B Option Pricing Intuition

This appendix provides additional intuition based on option pricing for the result in Proposition 2.

Consider the case of monopoly. At time one, the monopolist is selling a series of call options, or equivalently units bundled with put options, rather than units themselves. The marginal price charged for a unit \( q \) at time two is simply the strike price of the option sold on unit \( q \) at time one. The series of call options being sold are interrelated; a call option for unit \( q \) can’t be exercised unless the call option for unit \( q - 1 \) has already been exercised. However, it is useful to consider the market for each option independently. According to Proposition 2 (equation 8), when there is no pooling and the satiation constraint is not binding, the optimal marginal price for a unit \( q \) is:

\[
\hat{P}_q(q) = C_q(q) + V_q\phi(q, \hat{\theta}(q)) \frac{F*\hat{\theta}(q) - F(\hat{\theta}(q))}{f(\hat{\theta}(q))}
\]  

(21)

It turns out that this is exactly the strike price that maximizes the net value of a call or put option on unit \( q \) given the difference in priors between the two parties.\(^{37}\)

To show this explicitly, write the net value \( NV \) of a call option on minute \( q \) as the difference between the consumers’ value of the option \( CV \) and the firm’s cost of providing that option, \( FV \). The option will be exercised whenever the consumer values unit \( q \) more than the strike price \( p \), that is whenever \( V_q(q, \theta) \geq p \). Let \( \theta(p) \) denote the minimum type who exercises the call option, characterized by the equality \( V_q(q, \theta(p)) = p \).

\(^{37}\)This parallels Mussa and Rosen’s (1978) finding in their static screening model, that the optimal marginal price for unit \( q \) is identical to the optimal monopoly price for unit \( q \) if the market for unit \( q \) were treated independently of all other units.
The consumers’ value for the option is their expected value received upon exercise, less the expected strike price paid, where expectations are based on the consumers’ prior $F^*(\theta)$:

$$CV(p) = \int_{\theta_{(p)}}^{\bar{\theta}} V_q(q, \theta) f^*(\theta) d\theta - [1 - F^*(\theta(p))] p$$

The firm’s cost of providing the option is the probability of exercise based on the firm’s prior $F(\theta)$ times the difference between the cost of unit $q$ and the strike price received:

$$FV(p) = [1 - F(\theta(p))] (c - p)$$

Putting these two pieces together, the net value of the call option is equal to the consumers’ expected value of consumption less the firm’s expected cost of production plus an additional term due to the gap in perceptions:

$$NV(p) = \int_{\theta_{(p)}}^{\bar{\theta}} V_q(q, \theta) f^*(\theta) d\theta - [1 - F(\theta(p))] c + [F^*(\theta(p)) - F(\theta(p))] p$$

The additional term $[F^*(\theta(p)) - F(\theta(p))] p$ represents the difference between the exercise payment the firm expects to receive and the consumer expects to pay. The term $[F^*(\theta(p)) - F(\theta(p))]$ represents the disagreement between the parties about the probability of exercise.

As a monopolist selling call options on unit $q$ earns the net value $NV(p)$ of the call option by charging the consumer $CV(p)$ upfront, a monopolist should set the strike price $p$ to maximize $NV(p)$. By the implicit function theorem, $\frac{d}{dp} \theta(p) = \frac{1}{V_{q\theta}(q, \theta(p))}$, so the first order condition which characterizes the optimal strike price is:

$$\frac{f(\theta(p))}{V_{q\theta}(q, \theta(p))} [p - C_q(q)] = [F^*(\theta(p)) - F(\theta(p))]$$

(22)

As claimed earlier, this is identical to the characterization of the optimal marginal price $\hat{P}_q(q)$ for the complete nonlinear pricing problem when monotonicity and satiation constraints are not binding (equation 21).

Showing that the optimal marginal price for unit $q$ is given by the optimal strike price for a call option on unit $q$ is useful, because the first order condition $\Psi_q(q, \theta) = 0$ can be interpreted in the option pricing framework. Consider the choice of exercise price $p$ for an option on unit $q$. A small change in the exercise price has two effects. First, if a consumer is on the margin, it will change the consumers’ exercise decision. Second, it changes the payment made upon exercise by all infra-marginal consumers. In a common-prior model, the infra-marginal effect would net to zero as
the payment is a transfer between the two parties. This is not the case here, however, as the two parties disagree on the likelihood of exercise by \([F^* (\theta (p)) - F (\theta (p))]\).

Consider the first order condition as given above in equation (22). On the left hand side, the term \(\frac{f(\theta(p))}{q \theta(q, \theta(p))} [p - C_q (q)]\) represents the probability that the consumer is on the margin and that a marginal increase in the strike price \(p\) would stop the consumer exercising. The term \([p - C_q (q)]\) is the cost to the firm if the consumer is on the margin and no longer exercises. There is no change in the consumer’s value of the option by a change in exercise behavior at the margin, as the margin is precisely where the consumer is indifferent to exercise (\(V_q (q, \theta (p)) = p\)).

On the right hand side, the term \([F^* (\theta (p)) - F (\theta (p))]\) is the firm’s gain on infra-marginal consumers from charging a slightly higher exercise price. This is because consumers believe they will pay \([1 - F^* (\theta (p))]\) more in exercise fees, and therefore are willing to pay \([1 - F^* (\theta (p))]\) less upfront for the option. However the firm believes they will actually pay \([1 - F (\theta (p))]\) more in exercise fees, and the difference \([F^* (\theta (p)) - F (\theta (p))]\) is the firm’s perceived gain.

The first order condition requires that at the optimal strike price \(p\), the cost of losing marginal consumers \(\frac{f(\theta(p))}{q \theta(q, \theta(p))} [p - C_q (q)]\) is exactly offset by the "perception arbitrage" gain on infra-marginal consumers \([F^* (\theta (p)) - F (\theta (p))]\). Setting the strike price above or below marginal cost is always costly because it reduces efficiency.

In the discussion above, referring to \([F^* (\theta (p)) - F (\theta (p))]\) as a "gain" to the firm for a marginal increase in strike price implies that the term \([F^* (\theta (p)) - F (\theta (p))]\) is positive. This is the case for \(\theta (p) > \theta^*\), when consumers underestimate their probability of exercise. In this case, from the firm’s perspective, raising the strike price above marginal cost increases profits on infra-marginal consumers thereby effectively exploiting the perception gap. On the other hand, for \(\theta (p) < \theta^*\), the term \([F^* (\theta (p)) - F (\theta (p))]\) is negative and consumers overestimate their probability of exercise. In this case reducing the strike price below marginal cost exploits the perception gap between consumers and the firm.

Fixing \(\theta\) and the firm’s prior \(F(\theta)\), the absolute value of the perception gap is largest when the consumer’s prior is at either of two extremes, \(F^* (\theta) = 1\) or \(F^* (\theta) = 0\). When \(F^* (\theta) = 1\), the optimal marginal price reduces to the monopoly price for unit \(q\) where the market for minute \(q\) is independent of all other units. This is because consumers believe there is zero probability that they will want to exercise a call option for unit \(q\). The firm cannot charge anything for an option at time one; essentially the firm must wait to charge the monopoly price until time two when consumers realize their true value.

Similarly, when \(F^* (\theta) = 0\), the optimal marginal price reduces to the monopsony price for unit \(q\). Now rather than thinking of a call option, think of the monopolist as selling a bundled unit and
put option at time one. In this case consumers believe they will consume the unit for sure and exercise the put option with zero probability. This means that the firm cannot charge anything for the put option upfront, and must wait until time two when consumers learn their true values and buy units back from them at the monopsony price. The firm’s ability to do so is of course limited by free disposal which means the firm could not buy back units for a negative price.

Marginal price can therefore be compared to three benchmarks. For all quantities $q$, the marginal price will lie somewhere between the monopoly price $p^{ml}(q)$ and the maximum of the monopsony price $p^{mS}(q)$ and zero, hitting either extreme when $F^*(\theta) = 1$ or $F^*(\theta) = 0$, respectively. When $F^*(\theta) = F(\theta)$, marginal price is equal to marginal cost. To illustrate this point, the equilibrium marginal price for the running example with positive marginal cost $c = $0.035 and low overconfidence $\Delta = 0.25$ previously shown in Figure 4, plot C is replotted with the monopoly and monopsony prices for comparison in Figure 10.

