Appendix E: Mathematical Derivations for the equations of
“Housing Market Spillovers:
Evidence from an Estimated DSGE Model”

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1 The model

1.1 Patient households

Lifetime utility is given by:

\[ V_t = E_0 \sum_{t=0}^{\infty} (\beta G_C)^t z_t \left[ \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \log (c_t - \varepsilon c_{t-1}) + j_t \log h_t - \frac{\tau_t}{1 + \eta} \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{1+\eta}{1+\xi}} \right] \]

where the term in square brackets represents period utility. With this formulation, the marginal utility of consumption is given by:

\[ u_{ct} = z_t \left( \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \right) \left( \frac{1}{c_t - \varepsilon c_{t-1}} - \frac{\beta G_C \varepsilon}{c_{t+1} - \varepsilon c_t} \right) \]

the marginal utility of housing is:

\[ u_{ht} = \frac{z_j t}{h_t} \]

and the marginal disutility of working in the goods and housing sector:

\[ u_{nct} = z_j t (1 + \eta) n_{ct}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta - \xi}{1+\xi}} \]

\[ u_{nht} = z_j t (1 + \eta) n_{ht}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta - \xi}{1+\xi}} \]

Since along the balance growth path (BGP) consumption grows at the rate \( G_C \) every quarter, the marginal utility of consumption falls at this rate. Hence the transformed marginal utility \( \tilde{u}_{ct} = u_{ct} G_C^t \) is stationary around the steady state and equal to:

\[ \tilde{u}_{ct} = G_C^t u_{ct} = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{G_C^t}{c_{t+1} - \varepsilon c_t} - \frac{\beta G_C^{t+1} \varepsilon}{c_{t+1} - \varepsilon c_t} \right) \]

\[ = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{1}{\frac{c_t}{G_C} - \frac{\varepsilon}{G_C} \frac{c_{t-1}}{c_t}} - \frac{\beta \varepsilon}{\frac{c_{t+1}}{G_C} - \frac{\varepsilon}{G_C} \frac{c_t}{c_{t+1}}} \right) \]

\[ = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{1}{\frac{c_t}{G_C} - \frac{\varepsilon}{G_C} \frac{c_{t-1}}{c_t}} - \frac{\beta \varepsilon}{\frac{c_{t+1}}{G_C} - \frac{\varepsilon}{G_C} \frac{c_t}{c_{t+1}}} \right) \]

\[ = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{1}{\frac{c_t}{G_C} - \frac{\varepsilon}{G_C} \frac{c_{t-1}}{c_t}} - \frac{\beta \varepsilon}{\frac{c_{t+1}}{G_C} - \frac{\varepsilon}{G_C} \frac{c_t}{c_{t+1}}} \right) \]

Transformed consumption, \( \tilde{c}_t = c_t / G_t \), and the scaled marginal utility of consumption \( \tilde{u}_c \):

\[ \tilde{u}_c = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{1}{1 - \frac{\varepsilon}{G_C}} - \frac{\beta \varepsilon}{1 - \frac{\varepsilon}{G_C}} \right) \left( \frac{1}{\tilde{c}} \right) = \frac{G_C - \varepsilon}{G_C - \beta \varepsilon G_C} \left( \frac{G_C (1 - \beta \varepsilon)}{G_C - \varepsilon} \right) \left( \frac{1}{\tilde{c}} \right) = \frac{1}{\tilde{c}} \]
are both constant in steady state.

The marginal utility of housing $u_{ht} = \frac{h_{t}}{h_{t}}$ declines at the rate $G_{H}$. Therefore the transformed marginal utility $\bar{u}_{ht} = u_{ht}G_{H}^{t}$ is stationary around the steady state and equal to:

$$\bar{u}_{ht} = \frac{j_{t}z_{t}}{h_{t}}$$

In steady state it is equal to $\bar{u}_{h} = \frac{1}{h}$ since both $j_{t}$ and $z_{t}$ are equal to one.

Due to the assumptions on preferences and technology hours worked in the two sector are stationary already in the level economy.

The patient household’s budget constraint:

$$c_{t} + \frac{k_{ct}}{A_{kt}} + k_{ht} + k_{lt} + q_{t} \left[ h_{t} - (1 - \delta_{h}) h_{t-1} \right] + p_{lt}l_{t} = \frac{w_{ct}}{X_{wct}}n_{ct} + \frac{w_{ht}}{X_{wht}}n_{ht}$$

$$+ Div_{t} - \phi_{t} + \left( R_{ct}z_{ct} + \frac{1 - \delta_{k}}{A_{kt}} \right) k_{ct-1} + (R_{ht}z_{ht} + 1 - \delta_{k}) k_{ht-1} + p_{lt}k_{lt}$$

$$+ b_{t} - \frac{R-t_{b_{t-1}}}{\pi_{t}} + (R_{ht} + p_{lt}) l_{t-1} - a(z_{ct}) k_{ct-1} - a(z_{ht}) k_{ht-1}$$

where the adjustment costs on capital are

$$\phi_{t} = \frac{\phi_{kc}}{2} \left( \frac{k_{ct}}{k_{ct-1}} - G_{KC} \right)^{2} \frac{k_{ct-1}}{\Gamma_{Ak}^{t}} + \frac{\phi_{kh}}{2} \left( \frac{k_{ht}}{k_{ht-1}} - G_{C} \right)^{2} k_{ht-1}$$

where $\Gamma_{Ak}$ is the gross growth rate of the investment specific technology process in the goods sector and $G_{KC}$ is the BGP gross growth rate of capital in the goods sector. Adjustment costs on capacity utilisation are:

$$a(z_{ct}) = R_{c} \left( \frac{1}{2} \omega z_{ct}^{2} + (1 - \omega) z_{ct} + \left( \frac{\omega}{2} - 1 \right) \right)$$

$$a(z_{ht}) = R_{h} \left( \frac{1}{2} \omega z_{ht}^{2} + (1 - \omega) z_{ht} + \left( \frac{\omega}{2} - 1 \right) \right)$$

where $R_{c}$ and $R_{h}$ are the steady state levels of the rental rate of capital in, respectively, the goods and the housing sector.

