Notes: Goldman’s symplectic structure for surface representations

1 Invariant functions

All the below material is a discussion of Goldman’s paper on the Poisson structure associated to the symplectic structure he introduced.

We let $G$ be a reductive matrix Lie group with Lie algebra $\mathfrak{g}$. We let $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which is invariant under the Adjoint action of $G$. In these notes we will choose $\mathcal{B}(X,Y) = \text{tr}(XY)$.

An invariant function on $G$ is a smooth function $f : G \to \mathbb{R}$ which is invariant under conjugation. In particular we have $f(A) = \text{tr}(A)$ is an invariant function. Given an invariant function $f$ there is a natural map $F : G \to \mathfrak{g}$ defined as follows; The bilinear form $\mathcal{B}$ gives a pairing of $\mathfrak{g}$, $\mathfrak{g}^*$ denoted $\mathcal{B}^* : \mathfrak{g} \to \mathfrak{g}^*$ given by $\mathcal{B}^*(X)(Y) = \mathcal{B}(X,Y)$. As $f$ is smooth, we have for $A \in G$, $df(A) \in T_A^*(G)$ and $L_A^*(df(A)) \in T_e^*(G) \cong \mathfrak{g}^*$. Then $F(A)$ is given by $\mathcal{B}(F(A)) = L_A^*(df(A))$

Therefore we have the equation

$$B(F(A), X) = L_A^*(df(A))(X) = df(A)(L_A(X)) = \frac{d}{dt}(f(A \exp(tX)))$$

By invariance of $\mathcal{B}$ and $f$ we have for $P \in G$

$$F(PAP^{-1}) = Ad(P)F(A).$$

We note that if $G$ is a reductive subgroup of $GL(n, \mathbb{R})$ then $\mathcal{B}$ is the restriction of the form $\mathcal{B} : \mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}$ given by $\mathcal{B}(X,Y) = \text{tr}(XY)$. Furthermore we have a natural orthogonal projection map $p : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{g}$ with respect to $\mathcal{B}$.

**Lemma 1** Let $G$ be a reductive subgroup of $GL(n, \mathbb{R})$ and $f$ the invariant function $f = \text{tr}$. Then $F(A) = p(A)$ where $G \subseteq \mathfrak{gl}(n, \mathbb{R})$ is the natural inclusion of $G$ in the space of $n \times n$ matrices $M(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R})$.

**Proof:** The symmetric bilinear form $\mathcal{B}(X,Y) = \text{tr}(XY)$ is the restriction of $\mathcal{B}$ to $\mathfrak{g}$. By the above formula for all $X \in \mathfrak{g}$

$$\mathcal{B}(F(A), X) = \frac{d}{dt}(\text{tr}(A \exp(tX))) = \text{tr}(AX) = \mathcal{B}(A, X).$$

By definition of orthogonal projection we have for all $X \in \mathfrak{g}$

$$\mathcal{B}(A, X) = \mathcal{B}(p(A), X) = \mathcal{B}(p(A), X).$$

Therefore for all $X \in \mathfrak{g}$

$$\mathcal{B}(F(A), X) = \mathcal{B}(p(A), X).$$

As $\mathcal{B}$ is non-degenerate on $\mathfrak{g}$, we have $F(A) = p(A)$. □

**Examples:**
If \( G = GL(n, \mathbb{R}) \) then as \( p = id \), we have \( F(A) = A \).

If \( G = SL(n, \mathbb{R}) \) then \( p(A) = A - \frac{1}{n} tr(A).I \). Therefore

\[
F(A) = A - \frac{1}{n} tr(A).I.
\]

If \( G = O(n, \mathbb{R}) \) then \( p(A) = A - 1_n tr(A) \) giving

\[
F(A) = \frac{1}{2} (A - A^t).
\]

2 Representation varieties

We let \( S \) be a closed surface of genus \( g \geq 2 \) and \( \pi = \pi_1(S,x) \) for some \( x \). We define \( \mathcal{R}(\pi,G) = Hom(\pi,G)/G \) the space of representations up to conjugacy.