![Figure 10](image-url)

Figure 10: Equilibrium pricing for Example 1 given $c = $0.035 and $\Delta = 0.25$: Marginal price is plotted along with benchmarks: (1) marginal cost, (2) ex post monopoly price - the upper bound, and (3) ex post monopsony price - the lower bound.

C Pooling

As it was omitted from the main text, a characterization of pooling quantities when the monotonicity constraint binds is provided below in Lemma 4. This is useful because it facilitates the calculation of pooling quantities in numerical examples.

**Lemma 4** On any interval $[\theta_1, \theta_2]$ such that monotonicity (but not non-negativity) is binding inside, but not just outside the interval, the equilibrium allocation is constant at some level $\hat{q} > 0$ for all $\theta \in [\theta_1, \theta_2]$. Further, the pooling quantity $\hat{q}$ and bounds of the pooling interval $[\theta_1, \theta_2]$ must main-
tain continuity of \( \hat{q}(\theta) \) and satisfy the first order condition on average: 
\[
\int_{\theta_1}^{\theta_2} \Psi_q (\hat{q}, \theta) f (\theta) \, d\theta = 0.
\]
Non-negativity binds inside, but not just above, the interval \([\theta_1, \theta_2]\) only if \( \int_{\theta_1}^{\theta_2} \Psi_q (0, \theta) f (\theta) \, d\theta \leq 0 \) and \( q^R (\theta_2) = 0 \).

Proof. Given the result in Lemma 2, the proof is omitted as it closely follows ironing results for the standard screening model. The result follows from the application of standard results in optimal control theory (Seierstad and Sydsæter 1987). See for example the analogous proof given in Fudenberg and Tirole (1991), appendix to chapter 7. ■

Note that because the virtual surplus function is strictly quasi-concave, but not necessarily strictly concave, optimal control results yield necessary rather than sufficient conditions for the optimal allocation. For the special case \( V_{qq} = 0 \), virtual surplus is strictly-concave and applying the ironing algorithm suggested by Fudenberg and Tirole (1991) using the conditions in Lemmas 2 and 4 is sufficient to identify the uniquely optimal allocation.

Proposition 2 characterizes marginal pricing at quantities for which there is no pooling, and states that marginal price will jump discretely upwards at quantities where there is pooling. It is therefore interesting to know when \( q^R (\theta) \) will be locally decreasing, so that the equilibrium allocation \( \hat{q}(\theta) \) involves pooling. First, a preliminary result is helpful. Proposition 5 compares the relaxed allocation to the first best allocation, showing that the relaxed allocation is above first best whenever \( F (\theta) > F^* (\theta) \) and is below first best whenever \( F (\theta) < F^* (\theta) \):

**Proposition 5** Given maintained assumptions:

\[
q^R (\theta) \begin{cases} 
\geq q^{FB} (\theta) & F (\theta) > F^* (\theta) \\
= q^{FB} (\theta) & F (\theta) = F^* (\theta) \\
< q^{FB} (\theta) & F (\theta) < F^* (\theta)
\end{cases}
\]

(\( \diamond \)) strict iff \( \text{C}_q \left( q^{FB} (\theta) \right) > 0 \)

Proof. See Section C.1 at the end of this appendix. ■

Given that \( F^* (\theta) \) crosses \( F (\theta) \) once from below, the relationship between the relaxed allocation and the (strictly increasing) first best allocation given in Proposition 5 leads to the conclusion that \( q^R (\theta) \) is strictly increasing near the bottom \( \theta \) and near the top \( \theta \). More can be said about pooling when consumers either have nearly correct beliefs, or are extremely overconfident. When consumers’ prior is close to that of the firm, the relaxed solution is close to first best, and like first best is strictly increasing. In this case the equilibrium and relaxed allocations are identical.

When consumers are extremely overconfident such that their prior is close to the belief that \( \theta = \theta^* \) with probability one, the relaxed solution is strictly decreasing at or just above \( \theta^* \). In this
The intuition is that when the consumers’ prior is exactly the belief that $\theta = \theta^*$ with probability one, $[F(\theta) - F^*(\theta)]$ falls discontinuously below zero at $\theta^*$. Thus, by Proposition 5, the relaxed solution must drop discontinuously from weakly above first best just below $\theta^*$ to strictly below first best at $\theta^*$. These results and the notion of "closeness" are made precise in Proposition 6.

**Proposition 6** (1) There exists $\varepsilon > 0$ such that if $|f^*(\theta) - f(\theta)| < \varepsilon$ for all $\theta$ then $q^R(\theta)$ is strictly increasing for all $\theta$. (2) There exists a finite constant $\gamma$, such that if $f^*(\theta^*) > \gamma$ and $f^*(\theta)$ is continuous at $\theta^*$ then $q^R(\theta)$ is strictly decreasing just above $\theta^*$.

**Proof.** If neither non-negativity nor satiation constraints bind at $\theta$, Lemma 2 and the implicit function theorem imply that $\frac{d}{d\theta} q^R(\theta) = \frac{-\Psi_{q\theta}(q^R(\theta), \theta)}{\Psi_{qq}(q^R(\theta), \theta)}$. In this case the sign of the cross partial derivative $\Psi_{q\theta}$ determines whether $q^R(\theta)$ is increasing or decreasing. For any $\theta$ at which $F^*$ is sufficiently close to $F$ in both level and slope, $\Psi_{q\theta} > 0$, but when $f^*(\theta)$ is sufficiently large $\Psi_{q\theta} < 0$. See the proof at the end of this appendix for details.

### C.1 Pooling Appendix Proofs

#### C.1.1 Small Lemma 5

**Lemma 5** Satiation $q^S(\theta)$ and first best $q^{FB}(\theta)$ quantities are continuously differentiable, strictly positive, and strictly increasing. Satiation quantity is higher than first best quantity, and strictly so when marginal costs are strictly positive at $q^{FB}$.

**Proof.** Satiation and first best quantities are strictly positive because by assumption: $V_q(0, \theta) > C_q(0) \geq 0$. Therefore given maintained assumptions, $q^S(\theta)$ and $q^{FB}(\theta)$ exist, and are continuous functions characterized by the first order conditions $V_q(q^S(\theta), \theta) = 0$ and $V_q(q^{FB}(\theta), \theta) = C_q(q^{FB}(\theta))$ respectively. Zero marginal cost at $q^{FB}(\theta)$ implies $q^S(\theta) = q^{FB}(\theta)$. When $C_q(q^{FB}(\theta)) > 0$, $V_{qq}(q, \theta) < 0$ implies that $q^S(\theta) > q^{FB}(\theta)$. The implicit function theorem implies $\frac{d}{d\theta} q^S = -\frac{V_{q\theta}}{V_{qq}} > 0$ and $\frac{d}{d\theta} q^{FB} = -\frac{V_{q\theta}}{V_{qq} - C_{qq}} > 0$. ■

#### C.1.2 Proof of Proposition 5

**Proof.** The relaxed allocation maximizes virtual surplus $\Psi(q, \theta)$ within the constraint set $[0, q^S(\theta)]$ and $\Psi(q, \theta)$ is strictly quasi-concave in $q$ (See proof of Lemma 2). Moreover, $q^{FB} \in (0, q^S(\theta)]$ and $\Psi_q(q^{FB}(\theta), \theta) = V_{q\theta}(q^{FB}(\theta), \theta) \frac{F(\theta) - F^*(\theta)}{f(\theta)}$ since the first best allocation satisfies $V_q(q^{FB}(\theta), \theta) = C_q(q^{FB}(\theta))$ (Lemma 5). Therefore there are three cases to consider:

1. $F(\theta) = F^*(\theta)$: In this case virtual surplus and true surplus are equal so $q^R(\theta) = q^{FB}(\theta)$.
2. \( F(\theta) > F^*(\theta) \): In this case \( \Psi_q(q^{FB}(\theta), \theta) > 0 \) and therefore \( q^R(\theta) \geq q^{FB}(\theta) \). When \( C_q(q^{FB}(\theta)) = 0 \), Lemma 5 shows that \( q^S(\theta) = q^{FB}(\theta) \) and therefore the satiation constraint binds: \( q^R(\theta) = q^{FB}(\theta) = q^S(\theta) \). When \( C_q(q^{FB}(\theta)) > 0 \), Lemma 5 shows that satiation is not binding at first best, and therefore the comparison is strict: \( q^R(\theta) > q^{FB}(\theta) \).