The budget constraint can be transformed as follows:

$$\frac{c_{t}}{G_{C}^{t}} + \frac{k_{ct}}{A_{kt}G_{C}^{t}} + k_{ht} + k_{lt} + q_{t} \left( h_{t} - (1 - \delta_{h}) h_{t-1} \right) + p_{lt}l_{t} = \frac{w_{ct}}{X_{wct}G_{C}^{t}}n_{ct} + \frac{w_{ht}}{X_{wht}G_{C}^{t}}n_{ht}$$

$$+ Div_{t} - \phi_{t} + \frac{\Gamma_{Ak}^{t}}{\frac{\Gamma_{Ak}^{t-1}G_{C}^{t}}{G_{C}^{t}}} - R_{ct}z_{ct}k_{ct-1} - \frac{1}{G_{C}^{t-1}G_{C}^{t}} + (1 - \delta_{k}) k_{ct-1} - \frac{1}{A_{kt}G_{C}^{t}} + R_{ht}z_{ht} + 1 - \delta_{k}) k_{ht-1} + p_{lt}k_{lt}$$

$$+ b_{t} - \frac{R_{t-b_{t-1}}}{\pi_{t}} + (R_{ht} + p_{lt}) l_{t-1} - a(z_{ct}) k_{ct-1} - a(z_{ht}) k_{ht-1}$$

$$- \frac{\phi_{kc}}{2} \left( \frac{k_{ct}}{k_{ct-1}} - G_{KC} \right)^{2} \frac{k_{ct-1}}{\Gamma_{Ak}^{t}} + \frac{\phi_{kh}}{2} \left( \frac{k_{ht}}{k_{ht-1}} - G_{C} \right)^{2} k_{ht-1}$$
Using the definition of dividends:

\[ \text{Div} = \frac{c_t}{G_C} + \frac{k_{ct}}{A_{kt}G_C^t} + \frac{k_{ht}}{G_C^t} + \frac{k_{bt}}{G_C^t} + \frac{q_t}{G_C^t} \left( \frac{h_t}{G_C^t} - (1 - \delta_h) \frac{h_{t-1}G_C^{t-1}}{G_C^t} \right) + \frac{p_{lt}}{G_C^t} l_t = \frac{w_{ct}}{X_{wct}G_C^t}n_{ct} + \frac{w_{ht}}{X_{wht}G_C^t}n_{ht} \]

\[ + \frac{\text{Div}_{lt}}{G_C^t} - \frac{\phi_t}{G_C^t} + \frac{R_{ct}^l}{\Gamma_{AK}^lG_C^t} \frac{\tilde{k}_{ct-1}}{A_{kt}G_C^t} (1 - \delta_k) \frac{\tilde{k}_{ct-1}A_{kt-1}}{A_{kt}A_{kt-1}G_C^t} + (R_{ht}z_{ht} + 1 - \delta_h) \frac{k_{ht-1G_C^{-1}}}{G_C^t} + \frac{p_{ht}}{G_C^t} b_t - \frac{R_{t-1}b_{t-1}}{\pi_t} G_C^{t-1} + \frac{R_{lt} + p_{lt}}{G_C^t} l_{t-1} \]

\[ - a(z_{ct}) \frac{A_{kt-1}}{A_{kt}} G_C^t - a(z_{ht}) \frac{k_{ht-1G_C^{-1}}}{G_C^t} + (\frac{\tilde{h}_{t-1}}{G_H} + \tilde{p}_{lt}) = \frac{\bar{w}_{ct}}{X_{wct}^t} n_{ct} + \frac{\bar{w}_{ht}}{X_{wht}^t} n_{ht} + \bar{\text{Div}}_t - \bar{\phi}_t \]

\[ + \frac{\tilde{R}_{ct}^l}{G_{KC}^t} + \frac{(1 - \delta_h) \tilde{k}_{ct-1}}{a_{kt}} + (R_{ht}z_{ht} + 1 - \delta_h) \frac{k_{ht-1G_C^{-1}}}{G_C^t} + \frac{p_{ht}}{G_C^t} \tilde{k}_{bt} \]

\[ - a(z_{ct}) \frac{A_{kt-1}}{a_{kt}} - a(z_{ht}) \frac{k_{ht-1G_C^{-1}}}{G_C^t} + \tilde{b}_t - \frac{R_{t-1}b_{t-1}}{\pi_t} G_C^{t-1} + \left( \tilde{R}_{lt} + \tilde{p}_{lt} \right) l_{t-1} \]

where the following result, which will be derived later, has been used:

\[ G_{KC}^t = \Gamma_{AK}^l G_C^t \]

Adjustment costs for capital can be transformed as follows:

\[ \frac{\bar{\phi}_t}{G_C^t} = \frac{\phi_{kc}}{2} \left( \frac{k_{ct}}{\tilde{k}_{ct-1}} - G_{KC}^t \right)^2 \frac{k_{ct-1}}{G_C^{t-1}G_C^t\Gamma_{Ak}^l\Gamma_{Ak}^l} + \frac{\phi_{kh}}{2} \left( \frac{k_{ht}}{k_{ht-1}} - G_C^t \right)^2 \frac{k_{ht-1}}{G_C^{t-1}G_C^t} \]

\[ \tilde{\phi}_t = \frac{\phi_{kc}}{2G_{KC}^t} \left( G_{KC}^t \frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - G_{KC}^t \right)^2 \tilde{k}_{ct-1} + \frac{\phi_{kh}}{2G_C^t} \left( G_C^t \frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}} - G_C^t \right)^2 \tilde{k}_{ht-1} \]

Using the definition of dividends:

\[ \text{DIV}_t = \left( 1 - \frac{1}{X_{wct}} \right) w_{ct} n_{ct} + \left( 1 - \frac{1}{X_{wht}} \right) w_{ht} n_{ht} + \left( 1 - \frac{1}{X_t} \right) Y_t \]

the terms \( \frac{1}{X_{wct}} w_{ct} n_{ct} \) and \( \frac{1}{X_{wht}} w_{ht} n_{ht} \) cancel out in the budget constraint so that dividends to the patient households are given by:

\[ \text{DIV}_t = \left( 1 - \frac{1}{X_t} \right) Y_t \]

The final expression for the budget constraint is:
\[\begin{align*}
\bar{c}_t + \frac{k_{ct}}{a_{kt}} + \bar{k}_{ht} + \bar{k}_{bt} + \bar{q}_t \bar{h}_t - (1 - \delta_h) \bar{q}_t \bar{h}_{t-1} + \bar{p}_{lt} l_t = \bar{w}_{ct} n_{ct} + \bar{w}_{ht} n_{ht} + \left(1 - \frac{1}{X_t}\right) \bar{Y}_t \\
+ \left(\bar{R}_{ct} z_{ct} + \frac{(1 - \delta_k)}{a_{kt}}\right) \frac{k_{ct-1}}{G_{KC}} + (\bar{R}_{ht} z_{ht} + 1 - \delta_k) \frac{k_{ht-1}}{G_C} + p_{lt} \bar{k}_{bt} \\
- \bar{b}_t + \frac{R_{t-1}}{\pi_t} \frac{b_{t-1}}{G_C} + \left(\bar{R}_{lt} + \bar{p}_{lt}\right) l_{t-1} - a(z_{ct}) \frac{k_{ct-1}}{a_{kt}} - a(z_{ht}) \frac{k_{ht-1}}{G_C} - \frac{\phi_{kc}}{G_{KC}} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC}\right)^2 k_{ct-1} \\
+ \frac{\phi_{kh}}{2G_C} \left(\frac{k_{ht}}{k_{ht-1}} - G_C\right)^2 k_{ht-1}
\end{align*}\]

