K_{\sigma} = e^{-2u}(K_{\rho} - \Delta u)\) giving
\[
W(\sigma) = W(\rho) - \frac{1}{4} \int uK_{\rho}dA_{\rho} - \frac{1}{4} \int uK_{\sigma}dA_{\sigma}
\]
Therefore
\[
W(\sigma) - W(\rho) = -\frac{1}{4} (\Omega_{\rho} + \Omega_{\sigma})(u).
\]
\[
\square
\]
We note that both monotonicity and scaling follow from the prior lemma.

### 7.3 Integral of Mean Curvature

**Lemma 7.3** Let \(\Sigma \in P(X)\) and \(\rho \in Conf(X)\). Let \(Y_\rho\) be the Epstein surface of with mean curvature \(H_\rho\) and area \(dA_{Y_\rho}\). Then the integral of mean curvature of the Epstein surface satisfies
\[
\int H_\rho dA_{Y_\rho} = ||Q_\Sigma(\rho)||_2^2 - \frac{1}{4} ||K(\rho)||_2^2 + \frac{1}{4} Area(\rho)
\]

**Proof:** We let \(dA_\rho\) be the area form of \(\rho = e^{2\phi}|dz|^2\) and \(K(\rho)\) its curvature. From the above formula
\[
\int H_\rho dA_{Y_\rho} = \frac{1}{4} \int (1 - det(B^*))dA_\phi = \frac{1}{4} \left(A(\rho) - \int det(B^*)dA_\rho\right)
\]
We let \(\rho = e^{2\phi}|dz|^2\) where. We have that
\[
det(B^*) = det((I^*)^{-1} II^*) = e^{-4\phi} det(II^*) = e^{-4\phi} det(2B_\Sigma(\rho) - K(\rho)e^{2\phi}Id) = K(\phi)^2 - 4e^{-4\phi} |Q(\rho)|^2.
\]
Thus
\[
\int H_\rho dA_{Y_\rho} = \frac{1}{4} \left(A(\rho) - \int K(\rho)^2 dA_\rho + 4 \int \frac{|Q_\Sigma(\rho)|^2}{e^{4\phi}} dA_\rho\right)
\]
\[
20
\]
We have
\[ \int \frac{|Q_z(\rho)|^2}{e^{2\phi}} \, dA \rho = \int \frac{|Q_\Sigma(\rho)|^2}{e^{2\phi}} \, dz \, d\tau = \|Q_\Sigma(\rho)\|^2 \]
\[ \square \]

For example if \( \rho \) is the projective metric we get
\[ \|Q_\Sigma(\rho)\|^2 + L(\beta)/4 = \int H dA = L(\beta)/2 \]
giving
\[ \|Q_\Sigma(\rho)\|^2 = L(\beta)/4. \]

8 Osgood-Stowe Shape Operator

Given \( M \) a hyperbolic manifold with conformal boundary \( \partial_c M \) we consider the map from conformal metrics on \( \partial_c M \) to the associated shape operator at infinity. We let \( \Omega^{k,l}(X) \) be the forms of type \((k,l)\) on \( M \). We define \( F : Conf(\partial_c M) \to \Omega^{1,1}(\partial_c M) \) by
\[ F(\rho) = \rho^{-1} II^*(\rho) = B^*(\rho) = 2B_\Sigma(\rho)e^{-2\phi} - K(\rho).Id \]

Its is obvious that \( F \) is linear under scaling. The map is not onto as \( F(\rho) \) is diagonalizable at every point. Also \( F(\rho) = -K(\rho).Id \) if any only if \( \rho \) is constant curvature (i.e. \( F(\rho) = c.Id \)) and \( M \) has totally geodesic convex core boundary.

**Lemma 8.1** The map \( F \) is smooth and injective.

**Proof:** Smoothness is obvious. Let \( \rho_i = e^{2\phi_i}|dz|^2 \) and \( F(\rho_1) = F(\rho_2) \). Then we have by taking the splitting into the traceless part and remainder, we have
\[ Q(\phi_1)e^{-2\phi_1} = Q(\phi_2)e^{-2\phi_2} \quad K(\rho_1) = K(\rho_2). \]

By the first equation, \( Q(\phi_1) \) have the same zeros given by a discrete set \( Z \). Differentiating the first equation we have
\[ Q_z(\phi_1)e^{-2\phi_1} - 2(\phi_1)_zQ(\phi_1)e^{-2\phi_1} = Q_z(\phi_2)e^{-2\phi_2} - 2(\phi_2)_zQ(\phi_2)e^{-2\phi_2}. \]

We have by the Codazzi equation at infinity that \( Q_z(\phi_1)e^{-2\phi_1} = -K_z(\phi_1)/2 \) giving
\[ -2(\phi_1)_zQ(\phi_1)e^{-2\phi_1} = -2(\phi_2)_zQ(\phi_2)e^{-2\phi_2}. \]

Therefore in the complement the zeros \( Z \) we have
\[ (\phi_1)_z = (\phi_2)_z. \]

Therefore as \( \phi_i \) are real, \( \phi_1 = \phi_2 + c \) for some constant. As the curvatures are equal, we have \( c = 0. \) \( \square \)
8.1 Almost Fuchsian Quasifuchsian manifolds

We note that for $M$ quasifuchsian, $M$ is almost-fuchsian if and only if there is an immersed minimal surface with principal curvatures in the interval $(-1, 1)$. Taking the dual forms at infinity, this is equivalent to having non-positively curved metric $\rho$ such that $\text{det}(B^*) = 1$. Equivalently that $F(\rho)$ is in $\text{SL}(2, \mathbb{R})$. This allows us to write an equation for being almost fuchsian as follows:

We let $\rho = e^{2\phi} |dz|^2$ in projective coordinates. Then the equation for minimality is

$$\text{det}(B^*) = 4||Q_M(\rho)||^2 - K(\rho)^2 = 1.$$ 

Writing $Q_M(\rho) = 2(\phi_{zz} - (\phi_z)^2)dz^2$ and $K(\rho) = -4e^{-2\phi} \phi_{zz}$ we have

$$(\phi_{zz} - (\phi_z)^2)(\phi_{zz} - (\phi_z)^2) - \phi_{zz}^2 = e^{4\phi}.$$ 

Thus $M$ is almost fuchsian if there is a negatively curved metric $\rho$ solving the above equation. We note by rescaling the metric such that $\phi + c$, we can consider the equation

$$(\phi_{zz} - (\phi_z)^2)(\phi_{zz} - (\phi_z)^2) - \phi_{zz}^2 = e^{4\phi}.$$ 

**Conjecture:** $M$ is almost fuchsian if and only if it has an immersed surface with principal curvatures in $(-1, 1)$.

We note that $M$ having an immersed surface with principal curvatures in $(-1, 1)$ is equivalent to having a negatively curved metric $\rho = e^{2\phi} |dz|^2$ such that

$$(\phi_{zz} - (\phi_z)^2)(\phi_{zz} - (\phi_z)^2) - \phi_{zz}^2 = f(z)$$

We note that $f(z)$ is a $(2, 2)$ form on $\partial M$ and therefore it is of the form $g(z)e^{4\phi}$ for some function on $\partial M$.

References