3. \( F(\theta) < F^*(\theta) \): In this case \( \Psi_q(q^{FB}(\theta), \theta) < 0 \) and therefore \( q^R(\theta) < q^{FB}(\theta) \) since, by Lemma 5, non-negativity is not binding at first best \( (q^{FB}(\theta) > 0) \).

C.1.3 Proof of Proposition 6

Proof. If, over the interval \((\theta_1, \theta_2)\), neither non-negativity nor satiation constraints bind and \( \Psi_q(q, \theta) \) is continuously differentiable, then in the same interval \( \Psi_q(q^R, \theta) = 0 \) (Lemma 2) and the implicit function theorem implies \( \frac{d}{d\theta} q^R(\theta) = -\frac{\Psi_{q\theta}(q^R(\theta), \theta)}{\Psi_{qq}(q^R(\theta), \theta)} \). (Given maintained assumptions, \( \Psi_q(q, \theta) \) is continuously differentiable at \( \theta \) where \( f^*(\theta) \) is continuous.) Therefore, in the same interval \( q^R(\theta) \) will be strictly increasing if \( \Psi_{q\theta}(q^R(\theta), \theta) > 0 \) and strictly decreasing if \( \Psi_{q\theta}(q^R(\theta), \theta) < 0 \).

Part (1): As the non-negativity constraint is not binding at \( \theta \) \( (q^R(\theta) = q^{FB}(\theta) > 0) \), the upper bound \( q^S(\theta) \) is strictly increasing, and \( q^R(\theta) \) is continuous, \( q^R(\theta) \) will be strictly increasing for all \( \theta \) if the cross partial derivative \( \Psi_{q\theta}(q, \theta) \) is strictly positive for all \( (q, \theta) \in [0, q^S(\theta)] \times [\theta, \bar{\theta}] \).

Define \( \varphi(q, \theta) \) and \( \varepsilon \): (Note that \( \varepsilon \) is well defined since \( [0, q^S(\theta)] \times [\theta, \bar{\theta}] \) is compact and \( F \in C^2 \) and \( V(q, \theta) \in C^3 \) imply \( \varphi(q, \theta) \) is continuous.)

\[
\varphi(q, \theta) \equiv \frac{1}{f(\theta)} + \frac{(\theta - \bar{\theta})}{f(\theta)} \left( \frac{|V_{q\theta}(q, \theta)|}{V_{q\theta}(q, \theta)} + \frac{|\frac{d}{d\theta} f(\theta)|}{f(\theta)} \right) > 0
\]

\[
\varepsilon \equiv \min_{(q, \theta) \in [0, q^S(\bar{\theta})] \times [\theta, \bar{\theta}]} \frac{1}{\varphi(q, \theta)}
\]

By differentiation:

\[
\Psi_{q\theta}(q, \theta) = V_{q\theta\theta}(q, \theta) \frac{F(\theta) - F^*(\theta)}{f(\theta)} + V_{q\theta}(q, \theta) \left( 1 + \frac{f(\theta) - f^*(\theta)}{f(\theta)} - \frac{\frac{d}{d\theta} f(\theta)}{f^2(\theta)} [F(\theta) - F^*(\theta)] \right)
\]

As \( f(\theta) > 0 \) and \( V_{q\theta}(q, \theta) > 0 \):

\[
\Psi_{q\theta}(q, \theta) \geq -|V_{q\theta\theta}(q, \theta)| \frac{|F(\theta) - F^*(\theta)|}{f(\theta)} + V_{q\theta}(q, \theta) \left( 1 - \frac{|f(\theta) - f^*(\theta)|}{f(\theta)} - \frac{|\frac{d}{d\theta} f(\theta)|}{f^2(\theta)} |F(\theta) - F^*(\theta)| \right)
\]
The assumption \(|f(\theta) - f^*(\theta)| < \varepsilon\) implies that \(|F(\theta) - F^*(\theta)| < \varepsilon(\bar{\theta} - \underline{\theta})\) and therefore that:

\[
\Psi_{q\theta}(q, \theta) > V_{q\theta}(q, \theta) (1 - \varepsilon \varphi(q, \theta))
\]

By the definition of \(\varepsilon\), this implies \(\Psi_{q\theta}(q, \theta) > 0\) for all \((q, \theta) \in [0, q^S(\bar{\theta})] \times [\underline{\theta}, \bar{\theta}]\).

**Part (2):** The first step is to show that \(\Psi_{q\theta}(q, \theta^*) < 0\). Define \(\gamma\):

\[
\gamma \equiv \max_{q \in [0, q^S(\bar{\theta})]} \left( \frac{|V_{q\theta}(q, \theta^*)|}{V_{q\theta}(q, \theta^*)} + 2 f(\theta^*) + \frac{|d f(\theta^*)|}{f(\theta^*)} \right)
\]

As \(f(\theta) > 0\) and \(V_{q\theta}(q, \theta) > 0\):

\[
\Psi_{q\theta}(q, \theta) \leq |V_{q\theta}(q, \theta)| \frac{|F(\theta) - F^*(\theta)|}{f(\theta)} + V_{q\theta}(q, \theta) \left( 2 - \frac{f^*(\theta)}{f(\theta)} + \frac{|d f(\theta^*)|}{f(\theta^*)} |F(\theta) - F^*(\theta)| \right)
\]

since \(|F(\theta) - F^*(\theta)| \leq 1\):

\[
\Psi_{q\theta}(q, \theta) \leq \frac{V_{q\theta}(q, \theta)}{f(\theta)} \left( \frac{|V_{q\theta}(q, \theta)|}{V_{q\theta}(q, \theta^*)} + 2 f(\theta) + \frac{|d f(\theta)|}{f(\theta)} - f^*(\theta) \right)
\]

By definition of \(\gamma\) and \(f^*(\theta^*) > \gamma\), it follows that for all \(q \in [0, q^S(\bar{\theta})]\): \(\Psi_{q\theta}(q, \theta^*) < 0\). Given continuity of \(f^*(\theta)\) at \(\theta^*\), \(\Psi_{q\theta}(q, \theta)\) is also continuous at \(\theta^*\). Therefore for some \(\delta_1 > 0\), \(\Psi_{q\theta}(q, \theta) < 0\) just above \(\theta^*\) in the interval \(\theta \in [\theta^*, \theta^* + \delta_1]\).

The second step is to show that for some \(\delta_2 > 0\) neither satiation nor non-negativity constraints are binding in the interval \((\theta^*, \theta^* + \delta_2)\). First, satiation is not binding just above \(\theta^*\) as \(q^R\) is below first best here (Proposition 5), which is always below the satiation bound (Lemma 5). Second, non-negativity is not binding just to the right of \(\theta^*\) because \(q^R(\theta^*) = q^{FB}(\theta^*) > 0\) (Proposition 5 and Lemma 5) and \(q^R(\theta)\) is continuous (Lemma 2).