The choice variables for the patient household are the following: \(c_t, h_t, k_{ct}, k_{ht}, b_t, n_{ct}, n_{ht}, k_{bt}, z_{ct}\) and \(z_{ht}\). The first-order conditions of the patient household’s maximisation problem are:

\[\begin{align*}
\frac{u_{ct} q_{t}}{A_{kt}} &= u_{ht} + \beta G_C E_t \left[u_{ct+1} q_{t+1} (1 - \delta_h)\right] \\
\frac{u_{ct}}{A_{kt}} &= \beta G_C E_t \left(u_{ct+1} R_t / \pi_{t+1}\right) \\
\frac{u_{ct}}{A_{kt}} (1 + \frac{\partial \phi_{ct}}{\partial k_{ct}}) &= \beta G_C E_t \left[u_{ct+1} \left(R_{ct+1} z_{ct+1} - a(z_{ct}) + 1 - \delta_k A_{kt+1} - \frac{\partial \phi_{ct+1}}{\partial k_{ct}}\right)\right] \\
\frac{u_{ct}}{A_{kt}} (1 + \frac{\partial \phi_{ht}}{\partial k_{ht}}) &= \beta G_C E_t \left[u_{ct+1} \left(R_{ht+1} z_{ht+1} - a(z_{ht}) + 1 - \delta_k - \frac{\partial \phi_{ht+1}}{\partial k_{ht}}\right)\right] \\
\frac{u_{ct}}{X_{wct}} &= u_{ct} \frac{w_{ct}}{X_{wct}} \\
\frac{u_{ct}}{X_{wht}} &= u_{ct} \frac{w_{ht}}{X_{wht}} \\
u_{ct} (p_{bt} - 1) &= 0 \\
\frac{R_{ct}}{A_{kt}} &= \frac{a'(z_{ct})}{A_{kt}} \\
\frac{R_{ht}}{A_{kt}} &= \frac{a'(z_{ht})}{A_{kt}} \\
\frac{u_{ct} p_{lt}}{A_{kt}} &= \beta G_C E_t \left[u_{ct+1} (p_{t+1} + R_{lt+1})\right]
\end{align*}\]

where we have substituted away the Lagrange multiplier on the budget constraint. These optimality conditions must be transformed to take into account the fact that some of the variables are growing over time.

The first order condition with respect to \(h_t\) is transformed in the following way:

\[\begin{align*}
\frac{u_{ct} q_{t}}{G_{Q}} &= u_{ht} + \beta G_C E_t \left(u_{ct+1} q_{t+1} (1 - \delta_h)\right) \\
\frac{u_{ct} G_{t}^{c} q_{t}}{G_{Q}} &= \left(u_{ht} G_{t}^{c} H\right) + \beta G_C E_t \left[u_{ct+1} G_{t+1}^{c} \left(\frac{q_{t+1}}{G_{Q}}\right) (1 - \delta_h)\right] \\
\frac{u_{ct} G_{t}^{c} q_{t}}{G_{Q}} &= \left(u_{ht} G_{t}^{c} H\right) + \beta G_C E_t \left[u_{ct+1} G_{t+1}^{c} \left(\frac{1}{G_{C}}\right) \left(\frac{q_{t+1} G_{Q}}{G_{Q}}\right) (1 - \delta_h)\right] \\
\tilde{u}_{ct} q_{t} &= \tilde{u}_{ht} + \beta G_C E_t \left[\tilde{u}_{ct+1} \tilde{q}_{t+1} (1 - \delta_h)\right] \frac{G_{Q}}{G_C}
\end{align*}\]

where \(G_Q\) is the BGP growth rate of real house prices whose expression will be derived later.
The transformation that must be applied to the first order condition with respect to lending, \( b_t \), is:

\[
uc = \beta G_C E_t \left( \frac{u_{ct+1} R_t}{\pi_{t+1}} \right)
\]

\[
u_{ct} G_C' = \beta G_C E_t \left( u_{ct+1} \frac{G_C^{t+1} R_t}{G_C \pi_{t+1}} \right)
\]

\[
tbar_{ct} = \beta \bar{u}_{ct+1} \frac{R_t}{\pi_{t+1}}
\]

The first order condition with respect to \( k_t \) is:

\[
u_{ct} \left[ 1 + \frac{\phi_{kc}}{G_KC} \left( \frac{k_{ct}}{k_{ct-1}} - G_{KC} \right) \right] = \beta G_C E_t \left[ u_{ct+1} \left( R_{ct+1} z_{ct+1} - \frac{a(z_{ct+1})}{A_{kt+1}} + \frac{1 - \delta_k}{A_{kt+1}} \right) + \frac{1 - \delta_k}{A_{kt+1}} + \frac{\phi_{kc}}{2G_KC} \left( \frac{k_{ct+1}^2}{k_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{t+1}} \right]
\]

It can be transformed in the following way:

\[
G_C' u_{ct} \left[ 1 + \frac{\phi_{kc}}{G_KC} \left( G_{KC} \frac{\tilde{k}_{ct}}{k_{ct-1}} - G_{KC} \right) \right] = \beta G_C E_t \left[ G_C' u_{ct+1} \frac{G_C^{t+1} R_{ct+1} z_{ct+1} - \frac{a(z_{ct+1})}{A_{kt+1}} + \frac{1 - \delta_k}{A_{kt+1}}}{\Gamma_{t+1}} \right] + \frac{1 - \delta_k}{A_{kt+1}} + \frac{\phi_{kc}}{2G_KC} \left( \frac{k_{ct+1}^2}{k_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{t+1}}
\]

\[
tbar_{ct} \left[ 1 + \frac{\phi_{kc}}{G_KC} \left( \frac{\tilde{k}_{ct}}{k_{ct-1}} - 1 \right) \right] = \beta G_C E_t \left[ \bar{u}_{ct+1} \frac{G_KC^{t+1} R_{ct+1} z_{ct+1} - \frac{a(z_{ct+1})}{a_{kt+1}} + \frac{1 - \delta_k}{a_{kt+1}}}{G_KC + \frac{G_KC \phi_{kc}}{2} \left( \frac{k_{ct+1}^2}{k_{ct}^2} - 1 \right)} \right]
\]

\[
tbar_{ct} \left[ 1 + \phi_{kc} \left( \frac{\tilde{k}_{ct}}{k_{ct-1}} - 1 \right) \right] = \beta G_C E_t \left[ \bar{u}_{ct+1} \frac{G_KC^{t+1} R_{ct+1} z_{ct+1} - \frac{a(z_{ct+1})}{a_{kt+1}} + \frac{1 - \delta_k}{a_{kt+1}}}{G_KC + \frac{G_KC \phi_{kc}}{2} \left( \frac{k_{ct+1}^2}{k_{ct}^2} - 1 \right)} \right]
\]