Let \( \alpha \) be a free homotopy class of a closed curve. Then \( \alpha \) defines a conjugacy class in \( \pi \). If \( f \) is an invariant function then we define functions \( f_\alpha : \mathcal{R}(\pi,G) \to \mathbb{R} \) by

\[
f_\alpha([\rho]) = f(\rho(\alpha_x)).
\]

where \( \alpha_x \) is some element of the conjugacy class given by \( \alpha \). As \( T_{[\rho]}(\mathcal{R}(\pi,G)) = H^1(\pi, \mathfrak{g}_{Ad\rho}) \) we have \( df_\alpha \in T^*_{[\rho]}(\mathcal{R}(\pi,G)) \in H^1(\pi, \mathfrak{g}_{Ad\rho})^* \). We will now describe \( df_\alpha \).

3 Goldman’s symplectic form

We now define the Goldman symplectic form \( \omega \) on \( \mathcal{R}(\pi,\mathfrak{g}) \). Given \( X, Y \in T_{[\rho]}(\mathcal{R}(\pi,G) = H^1(\pi, \mathfrak{g}) \) then

\[
\omega_{[\rho]}(X, Y) = \mathcal{B}(X \cup Y) \cap [\pi].
\]

Equivalently we have the maps

\[
H^1(\pi, \mathfrak{g}) \times H^1(\pi, \mathfrak{g}) \xrightarrow{\mathcal{B} \cup} H^2(\pi, \mathbb{R}) \xrightarrow{\cap [\pi]} H_0(\pi, \mathbb{R}) = \mathbb{R}
\]

Goldman showed that it is non-degenerate, closed and thereby gives a symplectic structure on \( \mathcal{R}(\pi,G) \).

Given a function \( f : \mathcal{R}(\pi,G) \to \mathbb{R} \) we define its Hamiltonian flow \( Hf \) by the equation

\[
\omega(Hf, X) = df(X).
\]

Thus \( Hf([\rho]) \in H^1(\pi, \mathfrak{g}) \) dual to \( df([\rho]) \) under the pairing given by \( \omega_{[\rho]} \).

Let \( \hat{\mathcal{B}} : \mathfrak{g} \to \mathfrak{g}^* \) be the pairing coming from \( \mathcal{B} \). Then for \( \xi \in H^1(\pi, \mathfrak{g}) \), \( \hat{\mathcal{B}} \circ \xi \in H^1(\pi, \mathfrak{g}^*) \) and we define a map \( \hat{\mathcal{B}} : H^1(\pi, \mathfrak{g}^*)^* \to H^1(\pi, \mathfrak{g})^* \) by \( \hat{\mathcal{B}}(\nu)(\xi) = \nu(\hat{\mathcal{B}} \circ \xi). \)

We also consider the pairing \( H^1(\pi, \mathfrak{g}^*) \times H_1(\pi, \mathfrak{g}) \to \mathbb{R} \) given by the cap product and evaluation.

\[
H^1(\pi, \mathfrak{g}^*) \times H_1(\pi, \mathfrak{g}) \xrightarrow{\cap} H_0(\pi, \mathbb{R}) = \mathbb{R}
\]

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This pairing gives a map \( \eta : H_1(\pi, g) \to H^1(\pi, g^*)^* \) with \( \eta(Y)(X) = X \cap Y \).