Steps one and two imply that \(q^R(\theta)\) is strictly decreasing in the interval \((\theta^*, \theta^* + \min\{\delta_1, \delta_2\})\) just above \(\theta^*\). Therefore, \(q^R(\theta)\) is either strictly decreasing at \(\theta^*\) or has a kink at \(\theta^*\) and is strictly decreasing just above \(\theta^*\). In either case, monotonicity is violated at \(\theta^*\) and the equilibrium allocation will involve pooling at \(\theta^*\). ■

D  Monopoly Multi-Tariff Menu Extension

In this appendix, I extend the single tariff model explored in detail in the main paper to a multi-tariff monopoly model. The model is described in Section 5 of the paper. (Note that I replace equation (1) by the stricter assumption \(V_{q\theta} = 0\).) The model and solution methods are closely
related to Courty and Li (2000). To work with the new problem, first define:

**Definition 2** (1) Let \( q^c(s, \theta, \theta') = \min \{q(s, \theta'), q^S(\theta)\} \) be the consumption quantity of a consumer who chooses tariff \( s \), is of type \( \theta \), and reports type \( \theta' \). Consumption quantity of a consumer who honestly reports true type \( \theta \) is \( q^c(s, \theta) \equiv q^c(s, \theta, \theta) \). (2) \( u(s, \theta, \theta') \equiv V(q^c(s, \theta, \theta'), \theta) - P(s, \theta') \) is the utility of a consumer who chooses tariff \( s \), is of type \( \theta \), and reports type \( \theta' \). (3) \( u(s, \theta) \equiv u(s, \theta, \theta) \) is the utility of a consumer who chooses a tariff \( s \), and honestly reports true type \( \theta \). (4) \( U(s, s') = \int_{\theta} \min \{u(s', \theta) f(\theta|s) d\theta \} \) is the true expected utility of a consumer who receives signal \( s \), chooses tariff \( s' \), and later reports \( \theta \) honestly. \( U^*(s, s') \equiv \int_{\theta} \min \{u(s', \theta) f^*(\theta|s) d\theta \} \) is the analogous perceived expected utility. (5) \( U(s) \equiv U(s, s) \) is the expected utility of a consumer who honestly chooses the intended tariff \( s \) given signal \( s \), and later reports \( \theta \) honestly. \( U^*(s) \equiv U^*(s, s) \) is the analogous perceived expected utility.

Invoking the revelation principle, the monopolist’s problem may then be written as:

\[
\max_{q(s, \theta) \geq 0, P(s, \theta)} E \left[ \int_{\theta} (P(s, \theta) - C(q(s, \theta))) f(\theta|s) d\theta \right]
\]

such that

1. Global IC-2 \( u(s, \theta, \theta) \geq u(s, \theta, \theta') \) \( \forall s \in S, \forall \theta, \theta' \in \Theta \)
2. Global IC-1 \( U^*(s, s) \geq U^*(s, s') \) \( \forall s, s' \in S \)
3. Participation \( U^*(s) \geq 0 \) \( \forall s \in S \)

As the signal \( s \) does not enter the consumers value function directly, the second period incentive compatibility constraints may be handled just as they are in the single tariff model (See part 1 of the proof of Proposition 1). In particular, second period local incentive compatibility \( \left( \frac{\partial}{\partial \theta} u(s, \theta) = V_\theta(q^c(s, \theta), \theta) \right) \) and monotonicity \( \left( \frac{\partial}{\partial \theta} q^c(s, \theta) \geq 0 \right) \) are necessary and sufficient for second period global incentive compatibility. Moreover, Lemma 1 naturally extends to the multi-tariff setting. Applying the same satiation refinement \( q(s, \theta) \leq q^S(\theta) \), the distinction between \( q^c(s, \theta) \) and \( q(s, \theta) \) may be dropped.

The next step is to express payments in terms of consumer utility, and substitute in the second period local incentive constraint, just as was done in the single-tariff model (See parts 2 and 3 of the proof of Proposition 1). This yields analogs of equations (3, 15, 6):

\[
U^*(s, s') = u(s', \theta) + E \left[ V_\theta(q(s', \theta), \theta) \frac{1 - F^*(\theta|s)}{f(\theta|s)} \right] \tag{23}
\]

\[
U^*(s) - U(s) = E \left[ V_\theta(q(s, \theta), \theta) \frac{F(\theta|s) - F^*(\theta|s)}{f(\theta|s)} \right] \tag{24}
\]
\[ P(s, \theta) = V(q(s, \theta), \theta) - \int_{q}^{\theta} V_{\theta}(q(s, z), z) dz - u(s, \theta) \] (25)

In the single tariff model, the utility of the lowest type \( \theta \) was determined by the binding participation constraint. To pin down the utilities \( u(s, \theta) \) in the multi-tariff model requires consideration of both participation and first-period incentive constraints given an assumed ordering of the signal space \( S \). I will consider signal spaces \( S \) which are ordered either by FOSD, or a more general reverse second order stochastic dominance (RSOSD).\(^{38}\) Note that I assume the ordering, either FOSD or RSOSD, applies to consumer beliefs \( F^*(\theta|s) \).

In the standard single period screening model the participation constraint will bind for the lowest type, and this guarantees that it is satisfied for all higher types. The same is true here given the assumed ordering of \( S \), as stated in Lemma 6.

**Lemma 6** If \( S \) is ordered by FOSD then the participation constraint binds at the bottom \( (U^*(s) = 0) \). This coupled with first period incentive constraints are sufficient for participation to hold for all higher types \( s > s_0 \). The same is true if \( S \) is ordered by RSOSD and \( V_{\theta \theta} \geq 0 \).

**Proof.** It is sufficient to show that \( U^*(s, s') \) is non-decreasing in \( s \). If this is true, then IC-1 and \( U^*(s) \geq 0 \) imply participation is satisfied: \( U^*(s, s) \geq U^*(s, s') \geq 0 \). Hence if \( U^*(s) \geq 0 \) were not binding, profits could be increased by raising fixed fees of all tariffs. Now by definition, \( U^*(s, s') \equiv E[u(s', \theta)|s] \). By local IC-2 and \( V_{\theta} \geq 0 \) (which follows from \( V(0, \theta) = 0 \) and \( V_{\theta \theta} > 0 \)), it is clear that conditional on tariff choice \( s' \), consumers' utility is non-decreasing in realized \( \theta \):

\[ u_{\theta}(s', \theta) = V_{\theta}(q(s', \theta), \theta) \geq 0 \]

Given a FOSD ordering, this is sufficient for \( U^*(s, s') \) to be non-decreasing in \( s \) (Hadar and Russell 1969, Hanoch and Levy 1969).

Taking a second derivative of consumer second period utility shows that conditional on tariff choice \( s' \), consumers' utility is convex in \( \theta \) if \( V_{\theta \theta} \geq 0 \). This follows from increasing differences \( V_{\theta \theta} > 0 \), monotonicity \( q_{\theta}(s', \theta) \geq 0 \), and local IC-2:

\[ u_{\theta \theta}(s', \theta) = V_{\theta \theta}(q(s', \theta), \theta) q_{\theta}(s', \theta) + V_{\theta \theta}(q(s', \theta), \theta) \geq 0 \]

\(^{38}\) Courty and Li (2000) restrict attention primarily to orderings by first order stochastic dominance or mean preserving spread. However, in their two type model, they do mention that orderings can be constructed from the combination of the two which essentially allows for the more general reverse second order stochastic dominance orderings I consider here.
Under RSOSD ordering, this implies that \( U^* (s) \) is increasing in \( s \). (This result is analogous to the standard result that if \( X \) second order stochastically dominates \( Y \) then \( E [h (X)] \geq E [h (Y)] \) for any concave utility \( h \) (Rothschild and Stiglitz 1970, Hadar and Russell 1969, Hanoch and Levy 1969). The proof is similar and hence omitted.)

Given Lemma 6 and the preceding discussion, the monopolist’s problem can be simplified as described in Lemma 7.