The first order condition with respect to \( k_{ht} \) is:

\[
u_{ct} \left[ 1 + \frac{\phi_{kh}}{G_C} \left( \frac{k_{ht}}{k_{ht-1}} - G_C \right) \right] = \beta G_C E_t \left[ u_{ct+1} \left( R_{ht+1} z_{ht+1} - a(z_{ht+1}) + 1 - \delta_k \right) + \frac{\phi_{kh}}{2G_C} \left( \frac{k_{ht+1}^2}{k_{ht}^2} - G_C^2 \right) \right]
\]
This first order condition can be transformed into:

\[
G_C^t u_{ct} \left[ 1 + \frac{\phi_{kh}}{G_C} \left( G_C \frac{\bar{k}_{ht}}{k_{ht-1}} - G_C \right) \right] = \beta G_C E_t \left[ G_C^t u_{ct+1} \frac{G_C^{t+1}}{G_C} (R_{ht+1} z_{ht+1} - a(z_{ht+1}) + \\
+ 1 - \delta_k - \frac{\phi_{kh}}{2G_C} \left( G_C^2 \frac{\bar{k}_{ht+1}^2}{k_{ht}^2} - G_C^2 \right) \right]
\]

\[
\tilde{u}_{ct} \left[ 1 + \phi_{kh} \left( \frac{\bar{k}_{ht}}{k_{ht-1}} - 1 \right) \right] = \beta G_C E_t \left[ \tilde{u}_{ct+1} \frac{G_C}{G_C} (R_{ht+1} z_{ht+1} - a(z_{ct}) + \\
+ 1 - \delta_k - \frac{\phi_{kh} G_C}{2} \left( \frac{\bar{k}_{ht+1}^2}{k_{ht}^2} - 1 \right) \right]
\]

The first order conditions with respect to \(u_{nc}\) and \(u_{nh}\) are:

\[
u_{nc} = u_{ct} \frac{w_{ct}}{X_{wct}}
\]

\[
u_{nh} = u_{ct} \frac{w_{ht}}{X_{wht}}
\]

which can be transformed as follow:

\[
u_{nc} = z_{jt} (1 + \eta) n_{ct}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct} \frac{w_{ct}}{X_{wct} G_C^t}
\]

\[
u_{nh} = z_{jt} (1 + \eta) n_{ht}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct} \frac{w_{ht}}{X_{wht} G_C^t}
\]

\[
u_{nc} = z_{jt} (1 + \eta) n_{ct}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct} \frac{\tilde{w}_{ct}}{X_{wct}}
\]

\[
u_{nh} = z_{jt} (1 + \eta) n_{ht}^\xi \left( n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct} \frac{\tilde{w}_{ht}}{X_{wht}}
\]

The first order condition with respect to intermediate inputs \(k_{ht}\) is:

\[
u_{ct} (p_{bt} - 1) = 0
\]

which implies that their price is always equal to 1.

The first order conditions with respect to capacity utilisation are:

\[
R_{ct} = \frac{a' (z_{ct})}{A_{kt}}
\]

\[
R_{ht} = a' (z_{ht})
\]

which are transformed as:

\[
R_{ct} A_{kt} = a' (z_{ct})
\]

\[
R_{ht} = a' (z_{ht})
\]
where we have taken into account the definition $\tilde{R}_{ct} = R_{ct} \Gamma_k^t$.

The first order condition with respect to land $l_t$

$$u_{ct} p_{lt} = \beta G C E_t \left( u_{ct+1} \left( p_{lt+1} + R_{lt+1} \right) \right)$$

becomes after transformation:

$$u_{ct} G C^t \frac{p_{lt}}{G_C^t} = \beta G C E_t \left[ u_{ct+1} G_C^{t+1} \left( \frac{p_{lt+1}}{G_C^{t+1}} + \frac{R_{lt+1}}{G_C^{t+1}} \right) \right]$$

$$\tilde{u}_{ct} p_{lt} = \beta G C E_t \left[ \tilde{u}_{ct+1} \left( \tilde{p}_{lt+1} + \tilde{R}_{lt+1} \right) \right].$$

### 1.2 Impatient households

Lifetime utility is given by:

$$V_t = E_0 \sum_{t=0}^{\infty} \left( \beta' G C \right)^t z_t \left[ \frac{G_C - \varepsilon'}{G_C - \beta' \varepsilon G_C} \log \left( \frac{c_t'}{c_{t-1}'} \right) + j_t \log h_t' - \frac{\tau_t}{1 + \eta'} \left( n_t' \right)^{1+\varepsilon'} \left( n_{ht}' \right)^{1+\varepsilon'} \right]$$

With this formulation, the marginal utility of consumption is given by:

$$u'_t = \left( \frac{G_C - \varepsilon'}{G_C - \beta' \varepsilon G_C} \right) \left( \frac{1}{c_t'} - \varepsilon' c_{t-1}' - \frac{\beta' G C \varepsilon'}{c_{t+1}' - \varepsilon' c_t'} \right)$$

which can be made stationary using the same transformation employed for the patient households $\tilde{u}_t^t = u_t^t G_C^t$. The marginal utility of housing and the marginal disutilities of working are similar to those of the patient households with the exception that the household-specific variables and parameters are denoted with a prime.

The optimality conditions must be transformed to take into account the fact the some of the variables are growing over time. The first order condition with respect to housing, $h_t$:

$$u_{ct} q_t = u_{mt} + \beta' G C E_t \left( u_{ct+1} \left( q_{t+1} \left( 1 - \delta_h \right) \right) \right) + E_t \left( \lambda_t \frac{m_{qt+1}}{R_t} \right)$$

$$u_{ct} G_C^t \frac{q_t}{G_Q^t} = u_{mt} \frac{G_C^t}{G_Q^t} + \beta G C E_t \left( \left( u_{ct+1} G_C^{t+1} \frac{1}{G_C} \right) \left( \frac{q_{t+1}}{G_Q^{t+1}} G_Q \right) \left( 1 - \delta_h \right) \right)$$

$$+ \ E_t \left( \lambda_t \frac{m_{qt+1}}{R_t} \right) \frac{G_C^{t+1}}{G_Q^t}$$

$$\tilde{u}_{ct} \tilde{q}_t = \tilde{u}_{mt} + \beta G C E_t \left( \tilde{u}_{ct+1} \tilde{q}_{t+1} \left( 1 - \delta_h \right) \right) \frac{G_Q}{G_C} + E_t \left( \tilde{\lambda}_t \frac{m_{\tilde{q}t+1}}{R_t} \right) \frac{G_Q}{G_C}$$

where $G_Q$ is the BGP growth rate of real house prices whose expression will be derived later.
The first order condition with respect to lending, $b_t$:

\[
\begin{align*}
    u_{ct} &= \beta G_C E_t \left( \frac{u_{ct+1} R_t}{\pi_{t+1}} \right) + \lambda_t \\
    u_{ct} G_C^t &= \beta G_C E_t \left( \frac{G_{ct+1} R_t}{G_C \pi_{t+1}} \right) + \lambda_t G_C^t \\
    \tilde{u}_{ct} &= \beta \tilde{u}_{ct+1} \frac{R_t}{\pi_{t+1}} + \tilde{\lambda}_t
\end{align*}
\]

The budget constraint:

\[
\begin{align*}
    c_t' + q_t \left( h_t' - (1 - \delta_h) h_{t-1}' \right) &= \frac{w_{ct}'}{X_{wct}} n_{ct}' + \frac{w_{ht}'}{X_{wht}} n_{ht}' + Div_t' + b_t' - \frac{R_t b_{t-1}'}{\pi_t} \\
    \frac{c_t'}{G_C^t} + \frac{q_t}{G_C^t} \left( \frac{h_t'}{G_C^t} - (1 - \delta_h) \frac{h_{t-1}'}{G_C^t} \right) &= \frac{w_{ct}'}{X_{wct G_C^t}} n_{ct}' + \frac{w_{ht}'}{X_{wht G_C^t}} n_{ht}' + \frac{Div_t'}{G_C^t} + \frac{b_t'}{G_C^t} - \frac{R_t b_{t-1}'}{\pi_t} \frac{G_{ct+1}^t}{G_C^t}
\end{align*}
\]