We let \( \theta : H^1(\pi, g) \to H^1(\pi, g^*)^* \) be the isomorphism given by the pairing

\[
H^1(\pi, g) \times H^1(\pi, g^*)^* \ni (\omega, \eta) \mapsto \omega(\eta(\cdot)) = \eta(\omega(\cdot)),
\]

Thus for \( X \in H^1(\pi, g^*) \) then

\[
\theta(Y)(X) = (X \cup Y) \cap [\pi].
\]

**Lemma 2** The diagram below is commutative.

![Diagram](attachment:image.png)

**Proof:** The lower triangle commutes follows from the definition of \( \omega \). Let \( Y \in H^1(\pi, g) \). Therefore

\[
(\mathcal{B}(\theta(Y)))(X) = \theta(Y)(\mathcal{B} \circ X) = ((\mathcal{B} \circ X) \cup Y) \cap [\pi] = \mathcal{B}(Y \cup X) \cap [\pi] = \omega(Y, X).
\]

We now consider the upper triangle. Again let \( Y \in H^1(\pi, g) \) and \( X \in H^1(\pi, g^*) \). Then

\[
(\eta \circ (\cap[\pi]))(Y)(X) = \eta(Y \cap [\pi])(X) = X \cap (Y \cap [\pi]) = (X \cup Y) \cap [\pi] = \theta(Y)(X).
\]

\( \square \)

As \( S \) is a \( K(\pi, 1) \) space, we have the standard isomorphism between group cohomology \( H^k(\pi, g_{Ad \rho}) \) and cohomology with coefficients in the flat bundle \( H^k(S, g_{Ad \rho}) \). See appendix for explicit description for \( H^1 \).

**Lemma 3** \( d\alpha \in T^*_{[\rho]}(\mathcal{R}(\pi, G)) = H^1(\pi, g_{Ad \rho})^* = H^1(S, g_{Ad \rho})^* \) satisfies

\[
d\alpha = \mathcal{B} \circ \eta(\alpha \otimes F(\rho(\alpha_x))).
\]

**Proof:** Let \( [\xi] \in H^1(\pi, g) \) then

\[
\frac{d}{dt}f(\rho_t(\alpha_x)) = \frac{d}{dt}\alpha_x(\xi)
\]

for some choice of \( \alpha_x \) in conjugacy class given by \( \alpha \). We let \( A = \rho(\alpha_x) \) and \( X = \xi(\alpha_x) \). Then

\[
\frac{d}{dt}f(\exp(t\xi(\alpha_x))) = \frac{d}{dt}f(\exp(tX)) = f'(A)(XA)
\]

By definition of \( F \) we have \( \mathcal{B}(F(A), X) = f'(A)(AX) \). Therefore

\[
d\alpha([\xi]) = f'(A)(XA) = f'(A)(A(A^{-1}XA)) = \mathcal{B}(F(A), Ad(A^{-1})X) = \mathcal{B}(Ad(A)F(A), X) = \mathcal{B}(F(A), X)
\]
as \( Ad(A)F(A) = F(A) \). Thus
\[
df_{\alpha}([\xi]) = \mathcal{B}(F(\rho(\alpha_x)), \xi(\alpha_x)).
\]
We note that if \( \xi \in H^1(\pi, \mathfrak{g}) \) with \( \xi : \pi \to \mathfrak{g} \) then \( \eta(\sigma \otimes V)(\xi) = \xi(\sigma)(V) \). Therefore
\[
\left( ^t\mathcal{B} \circ \eta(\alpha \otimes F(\rho(\alpha_x))) \right)(\xi) = \eta(\alpha \otimes F(\rho(\alpha_x)))(\mathcal{B} \circ \xi)
\]
\[
= \eta(\alpha \otimes F(\rho(\alpha_x)))(\mathcal{B} \circ \xi) = ((\mathcal{B} \circ \xi)(\alpha_x))(F(\rho(\alpha_x)))
\]
\[
= \mathcal{B}(\xi(\alpha_x))(F(\rho(\alpha_x))) = \mathcal{B}(F(\rho(\alpha_x), \xi(\alpha_x)) = df_{\alpha}([\xi])
\]
\( \square \)

We use the above to prove the following:

**Theorem 1** The functions \( f_{\alpha} \) have Hamiltonian vector fields satisfying
\[
Hf_{\alpha} \cap [\pi] = \alpha \otimes F(\rho(\alpha_x)).
\]