**Lemma 7** Given a FOSD ordering of \( S \), or a RSOSD ordering of \( S \) and \( V_{0\theta} \geq 0 \), the monopolist’s problem reduces to the following constrained maximization over allocations \( q(s, \theta) \) and utilities \( u(s, \theta) \) for \( U^* (s) = U^* (s, s) \) and \( U^* (s, s') \) given by equation (23):

\[
\max_{q(s, \theta) \in [0, q^0(\theta)]} \frac{E[\Psi(s, q(s, \theta), \theta)] - E[U^*(s)]}{u(s, \theta)}
\]

such that

1. **Monotonicity** \( q(s, \theta) \) non-decreasing in \( \theta \)
2. **Global IC-1** \( U^*(s, s) \geq U^*(s, s') \) \( \forall s, s' \in S \)
3. **Participation** \( U^*(s) = 0 \)

\[
\Psi(s, q, \theta) \equiv V(q, \theta) - C(q) + V_{0\theta}(q, \theta) \frac{F(\theta | s) - F^*(\theta | s)}{f(\theta | s)}
\]

Payments \( P(s, \theta) \) are given as a function of the allocation \( q(s, \theta) \) and utilities \( u(s, \theta) \) by equation (25). Second period local incentive compatibility \( (\frac{\partial}{\partial \theta} u(q, \theta) = V_{0\theta}(q(s, \theta), \theta)) \) always holds for the described payments.

**Proof.** Firm profits can be re-expressed as shown in (equation 26).

\[
E[\Pi(s, \theta)] = E[S(s, \theta)] + E[U^*(s) - U(s)] - E[U^*(s)]
\]

Substituting equation (24) for fictional surplus gives the objective function. The participation constraint follows from Lemma 6. For handling of the satiation constraint and second period incentive compatibility, refer to discussion in the text and proofs of Lemma 1 and Proposition 1.

To characterize the solution to the monopolist’s problem described by Lemma 7, I now proceed to analyze a two type \( (s \in \{L, H\}) \) model and a continuum of types \( (s \in [s, \bar{s}] \) model separately.
D.1 Two-Tariff Menu

Assume that there are two possible first-period signals \( s \in \{L, H\} \), and that the probability of signal \( H \) is \( \alpha \). There are two first-period incentive constraints. Given Lemma 6, the downward incentive constraint \( U^*(H, H) \geq U^*(H, L) \) (IC-H) must be binding. Otherwise the monopolist could raise the fixed fee \( P(H, \theta) \), and increase profits. Together with equation (23) and the binding participation constraint \( U^*(L) = 0 \) (IR-L), this pins down \( U^*(H) \) as a function of the allocation \( q(L, \theta) \). Substituting the binding IR-L and IC-H constraints for \( u(L, \theta) \) and \( u(H, \theta) \) into both the monopolist’s objective function described in Lemma 7 and the remaining upward incentive constraint \( U^*(L, L) \geq U^*(L, H) \) (IC-L) delivers a final simplification of the monopolist’s problem in Proposition 7.

**Proposition 7** Define a new virtual surplus \( \Phi(q_L, q_H, \theta) \) and function \( \zeta(q_L, q_H, \theta) \) by equations (27) and (28) respectively.

\[
\Phi(q_L, q_H, \theta) \equiv \alpha \Psi(H, q_H, \theta) f_H(\theta) + (1 - \alpha) \Psi(L, q_L, \theta) f_L(\theta) - \alpha V_\theta(q_L, \theta) (F_L^*(\theta) - F_H^*(\theta))
\]

(27)

\[
\zeta(q_L, q_H, \theta) \equiv [F_L^*(\theta) - F_H^*(\theta)] [V_\theta(q_L, \theta) - V_\theta(q_H, \theta)]
\]

(28)

Given either a FOSD ordering of \( S \), or a RSOSD ordering and \( V_{\theta \theta} \geq 0 \):

1. A monopolist’s optimal two-tariff menu solves the reduced problem:

\[
\max_{q_L(\theta), q_H(\theta) \in [0, q(S(\theta))] \int_0^\theta \Phi(q_L(\theta), q_H(\theta), \theta) d\theta
\]

such that

IC-2 \( q_L(\theta) \) and \( q_H(\theta) \) non-decreasing in \( \theta \)

IC-L \( \int_0^\theta \zeta(q_L(\theta), q_H(\theta), \theta) d\theta \geq 0 \)

2. Payments are given by equations (25), and (29-30):

\[
u(L, \theta) = -E \left[ V_\theta(q_L(\theta), \theta) \frac{1 - F_L^*(\theta)}{f_L(\theta)} \right] L
\]

(29)

\[
u(H, \theta) = E \left[ [V_\theta(q_L(\theta), \theta) - V_\theta(q_H(\theta), \theta)] \frac{1 - F_H^*(\theta)}{f_H(\theta)} \right] H + u(L, \theta)
\]

(30)

**Proof.** Beginning with the result in Lemma 7, the simplification is as follows: Equation (23) and \( U^*(L) = 0 \) imply equation (29). Equation (23) and \( U^*(H, H) = U^*(H, L) \) imply equation (30). By equation (23) and equations (29-30) \( U^*(H) \) is \( \int_0^\theta V_\theta(q_L(\theta), \theta) [F_L^*(\theta) - F_H^*(\theta)] d\theta \). As \( U^*(L) = 0 \), this means \( E[U^*(s)] = \alpha \int_0^\theta V_\theta(q_L(\theta), \theta) [F_L^*(\theta) - F_H^*(\theta)] d\theta \), which leads to the new term in the
revised virtual surplus function $\Phi$. Further, as $U^*(L) = 0$, the upward incentive constraint (IC-L) is $-U^*(L,H) \geq 0$. By equation (23) and equations (29-30) $U^*(L,H)$ is $\int_{\theta}^{\hat{\theta}} \zeta(q_L(\theta), q_H(\theta), \theta) d\theta$ which gives the new expression for the upward incentive constraint. ■

Given the problem described by Proposition 7, a standard approach would be to solve a relaxed problem which ignores the upward incentive constraint, and then check that the constraint is satisfied. Given a FOSD ordering, a sufficient condition for the resulting relaxed allocation to solve the full problem is for $q(s, \theta)$ to be non-decreasing in $s$. ( Sufficiency follows from IC-H binding and $V_{\theta} > 0$.) This is not the most productive approach in this case, because the sufficient condition is likely to fail with high levels of overconfidence. An alternative approach is to directly incorporate the upward incentive constraint into the maximization problem using optimal control techniques.

**Proposition 8** Given either a FOSD ordering or a RSOSD ordering and $V_{\theta \theta} \geq 0$: The equilibrium allocations $\hat{q}_L(\theta)$ and $\hat{q}_H(\theta)$ are continuous and piecewise smooth. For fixed $\gamma \geq 0$, define relaxed allocations (which ignore monotonicity constraints):

$$q^R_L(\theta) = \arg \max_{q_L \in [0,q^S(\theta)]} \left\{ \Psi (L, q_L, \theta) - \frac{\gamma + \alpha}{1 - \alpha} V_{\theta} (q_L, \theta) \frac{F^*_L(\theta) - F^*_H(\theta)}{f_L(\theta)} \right\}$$

$$q^R_H(\theta) = \arg \max_{q_H \in [0,q^S(\theta)]} \left\{ \Psi (H, q_H, \theta) + \frac{\gamma}{\alpha} V_{\theta} (q_H, \theta) \frac{F^*_L(\theta) - F^*_H(\theta)}{f_H(\theta)} \right\}$$

The relaxed allocations $q^R_L(\theta)$ and $q^R_H(\theta)$ are continuous and piecewise smooth functions characterized by their respective first order conditions except where satiation or non-negativity constraints bind. Moreover, there exists a non-negative constant $\gamma \geq 0$ such that:

1. On any interval over which a monotonicity constraint is not binding, the corresponding equilibrium allocation is equal to the relaxed allocation: $\hat{q}_s(\theta) = q^R_s(\theta)$.

2. On any interval $[\theta_1, \theta_2]$ such that the monotonicity constraint is binding inside, but not just outside the interval for tariff $s$, the equilibrium allocation is constant at some level $\hat{q}_s(\theta) = q'$ for all $\theta \in [\theta_1, \theta_2]$. Further, the pooling quantity $q'$ and bounds of the pooling interval $[\theta_1, \theta_2]$ must satisfy $q^R_s(\theta_1) = q^R_s(\theta_2) = q'$ and meet the first order condition from the relaxed problem in expectation over the interval. For instance, for $q_L(\theta)$ this second condition is:

$$\int_{\theta_1}^{\theta_2} \left\{ \Psi (L, \hat{q}, \theta) - \frac{\gamma + \alpha}{1 - \alpha} V_{\theta} (\hat{q}, \theta) \frac{F^*_L(\theta) - F^*_H(\theta)}{f_L(\theta)} \right\} f_L(\theta) d\theta = 0$$

3. Complementary slackness: $\gamma = 0$ or IC-L binds with equality ($\int_{\theta}^{\hat{\theta}} \zeta (\hat{q}_L(\theta), \hat{q}_H(\theta), \theta) d\theta = 0$).