By substituting the expression for the dividends from the unions the transformed budget constraint becomes:

\[
\begin{align*}
    c_t' + q_t (h_t' - (1 - \delta_h) h_{t-1}') G_H &= \tilde{w}_{ct}' n_{ct}' + \tilde{w}_{ht}' n_{ht}' + \tilde{b}_t' - \frac{R_t b_{t-1}'}{\pi_t} G_C
\end{align*}
\]

The borrowing constraint:

\[
\begin{align*}
    b_t' &= m E_t \left( \frac{q_t h_t' \pi_{t+1}}{R_t} \right)
\end{align*}
\]

can be transformed as follows:

\[
\begin{align*}
    b_t' &= m E_t \left( \frac{q_t h_t' \pi_{t+1} G_C^t R_t}{G_C^t} \right) \\
    \tilde{b}_t &= m E_t \left( \frac{q_t h_t' \pi_{t+1} G_C^t G_Q^t R_t}{G_C^t G_Q^t G_H^t} \right) \\
    \tilde{b}_t &= m E_t \left( \frac{G_Q q_t h_t' \pi_{t+1} G_C^t G_Q^t G_H^t R_t}{G_C^t G_Q^t G_H^t R_t} \right) \\
    \tilde{b}_t &= m E_t \left( \frac{G_Q \tilde{q}_t h_t' \pi_{t+1} R_t}{R_t} \right)
\end{align*}
\]
1.3 Intermediate goods firms

Wholesale firms solve the following maximization problem:

$$\max \frac{Y_t}{X_t} + q_tIH_t - (\sum w_t n_{ct} + R_c z_{ct} k_{ct-1} + R_h z_{ht} k_{ht-1} + R_l z_{lt-1} + p_{bt} k_{bt})$$

The two production technologies are:

$$Y_t = [A_{ct} (n_{ct}^{\alpha} n_{ct}^{(1-\alpha)})]^{1-\mu_c} (z_{ct} k_{ct-1})^{\mu_c}$$

$$IH_t = [A_{ht} (n_{ht}^{\alpha} n_{ht}^{(1-\alpha)})]^{1-\mu_h - \mu_l} (z_{ht} k_{ht-1})^{\mu_h} k_{bt}^{\mu_l} k_{bt}^{\mu_l}$$

The first order condition with respect to $n_{ct}$ is:

$$(1 - \mu_c) \alpha \frac{Y_t}{X_t n_{ct}} = w_{ct}$$

which after taking into account that both $Y_t$ and $w_{ct}$ growth at rate $G_C$ along the BGP becomes:

$$(1 - \mu_c) \alpha \frac{Y_t}{G_C X_t n_{ct}} = \frac{w_{ct}}{G_C}$$

$$(1 - \mu_c) \alpha \frac{\dot{Y}_t}{X_t n_{ct}} = \dot{w}_{ct}$$

Similarly for $n'_{ct}$:

$$(1 - \mu_c) (1 - \alpha) \frac{Y_t}{X_t n'_{ct}} = w'_{ct}$$

$$(1 - \mu_c) (1 - \alpha) \frac{\dot{Y}_t}{X_t n'_{ct}} = \dot{w}'_{ct}$$

and for $n_{ht}$:

$$(1 - \mu_h - \mu_l) \alpha \frac{q_t IH_t}{n_{ht}} = w_{ht}$$

$$(1 - \mu_h - \mu_l) \alpha \frac{q_t IH_t}{G_Q G_H n_{ht}} = \frac{w_{ht}}{G_C}$$

$$(1 - \mu_h - \mu_l) \alpha \frac{q_t IH_t}{G_Q G_H n_{ht}} = \dot{w}_{ht}$$

and for $n'_{ht}$:

$$(1 - \mu_h - \mu_l) (1 - \alpha) \frac{q_t IH_t}{n'_{ht}} = w'_{ht}$$

$$(1 - \mu_h - \mu_l) (1 - \alpha) \frac{q_t IH_t}{G_Q G_H n'_{ht}} = \frac{w'_{ht}}{G_C}$$

$$(1 - \mu_h - \mu_l) (1 - \alpha) \frac{q_t IH_t}{G_Q G_H n'_{ht}} = \dot{w}'_{ht}$$

The first-order condition with respect to $k_{ct-1}$ is:

$$\mu_c \frac{Y_t}{X_t k_{ct-1}} = R_c z_{ct}$$
which is transformed into:

\[
\frac{\mu_c}{X_t} \frac{\hat{q} t}{k_{ht-1}} G_{Ct} G_{ct} \frac{1}{G_{C}} G_{et} G_{et} = R_{et},
\]

\[
\frac{\mu_c}{X_t} \frac{\hat{Y} t}{k_{ht-1}} G_{Ct} G_{ct} \frac{1}{G_{C}} G_{et} G_{et} = \tilde{R}_{et}.
\]

Similarly with respect to \(k_{ht-1}\) is

\[
\frac{\mu_h}{k_{ht-1}} \frac{q t IH t}{G_{Ct} G_{et}} = R_{ht},
\]

\[
\frac{\mu_h}{k_{ht-1}} \frac{q t IH t}{G_{Ct} G_{et}} = R_{ht},
\]

\[
\mu_h \frac{\tilde{q} t IH t}{k_{ht-1}} G_{Ct} G_{et} = \tilde{R}_{ht}.
\]

The first order condition with respect to \(l_t\), after setting \(l_t = 1\), is:

\[
\frac{\mu l q t IH t}{l} = R_{lt},
\]

\[
\frac{\mu l q t IH t}{l} = R_{lt},
\]

\[
\mu l \frac{\tilde{q} t IH t}{k_{ht}} G_{C} G_{et} = \tilde{R}_{lt},
\]

and with respect to \(k_{bt}\):

\[
\frac{\mu b q t IH t}{k_{bt}} = p_{bt},
\]

\[
\frac{\mu b q t IH t}{k_{bt}} = p_{bt},
\]

\[
\mu b \frac{\tilde{q} t IH t}{k_{bt}} = \tilde{p}_{bt}
\]