**Proof:** As the diagram commutes we have
\[
df_{\alpha} = ^t\mathcal{B} \circ \eta(Hf_{\alpha} \cap [\pi])
\]
Also by lemma 3 we have
\[
df_{\alpha} = ^t\mathcal{B} \circ \eta(\alpha \otimes F(\rho(\alpha_x)))
\]
Therefore we get the desired result
\[
Hf_{\alpha} \cap [\pi] = \alpha \otimes F(\rho(\alpha_x)).
\]
\( \square \)

## 4 Goldman bracket

Let \( S \) is an oriented surface and \( \alpha, \beta \) two oriented closed curves. If \( p \in \alpha \cap \beta \) is a transverse intersection point, we let \( \epsilon(p, \alpha, \beta) = \pm 1 \) be the orientation of the intersection point \( p \) with respect to the surface orientation. Further for any \( p \in \alpha \) we let \( \alpha_p \) be the oriented curve obtained by starting at \( p \).

We let \( \pi \) be the homotopy classes of closed curves on \( S \). We let \( \mathbb{Z}\pi \) be the \( \mathbb{Z} \)-module of formal sums of elements on \( \pi \). The Goldman bracket is the pairing \( [., .] : \mathbb{Z}\pi \times \mathbb{Z}\pi \to \mathbb{Z}\pi \) given by
\[
[[\alpha], \beta]] = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta)\alpha_p\beta_p
\]
Goldman proved the following:
Theorem 2 The bracket is well-defined and a Lie bracket on $\mathbb{Z}\pi$. The Poisson bracket with respect to $\omega$ satisfies
\[
\{f_\alpha, f_\beta\}([\rho]) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta)\mathcal{B}(F(\rho(\alpha_p)), F(\rho(\beta_p)))
\]
where $\rho : \pi_1(S, p) \to G$ is a choice of representation for $[\rho]$. Furthermore for $G = GL(n, K)$ $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the map $\alpha \to f_\alpha$ is a Lie algebra homomorphism between $\mathbb{Z}\pi$ and the space of smooth functions on $\mathcal{R}(\pi, G)$ with the Poisson bracket induced by $\omega$.

**Proof:** The proof that the bracket is well defined is an analysis of how it behaves under changes under homotopy and is straightforward. The skew-symmetry follows directly and the Jacobi identity by a simple calculation.

By definition of $\omega$ we have
\[
\{f_\alpha, f_\beta\}([\rho]) = \omega([\beta], (Hf_\alpha, Hf_\beta)) = \mathcal{B}(Hf_\alpha \cup Hf_\beta) \cap [\pi].
\]

From the prior section $Hf_\alpha \cap [\pi] = \alpha \otimes F(\rho(\alpha_x)) \in H_1(\pi, \mathfrak{g})$.

We have an intersection pairing $H_1(\pi, \mathfrak{g}) \times H_1(\pi, \mathfrak{g}) \to H_0(\pi, \mathbb{R}) = \mathbb{R}$ given by Poincare duality. The map $P = \cap[\pi] : H^1(\pi, \mathfrak{g}) \to H_1(\pi, \mathfrak{g})$ is the Poincare dual isomorphism. Then the geometric intersection is given by
\[
P(A).P(B) = \mathcal{B}(A \cup B) \cap [\pi]
\]
Thus
\[
\{f_\alpha, f_\beta\} = \mathcal{B}(Hf_\alpha \cup Hf_\beta) \cap [\pi] = (Hf_\alpha \cap [\pi]).(Hf_\beta \cap [\pi])
\]

From the prior section $Hf_\alpha = \alpha \otimes F(\rho(\alpha_x))$. Therefore
\[
\{f_\alpha, f_\beta\} = (\alpha \otimes F(\rho(\alpha_x))).(\beta \otimes F(\rho(\beta_x))).
\]