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Proof. I apply Seierstad and Sydsæter (1987) Chapter 6 Theorem 13, which gives sufficient conditions for a solution. To apply the theorem, I first restate the problem described by Proposition 7 in the optimal control framework. This includes translating the upward incentive constraint into a state variable $k(\theta) = \int^\theta_0 -\zeta(q_L(z), q_H(z), z) \, dz$ with endpoint constraints $k(\bar{\theta}) = 0$ (automatically satisfied) and $k(\theta) \geq 0$. Remaining state variables are $q_L(\theta)$ and $q_H(\theta)$, which have free endpoints. Control variables are $c_L(\theta)$, $c_H(\theta)$, and $c_k(\theta)$, and the control set is $\mathbb{R}^3$. Costate variables are $\lambda_L(\theta)$, $\lambda_H(\theta)$, and $\lambda_k(\theta)$.

$$\max_{q_L, q_H} \int^\theta_0 \Phi(q_L(\theta), q_H(\theta), \theta) \, d\theta$$

<table>
<thead>
<tr>
<th>State</th>
<th>Control</th>
<th>Costate</th>
<th>Control - State Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_L(\theta)$</td>
<td>$c_L(\theta)$</td>
<td>$\lambda_L(\theta)$</td>
<td>$\dot{q}_L(\theta) = c_L(\theta)$</td>
</tr>
<tr>
<td>$q_H(\theta)$</td>
<td>$c_H(\theta)$</td>
<td>$\lambda_H(\theta)$</td>
<td>$\dot{q}_H(\theta) = c_H(\theta)$</td>
</tr>
<tr>
<td>$k(\theta)$</td>
<td>$c_k(\theta)$</td>
<td>$\lambda_k(\theta)$</td>
<td>$\dot{k}(\theta) = -\zeta(q_L, q_H, \theta)$</td>
</tr>
</tbody>
</table>

End Point Constraints Lagrangian Multipliers
(by definition) $k(0) = 0$ $\gamma$
IC-L $k(\bar{\theta}) \geq 0$ $\bar{\gamma}$

Control Constraints Lagrangian Multipliers
Monotonicity $c_L(\theta) \geq 0$ $\mu_L(\theta)$
$c_H(\theta) \geq 0$ $\mu_H(\theta)$

State Constraints Lagrangian Multipliers
Non-negativity $q_L(\theta) \geq 0$ $\eta_L(\theta), \eta_L, \bar{\eta}_L$
$q_H(\theta) \geq 0$ $\eta_H(\theta), \eta_H, \bar{\eta}_H$
Satiation $\left[q_S^S(\theta) - q_L(\theta)\right] \geq 0$ $\sigma_L(\theta), \sigma_L, \bar{\sigma}_L$
$\left[q_S^H(\theta) - q_H(\theta)\right] \geq 0$ $\sigma_H(\theta), \sigma_H, \bar{\sigma}_H$

The Hamiltonian and Lagrangian for this problem are:

$$H = \Phi(q_L, q_H, \theta) + \lambda_L(\theta) c_L(\theta) + \lambda_H(\theta) c_H(\theta) - \lambda_k(\theta) \zeta(q_L, q_H, \theta)$$

$$L = H + \mu_L(\theta) c_L(\theta) + \mu_H(\theta) c_H(\theta) + \eta_L(\theta) q_L(\theta) + \eta_H(\theta) q_H(\theta)$$
$$+ \sigma_L(\theta) \left[q_S^S(\theta) - q_L(\theta)\right] + \sigma_H(\theta) \left[q_S^H(\theta) - q_H(\theta)\right]$$

The following 7 conditions are sufficient conditions for a solution:

1. Control and state constraints are quasi-concave in states $q_L$, $q_H$, and $k$, as well as controls $c_L$,.
\(c_H\), and \(c_k\) for each \(\theta\). Endpoint constraints are concave in state variables. This is satisfied because all constraints are linear.

2. The Hamiltonian is concave in states \(q_L, q_H\), and \(k\), as well as controls \(c_L, c_H\), and \(c_k\) for each \(\theta\). This is satisfied because \(H\) is constant in \(k\) and \(c_k\), linear in \(c_L\) and \(c_H\), strictly concave in \(q_L\) and \(q_H\) (equations 33-34), and all other cross partials are zero. This relies on \(V_{qq\theta} = 0\) and \(S_{qq}(q, \theta) < 0\).

\[
\frac{\partial^2 H}{\partial q_L^2} = (1 - \alpha) S_{qq}(q_L, \theta) < 0 \quad \text{(33)}
\]

\[
\frac{\partial^2 H}{\partial q_H^2} = \alpha S_{qq}(q_H, \theta) < 0 \quad \text{(34)}
\]

3. State and costates are continuous and piecewise differentiable in \(\theta\). Lagrangian multiplier functions as well as controls are piecewise continuous in \(\theta\). All constraints are satisfied.

4. \[
\frac{\partial \lambda_L}{\partial c_L} = \frac{\partial \lambda_L}{\partial c_H} = \frac{\partial \lambda_L}{\partial c_k} = 0
\]

5. \[
d \frac{\partial \lambda_L}{\partial q_L}; \quad \frac{d}{d \theta} \lambda_L = -\frac{\partial \lambda_L}{\partial q_L}; \quad \frac{d}{d \theta} \lambda_H = -\frac{\partial \lambda_H}{\partial q_H}; \quad \frac{d}{d \theta} \lambda_k = -\frac{\partial \lambda_k}{\partial q_k}
\]

6. Complementary Slackness conditions

(a) For all \(\theta\):

\[
\mu_L, \mu_H, \eta_L, \eta_H, \sigma_L, \sigma_H \geq 0
\]

\[
\mu_L c_L = \mu_H c_H = \eta_L q_L = \eta_H q_H = \sigma_L [q_s - q_L] = \sigma_H [q_s - q_H] = 0
\]

(b) For \(\bar{\theta}\)

\[
\bar{\eta}_L, \bar{\eta}_H, \bar{\sigma}_L, \bar{\sigma}_H \geq 0
\]

\[
\bar{\eta}_L q_L(\bar{\theta}) = \bar{\eta}_H q_H(\bar{\theta}) = \bar{\sigma}_L [q_s(\bar{\theta}) - q_L(\bar{\theta})] = \bar{\sigma}_H [q_s(\bar{\theta}) - q_H(\bar{\theta})] = 0
\]

(c) For \(\underline{\theta}\)

\[
\underline{\eta}_L, \underline{\eta}_H, \underline{\sigma}_L, \underline{\sigma}_H \geq 0
\]

\[
\underline{\eta}_L q_L(\underline{\theta}) = \underline{\eta}_H q_H(\underline{\theta}) = \underline{\sigma}_L [q_s(\underline{\theta}) - q_L(\underline{\theta})] = \underline{\sigma}_H [q_s(\underline{\theta}) - q_H(\underline{\theta})] = 0
\]

(d) For IC-L

\[
\dot{\gamma} \geq 0
\]
\[ \gamma k(\theta) = \bar{\gamma}k(\bar{\theta}) = 0 \]

7. Transversality Conditions

\[
\begin{align*}
\lambda_L(\theta) &= \bar{\eta}_L - \bar{\sigma}_L; \\
\lambda_H(\theta) &= \bar{\eta}_H - \bar{\sigma}_H; \\
\lambda_k(\theta) &= \bar{\gamma} \\
\lambda_L(\bar{\theta}) &= -\bar{\eta}_L + \bar{\sigma}_L; \\
\lambda_H(\bar{\theta}) &= -\bar{\eta}_H + \bar{\sigma}_H \\
\lambda_k(\bar{\theta}) &= -\bar{\gamma}
\end{align*}
\]

Fix \( \gamma \geq 0 \). Let \( \lambda_k(\theta) = \bar{\gamma} = -\bar{\gamma} = \gamma \). As \( \lambda_k(\theta) \) is constant, it is continuous and differentiable, and \( \frac{\partial}{\partial \theta} \lambda_k = 0 \). Neither the state \( k(\theta) \) nor the control \( c_k(\theta) \) enter the Lagrangian, so \( \frac{\partial L}{\partial c_k} = -\frac{\partial H}{\partial c_k} = 0 \).