1.4 Wage stickiness

Patient and impatient households supply their homogeneous labor services to labor unions. There are four unions, two for each sector, each one acting in the interest of either patient or impatient households. The unions differentiate labor services, set nominal wages subject to a Calvo scheme and offer labor services to intermediate labor packers who assemble the differentiated labor services into the homogeneous labor composites \(n_c, n_h, n_t, n_{ht}\). The probability of unions being allowed to change nominal wages in each sector is common to both households. Wholesale firms hire labor services from the labor packers. Under partial indexation of nominal wages to past inflation, the
wage-setting rules set by the union imply four wage Phillips curves that are isomorphic to the one in the goods sector:

\[
\begin{align*}
\ln \omega_{c,t} - \tau_{wc} \ln \pi_{t-1} &= \beta G_{c} \left( E_{t} \ln \omega_{c,t+1} - \tau_{wc} \ln \pi_{t} \right) - \varepsilon_{wc} \ln \left( X_{wc,t} / X_{wc} \right) \\
\ln \omega_{t} - \tau_{wc} \ln \pi_{t-1} &= \beta' G_{c} \left( E_{t} \ln \omega_{c,t+1} - \tau_{wc} \ln \pi_{t} \right) - \varepsilon'_{wc} \ln \left( X_{wc,t} / X_{wc} \right) \\
\ln \omega_{h,t} - \tau_{wh} \ln \pi_{t-1} &= \beta G_{c} \left( E_{t} \ln \omega_{h,t+1} - \tau_{wh} \ln \pi_{t} \right) - \varepsilon_{wh} \ln \left( X_{wh,t} / X_{wh} \right) \\
\ln \omega'_{h,t} - \tau_{wh} \ln \pi_{t-1} &= \beta' G_{c} \left( E_{t} \ln \omega'_{h,t+1} - \tau_{wh} \ln \pi_{t} \right) - \varepsilon'_{wh} \ln \left( X_{wh,t} / X_{wh} \right)
\end{align*}
\]

with \( \omega_{i,t} \) nominal wage inflation, that is, \( \omega_{i,t} = \frac{w_{i,t} \varepsilon_{t}}{u_{i,t-1}} \) for each sector/household pair, and

\[
\begin{align*}
\varepsilon_{wc} &= (1 - \theta_{wc}) (1 - \beta G_{c} \theta_{wc}) / \theta_{wc} \\
\varepsilon'_{wc} &= (1 - \theta_{wc}) (1 - \beta' G_{c} \theta_{wc}) / \theta_{wc} \\
\varepsilon_{wh} &= (1 - \theta_{wh}) (1 - \beta G_{c} \theta_{wh}) / \theta_{wh} \\
\varepsilon'_{wh} &= (1 - \theta_{wh}) (1 - \beta' G_{c} \theta_{wh}) / \theta_{wh}
\end{align*}
\]

define the slope of the wage equations.

1.5 Price stickiness

Price stickiness in the consumption-business investment sector is introduced by assuming monopolistic competition at the retail level, implicit costs of adjusting nominal prices following Calvo-style contracts and partial indexation to lagged inflation of those prices that can not be reoptimized. The resulting inflation equation is:

\[
\log \pi_{t} - \tau_{\pi} \log \pi_{t-1} = \beta \left( E_{t} \log \pi_{t+1} - \tau_{\pi} \log \pi_{t} \right) - \varepsilon_{\pi} \log \left( \frac{X_{t}}{X} \right) + \log u_{p,t}
\]

where the parameter \( \varepsilon_{\pi} \) is equal to \( \varepsilon_{\pi} = \frac{(1-\theta_{\pi})(1-\beta G_{c} \theta_{\pi})}{\theta_{\pi}} \).

1.6 Monetary policy

\[
R_{t} = (R_{t-1})^{R_{t}} \left[ \pi_{t}^{\frac{GDP_{t}}{G_{C} GDP_{t-1}}} \right]^{1-R_{t}} \frac{E_{Rt}}{A_{St}}
\]

where \( GDP_{t} \) is the sum of the value added of the two sectors, that is \( GDP_{t} = Y_{t} + \bar{q} IH_{t} + IK_{t} \).

1.7 Market clearing

The market clearing conditions are:

\[
\begin{align*}
C_{t} + IK_{ct}/A_{kt} + IK_{ht} + k_{ht} &= Y_{t} - \frac{\phi_{kc}}{2} \left( \frac{k_{ct}}{k_{ct-1}} - G_{KC} \right)^{2} \frac{k_{ct-1}}{I^{A_{K}}} - \frac{\phi_{kh}}{2} \left( \frac{k_{ht}}{k_{ht-1}} - G_{C} \right)^{2} k_{ht-1} \\
h_{t} + h'_{t} - (1 - \delta_{h}) (h_{t-1} + h'_{t-1}) &= IH_{t} \\
b_{t} + b'_{t} &= 0
\end{align*}
\]

\(^{1}\)Here we make use of the result that the price-setter stochastic discount factor for nominal payoffs (the ratio between future and current marginal utility of consumption) cancels out in the linearization of the Phillips curve itself, so that the effective discount factor is simply \( \beta G_{C} \), rather than \( \beta G_{C} E_{t} u_{c,t+1}/u_{c,t} \).
which are transformed as follows:

\[
\frac{C_t}{G_C} + \frac{IK_{ct}/A_{ht}}{G_C} + \frac{IK_{ht}}{G_C} + \frac{k_{ht}}{G_C} = \frac{Y_t}{G_C} - \phi_{ke} \left( \frac{k_{ct} G_{KC} - G_{KC}}{k_{ct-1} G_{KC}} \right)^2 \frac{k_{ct-1} A_{kC}}{G_C G_{HC}}
\]

\[-\frac{\phi_{kh}}{2} \left( \frac{k_{ht}}{G_C} - \frac{G_C}{G_C} \right)^2 \frac{k_{ht-1}}{G_C} \]

\[
\frac{h_t}{G_C} + \frac{h_t'}{G_H G_{HC}^{-1}} - (1 - \delta_h) \left( \frac{h_{t-1}}{G_H G_{HC}^{-1}} + \frac{h_{t-1}'}{G_H G_{HC}^{-1}} \right) = \frac{IH_t}{G_H}
\]

\[
\frac{b_t}{G_C} + \frac{b_t'}{G_C} = 0
\]

\[
\tilde{C}_t + \frac{IK_{ct}}{A_{kt}} + \tilde{IK}_{ht} + \tilde{k}_{ht} = \tilde{Y}_t - \frac{\phi_{ke}}{2 G_{KC}} \left( \frac{\tilde{k}_{ct} G_{KC} - G_{KC}}{\tilde{k}_{ct-1} G_{KC}} \right)^2 \tilde{k}_{ct-1}
\]

\[-\frac{\phi_{kh}}{2 G_C} \left( \frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}} - \frac{G_C}{G_C} \right)^2 \tilde{k}_{ht-1} \]

\[
\tilde{h}_t + \tilde{h}_t' - (1 - \delta_h) \left( \frac{\tilde{h}_{t-1}}{G_H} + \frac{\tilde{h}_{t-1}'}{G_H} \right) = \tilde{IH}_t
\]

\[
\tilde{b}_t + \tilde{b}_t' = 0
\]

2 Linear deterministic trends

Suppose there are linear deterministic trends in the technologies \(A_c, A_h, A_k\). Let the corresponding gross growth rates be respectively:

\[
\gamma_C, \gamma_H, \gamma_K
\]

Because of these trends, the variables:

\[
Y, c, c', \frac{k_c}{A_k}, k_h, k_b, qI
\]

all grow at a common rate along the balanced growth path. This result stems from the form of the utility function and the assumption of constant returns to scale in the production functions, which implies common expenditure shares. To compute the net growth rate (\(x\)) of \(Y\), we observe from the production function that \(x_Y = (1 - \mu_c) \gamma_C + \mu_c x_{KC}\). We also know that \(x_Y = x_{KC} - \gamma_K\). It then follows that

\[
x_Y = \gamma_C + \frac{\mu_c}{1 - \mu_c} \gamma_K
\]

\[
x_{KC} = \gamma_C + \frac{1}{1 - \mu_c} \gamma_K
\]

\[
x_{KH} = \gamma_C + \frac{\mu_c}{1 - \mu_c} \gamma_K
\]
In order to disentangle $q$ and $I$ separately, we use the formula for $I$ to obtain the steady state growth rate of $I$ as $x_I = (1 - \mu_h - \mu_l - \mu_b) \gamma_H + \mu_h x_{KH} + \mu_b x_{KB}$.