The generalized geometric intersection number for cycles with coefficients in a vector bundle is
\[
(\alpha \otimes V). (\beta \otimes W) = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta)\mathcal{B}(V(p), W(p))
\]
where $V, W$ are flat sections over $\alpha, \beta$ respectively. To calculate $V_p, W_p$ in our case, let $p \in \alpha \cap \beta$. Let $V_x = F(\rho(\alpha_x)), W_x = F(\rho(\beta_x))$. Then $\alpha_x = v\alpha_p v^{-1}$ and $\beta_x = w\beta_p w^{-1}$, where $v, w$ are paths joining $x, p$. Lifting the flat sections over $v, w$ to the cover $\tilde{M} \times \mathfrak{g}$ we obtain maps $s_v, s_w : [0, 1] \to \tilde{M} \times \mathfrak{g}$ given by
\[
s_v(t) = (\tilde{v}(t), V_x) \quad s_w(t) = (\tilde{w}(t), W_x).
\]

Therefore $\tilde{V}_p = s_v(1) = (p_1, V_x)$, and $\tilde{W}_p = s_w(1) = (p_2, W_x)$ are the lifts of $V_p$ and $W_p$. We let $\gamma = [v w^{-1}] \in \pi_1(X, x)$ then $\gamma(p_1) = p_2$. Then
\[
\gamma(\tilde{V}_p) = \gamma(p_1, V_x) = (\gamma p_1, Ad(\rho(\gamma)V_x)) = (p_2, Ad(\rho(\gamma)V_x)
\]

is a lift of $V_p$ in the same fiber as $\tilde{W}_p = (p_2, W_x)$. Thus
\[
\mathcal{B}(V_p, W_p) = \mathcal{B}(Ad(\rho(\gamma)V_x), W_x) = \mathcal{B}(Ad(\rho(\gamma)F(\rho(\alpha_x)), F(\rho(\beta_x))) = \mathcal{B}(F(\rho(\gamma(\alpha_x)\gamma^{-1})), F(\rho(\beta_x)))
\]

We fix the representation $\hat{\rho} : \pi_1(S, p) \to G$ by $\hat{\rho}([\gamma]) = \rho([v w^{-1}])$. Then
\[
\mathcal{B}(V_p, W_p) = \mathcal{B}(F(\rho(w^{-1}\alpha_x v^{-1}w), F(\rho(\beta_x)))) = \mathcal{B}(\hat{\rho}(v^{-1}\alpha_x v), F(\hat{\rho}(w^{-1}\beta_x w))) = \mathcal{B}(F(\alpha_p), F(\beta_p)).
\]
This finishes the proof of the formula.

To show that the map $\alpha \rightarrow f_\alpha$ is a Lie algebra homomorphism, we note that for $G = GL(n, \mathbb{R})$, $f = \text{tr}$, then $F(A) = A$. Therefore

$$B(F(\rho(\alpha_p)), \rho(\beta_p)) = \text{tr}(\rho(\alpha_p\beta_p)).$$

Then by linearity

$$f_{\{\alpha,\beta\}} = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) f_{\alpha_p\beta_p} = \sum_{p \in \alpha \cap \beta} \epsilon(p, \alpha, \beta) \text{tr}(\rho(\alpha_p\beta_p)) = \{f_\alpha, f_\beta\}.$$
5 Appendix on Cohomology

5.1 Group Cohomology

Let $\pi$ be a group and $\rho: \pi \to \text{End}(V)$ be a representation into vector space $V$. We let $C_k(\pi, V)$ be the set freely generated by elements $V \otimes \pi^k$. We define a boundary operator $d_k: C_k(\pi, V)$ by

$$d_k(v \otimes (g_1, \ldots, g_n)) = \rho(g_1)(v) \otimes (g_2, \ldots, g_n) + \sum_{i=1}^{k} (-1)^i v \otimes (g_1, \ldots, g_ig_{i+1}, \ldots, g_n) + (-1)^n v \otimes (g_2, \ldots, g_n).$$

It is easy to check that $d_k \circ d_{k+1} = 0$. We define $Z_k(\pi, V_\rho) = \ker(d_k)$, $B_k(\pi, V_\rho) = \text{im}(d_k)$ and $H_k(\pi, V_\rho)$ as the associated group homology of the representation $\rho$.