For any allocation, \( k(\theta) = 0 \) by definition, so \( \gamma k(\theta) = 0 \). Putting aside for now the constraint \( k(\bar{\theta}) \geq 0 \) and the complementary slackness condition \( \gamma k(\bar{\theta}) = 0 \), all other conditions concerning state \( k(\theta) \) are satisfied. Moreover, the Hamiltonian and Lagrangian are both additively separable in \( q_L \) and \( q_H \).

Hence the remaining conditions are identical to those for two independent maximization problems. Namely:

\[
\max_{q_L \in [0,q^\gamma(\theta)]} \left\{ \Psi(L,q_L,\theta) - \frac{\gamma + \alpha}{1 - \alpha} V_\theta(q_L,\theta) \frac{F^*_L(\theta) - F^*_H(\theta)}{f_L(\theta)} \right\}
\]

\[
\max_{q_H \in [0,q^\gamma(\theta)]} \left\{ \Psi(H,q_H,\theta) + \frac{\gamma}{\alpha} V_\theta(q_H,\theta) \frac{F^*_L(\theta) - F^*_H(\theta)}{f_H(\theta)} \right\}
\]

For fixed \( \gamma \geq 0 \), the solution to these problems exists, satisfies Proposition 8 parts 1-2, and meets all the relevant conditions in 3-7 above. The proof for this statement is omitted, because the solution for each subproblem given any fixed \( \gamma \geq 0 \), is similar to the single-tariff case, and closely parallels standard screening results. The idea is that for regions \( (\theta_1, \theta_2) \) where monotonicity is not binding for allocation \( \hat{q}_s(\theta) \), the \( \hat{q}_s(\theta) \) subproblem conditions are the Kuhn-Tucker conditions for the relaxed solution \( q^R_s(\theta) \). For regions in which monotonicity is binding, the characterization closely parallels Fudenberg and Tirole’s (1991) treatment of ironing in the standard screening model.

For the nice properties of the relaxed solution stated in the proposition, refer to the analogous proof of Proposition 2 part 1.

All that remains to show is that there exists a \( \gamma \geq 0 \), such that \( k(\bar{\theta}) \geq 0 \) and \( \gamma k(\bar{\theta}) = 0 \) given \( q_L(\theta) \) and \( q_H(\theta) \) that solve the respective subproblems for that \( \gamma \). If \( k(\bar{\theta}) \geq 0 \) for \( \gamma = 0 \) there is no problem. If \( k(\bar{\theta}) < 0 \) for \( \gamma = 0 \), the result follows from the intermediate value theorem and two observations: (1) \( k(\bar{\theta}) \) varies continuously with \( \gamma \), and (2) for \( \gamma \) sufficiently large \( k(\bar{\theta}) > 0 \).

The latter point follows because each subproblem is a maximization of \( E[\Phi(q_L,q_H,\theta)] + \gamma k(\bar{\theta}) \). As \( \gamma \) increases, the weight placed on \( k(\bar{\theta}) \) in the objective increases, and \( k(\bar{\theta}) \) moves towards its
maximum. Given FOSD, \( k(\theta) \) is maximized (given non-negativity, satiation, and monotonicity) at 
\( \{q_L(\theta), q_H(\theta)\} = \{0, q^S(\theta)\} \), for which \( q_H > q_L \) and therefore, by binding IC-II and \( V_{q\theta} > 0 \), IC-L must be strictly satisfied. The argument is more involved, but the constrained maximum of \( k(\theta) \) is also strictly positive given RSOSD. ■

**D.2 A Continuum of Tariffs**

For a continuum of first-period signals, Courty and Li (2000) characterize a necessary local condition for first period incentive compatibility. Its analog here is based on consumers perceived utility \( U^*(s) \) (Lemma 8).

**Lemma 8** Tariff \( \{q(s, \theta), P(s, \theta)\} \) satisfies period 1 incentive constraints only if

\[
\frac{dU^*(s)}{ds} = -\int_\theta^0 V_\theta(q(s, \theta), \theta) \frac{\partial F^*(\theta|s)}{\partial s} d\theta
\]

**Proof.** By the envelope theorem \( \frac{dU^*(s)}{ds} = \frac{\partial U^*(s)}{\partial s} \). Thus \( \frac{dU^*(s)}{ds} = \int_\theta^0 u(s, \theta) \frac{\partial f^*(\theta|s)}{\partial s} d\theta \). Integrating by parts and substituting the second period local incentive condition \( \frac{\partial u(s, \theta)}{\partial \theta} = V_\theta(q(s, \theta), \theta) \) then yields the desired result. ■

Now, applying the fundamental theorem of calculus, substituting the results of Lemmas 6 and 8, taking expectations, and integrating by parts yields the following expression for expected perceived utility \( E[U^*(s)] \):

\[
E[U^*(s)] = \int_\theta^0 \int_s^\theta V_\theta(q(s, \theta), \theta) (1 - G(s)) \frac{\partial}{\partial s} (1 - F^*(\theta|s)) d\theta ds \quad (35)
\]

Finally, combining equation (35) with Lemma 7, the monopolist’s problem may now be re-written as given in Proposition 9:

**Proposition 9** The monopolist’s problem may be re-expressed as the constrained maximization of expected virtual surplus \( \Phi(q(s, \theta)) \):

\[
\max_{q(s, \theta) \in [0, q^S(\theta)]} \int_\theta^0 \int_s^\theta \Phi(q(s, \theta), \theta) f(\theta|s) g(s) d\theta ds
\]

such that

1. monotonicity \( \frac{\partial}{\partial \theta} q(s, \theta) \geq 0 \)
2. Global IC-1 \( U^*(s, s') \geq U^*(s, s') \forall s, s' \in S \)

\[
\Phi(q(s, \theta)) = \Psi(q(s, \theta), V_\theta(q(s, \theta)) \frac{1 - G(s)}{g(s)} \frac{\partial}{\partial \theta} (1 - F^*(\theta|s)) \frac{f(\theta|s)}{f(\theta|s)}
\]

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Proof. Outlined in text. ■

One can see that virtual surplus now includes the information rent term \( \frac{1-G(s)}{g(s)} \frac{\partial}{\partial s} [1-F^*(\theta|s)] \) which arises in Courty and Li’s (2000) model, as well as the fictional surplus term \( \frac{F^*(\theta|s)-F(\theta|s)}{f(\theta|s)} \) which arose in the single tariff model with overconfidence. Ignoring the three remaining constraints, the relaxed solution is characterized by the first order condition \( \Phi_q = 0 \) because virtual surplus is concave in \( q \) (\( \Phi_{qq} < 0 \)) under the maintained assumption \( V_{qq\theta} = 0 \).