Hence the steady state growth rate of $I$ is:

$$x_I = (1 - \mu_h - \mu_l - \mu_b) \gamma_H + \mu_h \left( \gamma_C + \frac{\mu_c}{1 - \mu_c} \gamma_K \right) + \mu_b \left( \gamma_C + \frac{\mu_c}{1 - \mu_c} \gamma_K \right)$$

$$x_I = x_H = (\mu_h + \mu_b) \gamma_C + \frac{\mu_h + \mu_b}{1 - \mu_c} \gamma_K + (1 - \mu_h - \mu_l - \mu_b) \gamma_H$$

and the growth rate of $q$ is

$$x_Q = (1 - \mu_h - \mu_b) \gamma_C + \frac{(1 - \mu_h - \mu_b) \mu_c}{1 - \mu_c} \gamma_K - (1 - \mu_h - \mu_l - \mu_b) \gamma_H$$

$$x_Q = x_Y - x_I$$
3 Steady state of the model

We are interested in finding the steady state of the transformed model. In the transformed model, each variable is scaled by its long-run growth rate, e.g.

\[
\tilde{c}_t = \frac{c_t}{G_C}
\]

\[
\tilde{c}_{t-1} = \frac{c_{t-1}}{G_C^{-1}}
\]

hence in each equation we perform the necessary replacements such as the following:

\[
c_t = \tilde{c}_t G_C^t
\]

\[
c_{t-1} = \tilde{c}_{t-1} G_C^{t-1}
\]

\[
q_t = \tilde{q}_t G_Q^t
\]

\[
u_{ct} = \tilde{u}_{ct} G^{-t}
\]

3.1 Calculations

Marginal utility of consumption and housing are equal, respectively, to $1/c$ and $j/h$ in steady state. From the transformed consumption Euler equation:

\[
u_{ct} = \beta G_C u_{ct+1} \frac{R_t}{\pi_{t+1}}
\]

the $G_C$ term disappears and we can derive the steady state level of the real interest rate once we have imposed $\pi = 1$:

\[
R = \frac{1}{\beta}
\]

From the Euler equations for the two capital stocks we can derive the steady state values for the rental rates:

\[
R_{kc} = \frac{\Gamma_K}{\beta} - (1 - \delta_k)
\]

\[
R_{kh} = \frac{1}{\beta} - (1 - \delta_k)
\]

\[
r = \frac{R}{G_C} - 1
\]

Combining the Euler equation for $k_c$ and the expression for $R_{kc}$ (from the optimal demand for capital by firms in the good sector) the following ratio is obtained:

\[
\zeta_0 = \frac{k_c}{Y} = \frac{\beta G_K \mu_c}{(\Gamma_K - \beta (1 - \delta_{kc}))} \frac{1}{X}
\]

Combining the Euler equation for $k_h$ and the expression for $R_{kh}$ from the optimal demand for capital by firms in the good sector the following ratio is obtained:

\[
\zeta_1 = \frac{k_h}{qI} = \frac{\beta G_C \mu_h}{1 - \beta (1 - \delta_{kh})}
\]
From the Euler equation for $h$:

$$\zeta_2 = \frac{qh}{c} = \frac{j}{1 - \beta G_Q (1 - \delta_h)}$$

while from the Euler equation for $h'$ and $b'$:

$$\zeta_3 = \frac{j}{1 - \beta' G_Q (1 - \delta_h) - G_Q (\beta' - \beta) m}$$

$$\lambda = \frac{1 - \beta'/\beta}{c'}$$

$$f = \frac{X - l Y}{X}$$

For land, let $l = 1$, so that

$$R_l = \mu_l q I$$

The following equations describe the steady state (using $b + b' = 0$, where $b = m G_Q q h'/R$ and steady state repayment is $(\frac{R}{RC} - 1) b$, so that repayment equals $(\frac{R}{RC} - 1) \frac{mgQ}{R} q h' = \zeta_4 q h'$):

Define the adjusted depreciation rates:

$$\delta'_h = 1 - \frac{1 - \delta_h}{G_H}$$

$$\delta'_k = 1 - \frac{1 - \delta_k}{G_{KC}}$$

From the above ratios and using the budget constraints of the two types of households, we have:

$$k_c = \zeta_0 Y$$

$$k_h = \zeta_1 q I$$

$$qh = \zeta_2 c$$

$$qh' = \zeta_3 c'$$

$$\delta'_h qh + q h' = f + rk_c + rk_h + \mu_l q I + \sum wn + \zeta_4 q h' + \text{div}$$

$$c' + \delta'_h qh' = \sum wn - \zeta_4 q h' + \text{div}$$

Simple algebra yields:

$$\delta'_h (\zeta_2 c + \zeta_3 c') = q I$$

$$c' + \delta'_k (\zeta_0 Y + \zeta_1 q I) = Y$$

$$c + \delta'_h \zeta_0 Y = f + r \zeta_0 Y + \zeta_1 q I + \mu_l q I + \sum wn + \zeta_4 \zeta_3 c' + \text{div}$$

$$c' + \delta'_h \zeta_3 c' = \sum wn - \zeta_4 \zeta_3 c' + \text{div}$$
The equations in labor market satisfy from the demand side:

\[
(1 - \mu_c) \alpha \frac{Y}{Xn_c} = w_c \\
(1 - \mu_c) (1 - \alpha) \frac{Y}{Xn'_c} = w'_c \\
(1 - \mu_h - \mu_b) \alpha \frac{qI}{n_h} = w_h \\
(1 - \mu_h - \mu_b) (1 - \alpha) \frac{qI}{n'_h} = w'_h
\]

To compute the steady state, we simply need to know the total wage bill plus union dividends earned by each group, which equals

\[
w_c n_c + w_h n_h = \alpha \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) qI \right) \\
w'_c n'_c + w'_h n'_h = (1 - \alpha) \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) qI \right)
\]

Using \(\phi = (X - 1)/X\), we have

\[
delta'_h (\zeta_2 c + \zeta_3 c') = qI \\
c + c' + \delta'_h (\zeta_0 Y + \zeta_1 qI) = Y \\
c + \delta'_h \zeta_2 c = \phi Y + r\zeta_0 Y + r\zeta_1 qI + \mu_l qI + \alpha \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) qI \right) + \zeta_4 \zeta_3 c' \\
c' + \delta'_h \zeta_3 c' = (1 - \alpha) \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) qI \right) - \zeta_4 \zeta_3 c'
\]