Similarly we let $C^k(\pi, V)$ be the collection of functions $\phi: \pi^k \to V$ and define the coboundary operator $d^k: C^k(\pi, V) \to C^{k+1}(\pi, V)$ is given by

$$d^k \phi(g_1, \ldots, g_{k+1}) = \rho(g_1) \phi(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k} (-1)^i \phi(g_1, \ldots, g_ig_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} \phi(g_1, \ldots, g_k).$$

Again $d^{k+1} \circ d^k = 0$ and we define $Z^k(\pi, V_\rho) = \ker(d^k)$, $B^k(\pi, V_\rho) = \text{im}(d^{k-1})$ and $H^k(\pi, V_\rho)$ as the associated cohomology groups.

We identify $T_{[\rho]}(\mathcal{R}(\pi, G)) = H^1(\pi, g_{Ad\rho})$ as follows; If $\rho_t$ is a smooth path of representations with $\rho_0 = \rho$ then we define the cocycle $\xi: \pi \to g$ by

$$\xi(g) = \frac{d}{dt}\rho_t(g)\rho(g)^{-1}.$$

As $\rho_t(gh) = \rho_t(g)\rho_t(h)$ differentiating we have that

$$\xi(gh)\rho(gh) = \xi(g)\rho(g)\rho(h) + \rho(g)\xi(h)\rho(h)$$

giving

$$\xi(gh) = \xi(g) + \rho(g)\xi(h)\rho(g)^{-1} = \xi(g) + \text{Ad}(\rho(g))\xi(h).$$

This is equivalent to $d\xi = 0$ with respect to the coboundary operator and therefore $\xi \in Z^1(\pi, g_{Ad\rho})$. Furthermore if $\rho_t = g_t\rho_t^{-1}$ then the associated cocycle $\xi = \partial \nu$ for some 0-cocycle. This gives the identification $T_{[\rho]}(\mathcal{R}(\pi, G)) = H^1(\pi, g_{Ad\rho})$. For simplicity we will drop reference to the representation and refer to $H^1(\pi, g)$ with the action implicit.

5.2 Cohomology with coefficients in a flat bundle

Let $M$ be a manifold and $p: E \to M$ be a flat vector bundle over $M$ (i.e. a vector bundle with transition maps preserving horizontals). In particular if $\rho: \pi_1(M, x) \to \text{End}(V)$ is a representation then we have $\pi_1(M, x)$ acting on $M \times V$ by $\gamma(x, v) = (\gamma(x), \rho(\gamma)(v))$ and the quotient by this action $E = M \times \rho V$ is the flat bundle over $M$ given by the representation.

If $U$ is an open subset of $M$ then a flat section over $U$ is a continuous map $s: U \to E$ such that it is a section of $p$ over $U$ and is locally flat, i.e. restricted to a chart it is a map to a horizontal slice on each connected component.
We now extend singular homology and cohomology for $M$. If $\sigma : \Delta_k \to M$ is a singular $k$-simplex, then if $s$ is a flat section with domain containing the image of $\sigma$ then we let $\sigma \otimes s$ be a singular $k$-simplex in $M$ we values in $E$. We then let $C_k(M,E)$ be free sums of these and define a boundary operator $d_k : C_k(M,E) \to C_{k-1}(M,E)$ by

$$d_k(\sigma \otimes s) = \sum_{i=0}^{k} (-1)^i(\partial_i \sigma) \otimes s_i$$

where $s_i$ is the restriction of $s$ to $\partial_i \Delta_k$. Similarly we can define singular co-chains on $M$ with values in $E$ as maps $\xi$ which assign to $k$-simplexes $\sigma$ a flat section $\xi(\sigma)$ over $\sigma$. We then define the co-boundary operator $d^k$ as the usual dual operator given by $d^{k-1}(\xi)(\sigma) = \xi(d_k(\sigma))$.

It can be checked that $d_{k-1} \circ d_k = 0$ and that these define homology and cohomology groups $H_k(M,E), H^k(M,E)$.

Returning to the flat bundle $E = M \times_\rho V$, we denote $H_k(M,E) = H_k(M,V_\rho)$ (similarly for cohomology).

Let $\alpha$ be a free homotopy class of closed curve in $M$. We let $\alpha_x : [0,1] \to M$ be a choice of element in $\alpha$ based at $x$.

The map $\alpha$ has a unique lift $\tilde{\alpha}_x : \tilde{M}$ such that $\tilde{\alpha}_x(0)$ is the unique base point of $\tilde{M}$ corresponding to the fixed path at $x$. Then a flat section $s$ over $\alpha_x$ lifts to a map $s : \mathbb{R} \to \tilde{M} \times V$. As $s$ is a flat section $s(t) = (\tilde{\alpha}_x(t), v)$ for some $v \in V$. As $s(0) = s(1)$ then $\rho([\alpha_x])(v) = v$ giving

$$d(\alpha_x \otimes s) = \alpha_x(0) \otimes s(0) - \alpha_x(1) \otimes s(1) = 0$$

Thus $\alpha_x \times s$ is closed.

A vector $v \in g$ is an invariant vector for $g \in \pi_1$ if $\rho(g)(v) = v$ and $v$.

We identify $g$ with the fiber over $x$ and let $v = s(\alpha_x(0)) \in g$. We let $\alpha_x \otimes v$ be the cycle with invariant vector $v$ over $x$. Therefore we have $[\alpha_x \otimes s] \in H_1(M,V_\rho)$.

It follows easily that the cycles over free homotopy class $[\alpha] \otimes v$ are in one to one correspondence with the invariant vectors of $[\alpha_x]$.

The choice $[\alpha_x] \in \pi_1(M,x)$ is only defined up to conjugacy but we have that $v$ is an invariant vector for $\gamma \in \pi_1(M,x)$ if and only if $\rho(g)v$ is an invariant vector for $g\gamma g^{-1}$. Thus the cycle $\gamma \otimes v$ and $(g\gamma g^{-1}) \otimes \rho(g)v$ define the same flat section over $\alpha$ and thus give equal cycles.

We identify $H_1(M,V_\rho)$ with linear sums of $[\alpha_x \otimes v]$ where $v \in V$ is an invariant vector for $[\alpha_x]$.

If $M$ is a $K(\pi,1)$ space, there are natural isomorphism between $H^k(\pi,g_{Ad\rho})$ and $H^k(M,g_{Ad\rho})$ and between $H_k(\pi,g_{Ad\rho})$ and $H_k(M,g_{Ad\rho})$. We describe it explicitly for $H_1$. Let $[\alpha_x \otimes v] \in H_1(M,g_{Ad\rho})$ where $[\alpha]$ is a homotopy class of closed curve and $\alpha_x$ a curve base at $x$ with $[\alpha_x] \in \pi_1(M,x)$ in the conjugacy class of $[\alpha]$. Let $v$ be an invariant vector for $[\alpha_x]$. Then we consider $[\alpha_x] \otimes v \in C_1(\pi,V)$. Then as $d_1([g] \otimes v) = \rho(g)(v) - v$ we have

$$d_1([\alpha_x] \otimes v) = \rho([\alpha_x])(v) - v = 0.$$  

Thus $[\alpha_x] \otimes v$ is a cycle.

The canonical isomorphism $H_1(M,g_{Ad\rho}) \to H_1(\pi,g_{Ad\rho})$ maps

$$[\alpha_x \otimes v] \to [\alpha_x] \otimes v.$$