Eliminating one redundant equation (for example the second) and using the formula for \(qI\)

\[
c + \delta'_h \zeta_2 c = (\phi + r\zeta_0) Y + r\zeta_1 \delta_h (\zeta_2 c + \zeta_3 c') + \alpha \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) \delta_h (\zeta_2 c + \zeta_3 c') \right) + \zeta_4 \zeta_3 c' \\
c' + \delta'_h \zeta_3 c' = (1 - \alpha) \left( (1 - \mu_c) \frac{Y}{X} + (1 - \mu_h - \mu_b - \mu_l) \delta'_h (\zeta_2 c + \zeta_3 c') \right) - \zeta_4 \zeta_3 c'
\]

Hence the consumption-output ratios \(c/Y\) and \(c'/Y\) solve:

\[
(1 + \delta'_h \zeta_2 (1 - r\zeta_1 - \mu_l - \alpha (1 - \mu_b - \mu_h - \mu_l))) c - ((r\zeta_1 + \mu_l + \alpha (1 - \mu_h - \mu_b - \mu_l)) \delta'_h \zeta_3 + \zeta_4 \zeta_3) c' \\
= \left( \frac{X - 1}{X} + r\zeta_0 X + \alpha \frac{1 - \mu_c}{X} \right) Y \\
(1 + \delta'_h \zeta_3 - (1 - \alpha) (1 - \mu_h - \mu_b - \mu_l) \delta'_h \zeta_3 + \zeta_4 \zeta_3) c' - (1 - \alpha) (1 - \mu_h - \mu_b - \mu_l) \delta'_h \zeta_2 c
\]

\[
= (1 - \alpha) (1 - \mu_c) \frac{1}{X} Y
\]
Solve for $c/Y$, $c'/Y$ and $qI/Y$. Defining the following variables:

\[
\begin{align*}
\chi_1 &= 1 + \delta_h^t \zeta_2 (1 - r \zeta_1 - \mu_t - \alpha (1 - \mu_h - \mu_b - \mu_l)) \\
\chi_2 &= (r \zeta_1 + \mu_t + \alpha (1 - \mu_h - \mu_b - \mu_l)) \delta_h^t \zeta_3 + \zeta_4 \zeta_3 \\
\chi_3 &= \frac{X - 1}{X} + r \zeta_0 X + \alpha \frac{(1 - \mu_c)}{X} \\
\chi_4 &= 1 + \delta_h^t \zeta_3 - (1 - \alpha) (1 - \mu_h - \mu_b - \mu_l) \delta_h^t \zeta_3 + \zeta_4 \zeta_3 \\
\chi_5 &= (1 - \alpha) (1 - \mu_h - \mu_b - \mu_l) \delta_h^t \zeta_2 \\
\chi_6 &= (1 - \alpha) (1 - \mu_c) \frac{1}{X}
\end{align*}
\]

delivers the following solution:

\[
\begin{align*}
\frac{c}{Y} &= \frac{\chi_3 \chi_4 + \chi_2 \chi_6}{\chi_1 \chi_4 - \chi_2 \chi_5} \\
\frac{c'}{Y} &= \frac{\chi_1 \chi_6 + \chi_3 \chi_5}{\chi_1 \chi_4 - \chi_2 \chi_5} \\
\frac{qI}{Y} &= \delta_h^t \left( \zeta_2 c + \zeta_3 c' \right)
\end{align*}
\]

### 3.2 Levels

In order to compute the levels of the variables in steady state we need to find first the value of hours worked. It is useful to normalize $\tau$ to 1. The labor market equilibrium is of the kind:

\[
(1 - \mu_c) \alpha \frac{Y}{X} = X_w c \left( n_c^{1+\xi} + n_h^{1+\xi} \right) \frac{q - \varepsilon}{1+\xi} n_c^{1+\xi}
\]

\[
(1 - \mu_h - \mu_b - \mu_l) \alpha qI = X_w c \left( n_c^{1+\xi} + n_h^{1+\xi} \right) \frac{q - \varepsilon}{1+\xi} n_h^{1+\xi}
\]

so that the ratio of hours worked is:

\[
\frac{n_h}{n_c} = \left( \frac{(1 - \mu_h - \mu_b - \mu_l) qI X}{(1 - \mu_c) Y} \right) \frac{1}{1+\xi}
\]

plug back to get

\[
(1 - \mu_c) \alpha \frac{Y}{X} = X_w \left( 1 + \frac{(1 - \mu_h - \mu_b - \mu_l) qI X}{(1 - \mu_c) Y} \right) \frac{q - \varepsilon}{1+\xi} n_c^{1+\eta}
\]

so knowing $\frac{Y}{c}$ and $\frac{qI}{Y}$, this can be solved for $n_c$, and consequently for all the variables of the model:

\[
n_c = \left( \frac{(1 - \mu_c) \alpha \frac{Y}{X} X_w}{1 + \frac{(1 - \mu_h - \mu_b - \mu_l) qI X}{(1 - \mu_c) Y} \frac{q - \varepsilon}{1+\xi}} \right)^{\frac{1}{1+\eta}}
\]

Similar formulas apply to $n_h, n_c'$ and $n_h'$. Once we know the levels of hours worked by the two households in the two sectors, we can compute $Y, c, c', k_c, k_h$ and the product $qI$. To find $q$ and $I$ separately we use:

\[
k_b = \mu_b q I
\]
and
\[ I = \left( A_h n_h^{\alpha} n_h^{1-\alpha} \right) (\zeta_1 q)^{1-\mu_h-\mu_b-n} \left( \mu_0 q \right)^{1-\mu_h-\mu_b-n} \]

Let \( qI \) be equal to \( \theta \), then:
\[
\begin{align*}
qI & = \theta \\
I & = (A_h n_h^{\alpha} n_h^{1-\alpha})^{1-\mu_h-\mu_i} (\zeta_1 \theta)^{\mu_h} (\mu_0 \theta)^{\mu_b}
\end{align*}
\]

Given the values of hours worked, we can use the production function in the goods sector to compute output \( Y \):
\[
Y =
\]

The levels of capital stock in the two sectors are respectively:
\[
\begin{align*}
k_c & = \zeta_0 Y \\
k_c & = \zeta_1 q I
\end{align*}
\]

the levels of consumption of the two agents:
\[
\begin{align*}
c & = \left( \frac{\chi_3 \chi_4 + \chi_2 \chi_6}{\chi_1 \chi_4 - \chi_2 \chi_5} \right) Y \\
c' & = \left( \frac{\chi_1 \chi_6 + \chi_3 \chi_5}{\chi_1 \chi_4 - \chi_2 \chi_5} \right) Y
\end{align*}
\]

and their stock of housing:
\[
\begin{align*}
h & = \zeta_2 \frac{c}{q} \\
h' & = \zeta_3 \frac{c'}{q}
\end{align*}
\]

Finally, the level of loans is:
\[
b = m q G Q \frac{h'}{r}
\]