Given a FOSD ordering, as in Courty and Li’s (2000) model, allocation \( q(s, \theta) \) non-decreasing in signal \( s \) is sufficient but not necessary for first period global incentive compatibility. Now, however, general examples for which Courty and Li (2000) were able to show the relaxed solution was non-decreasing in signal \( s \) may violate this sufficient condition given high enough levels of overconfidence. This is because for high levels of overconfidence, the relaxed solution involves marginal prices near the monopoly price for each particular minute as discussed in Web Appendix B. As monopoly price increases with demand, this implies types with higher signals should face higher marginal prices at a given quantity, and therefore consume less for a given \( \theta \). This is unfortunate, because in these cases it is not known what the optimal tariff will look like.\(^{39}\)

For specific cases in which the allocation of the relaxed solution is non-decreasing in signal \( s \), marginal price is given by equation (36) if it is well defined, where \( \theta = \theta (s, q) \) is the inverse of \( \hat{q} (s, \theta) \) if the latter is invertible.

\[
P_q(s, q) = \max \left\{ 0, C_q(q) + V_{q\theta}(q, \theta) \left\{ \frac{1-G(s)}{g(s)} \frac{\partial}{\partial s} [1-F^*(\theta|s)] + \frac{F^*(\theta|s)-F(\theta|s)}{f(\theta|s)} \right\} \right\}
\]

Marginal pricing for the top tariff intended for consumers with signal \( s \) is the same as in the single tariff model. Marginal prices for all lower tariffs on the menu are distorted upwards by the information rent term. Whether these tariffs continue to offer initial quantities at zero marginal price depends on the relative size of the information rent and fictional surplus. Initial quantities are more likely to be offered at zero marginal price for all tariffs on the menu when overconfidence is high, and ex ante heterogeneity is low.

E Price Discrimination with Common Priors

The model description and analysis follow directly on from where appendix D.2 left off above, simply by assuming that consumers’ priors are correct: \( F^* (\theta|s) = F (\theta|s) \), and leaving off all superscript

\(^{39}\)Global incentive compatibility would need to be checked directly, and if it failed an ironing procedure could not be used to find the optimal tariff because monotonicity is not necessary.
In this case the virtual surplus function characterized in Proposition 9 reduces to:

$$
\Phi(s, q, \theta) \equiv V(q, \theta) - C(q) - V_\theta(q, \theta) \frac{1 - G(s) \frac{\partial}{\partial s}(1 - F(\theta|s))}{f(\theta|s)}
$$

It is now straightforward to characterize a solution to the relaxed problem in which the additional global incentive constraints are ignored. Virtual surplus is concave in quantity (Equation 38) so the relaxed solution $$q^R(s, \theta)$$ is characterized by the first order condition (Equation 37).

$$
\Phi_q(s, q, \theta) = V_q(q, \theta) - C_q(q) - V_{q\theta}(q, \theta) \frac{1 - G(s) \frac{\partial}{\partial s}(1 - F(\theta|s))}{g(s)} f(\theta|s) = 0
$$

$$
\Phi_{qq}(s, q, \theta) = V_{qq}(q, \theta) - C_{qq}(q) - V_{q\theta}(q, \theta) \frac{1 - G(s) \frac{\partial}{\partial s}(1 - F(\theta|s))}{g(s)} f(\theta|s) \leq 0
$$

Unfortunately, if the relaxed solution does not satisfy global incentive compatibility, no characterization of the true solution is given. This is because there is no known necessary and sufficient condition for first period incentive compatibility analogous to the monotonicity requirement $$q_\theta(s, \theta) \geq 0$$ which can be imposed in an ironing procedure.

Nevertheless, there are conditions sufficient to verify that the relaxed solution is globally incentive compatible. In particular, given the first order local incentive condition $$\frac{dU}{ds} = -\int_\theta^\theta V_\theta(q(s, \theta), \theta) \frac{\partial F(\theta|s)}{\partial s} d\theta$$, a positive cross partial derivative $$\frac{\partial^2 U(s, s')}{\partial s \partial s'} \geq 0$$ is a sufficient second order condition for guarantee global period one incentive compatibility. This gives the following lemma.

**Lemma 9** If the relaxed solution $$q^R(s, \theta)$$ satisfies (1) monotonicity in $$\theta$$: $$\frac{\partial}{\partial \theta} q^R(s, \theta) \geq 0 \ \forall s, \theta$$ and (2) $$\frac{\partial}{\partial s} q^R(s, \theta) \cdot \frac{\partial}{\partial s} [1 - F(\theta|s)] \geq 0 \ \forall s, \theta$$ then $$q^R(s, \theta)$$ and the corresponding implicit payment function solve the original problem.

**Proof.** Differentiating equation (23), the cross partial is:

$$
\frac{\partial^2 U(s, s')}{\partial s \partial s'} = \int_\theta^\theta V_{q\theta}(q(s', \theta), \theta) \frac{\partial}{\partial s} q(s', \theta) \frac{\partial}{\partial s} [1 - F(\theta|s)] d\theta
$$

By assumption $$V_{q\theta}$$ is positive, so $$\frac{\partial}{\partial s} q(s', \theta) \frac{\partial}{\partial s} [1 - F(\theta|s)] \geq 0$$ guarantees that $$\frac{\partial^2 U(s, s)}{\partial s \partial s} \geq 0$$. Given the first period local incentive constraint, this serves as a sufficient second order condition for first period global incentive compatibility. Finally, $$\frac{\partial}{\partial \theta} q^R(s, \theta) \geq 0$$ guarantees second period global incentive compatibility. ■

The conditions given in Lemma 9 are given in terms of the relaxed solution. Using standard monotone comparative statics results, sufficient conditions based on the primitives of the model can
be derived from the cross partials $\Phi_{q\theta}$ and $\Phi_{qs}$. Following definition 3, these are given in Lemma 10.

**Definition 3**

$$\lambda(s, \theta) = \frac{\partial}{\partial s} \left( 1 - F(\theta | s) \right) \frac{f(\theta | s)}{g(s)}$$

(*Baron and Besanko (1984) refer to this as an "informative measure"*)

**Lemma 10** Condition (1) $\frac{\partial}{\partial s} R(s, \theta) \geq 0 \forall s, \theta$ holds if and only if $\lambda(s, \theta)$ does not increase in $\theta$ too quickly:

$$\frac{1 - G(s)}{g(s)} \cdot \frac{\partial}{\partial \theta} \lambda(s, \theta) \leq 1 \forall s, \theta$$

If $G(s)$ satisfies the monotone hazard condition $\left( \frac{\partial}{\partial s} \left( \frac{1 - G(s)}{g(s)} \right) \leq 0 \right)$ then condition (2) $\frac{\partial}{\partial s} \lambda(s, \theta) \cdot \frac{\partial}{\partial s} (1 - F(\theta | s)) \geq 0 \forall s, \theta$ holds if $\lambda(s, \theta)$ and $\frac{\partial}{\partial s} \lambda(s, \theta)$ have opposite signs:

$$\lambda(s, \theta) \cdot \frac{\partial}{\partial s} \lambda(s, \theta) \leq 0$$

**Proof.** Follows from inspection of $\Phi_{q\theta}$ and $\Phi_{qs}$, $V_{q\theta} > 0$, and standard monotone comparative statics results. ■

Given the analysis thus far, it is now possible to consider whether or not menus of three-part tariffs of the type offered by cell phone service providers would be predicted by the relaxed solution of the model.

As absent pooling, the consumer will consume until his marginal valuation is equal to marginal price ($P_q(s, q) = V_q(q, \theta)$), marginal prices under the relaxed solution fall out of the first order condition in equation (37). In particular:

$$P_q(s, q) = C_q(q) + V_{q\theta}(q, \theta) \frac{1 - G(s)}{g(s)} \frac{\partial}{\partial s} \left( 1 - F(\theta | s) \right)$$

By inspection, marginal price is above marginal cost whenever $\frac{\partial}{\partial s} \left( 1 - F(\theta | s) \right)$ is positive, and marginal price is below marginal cost whenever $\frac{\partial}{\partial s} \left( 1 - F(\theta | s) \right)$ is negative.

First consider the special case of RSOSD where $S$ is ordered by first order stochastic dominance (FOSD) so that $F(\theta | s_H) \leq F(\theta | s_L)$ for all $\theta$ and for all $s_H \geq s_L$. In this case $\frac{\partial}{\partial s} \left( 1 - F(\theta | s) \right) \geq 0$ and therefore marginal prices are everywhere weakly above marginal cost.

The general sufficient conditions for global incentive compatibility given in Lemma 10 remain somewhat opaque. That being said, for a simple parametrization of $\theta$ it is easy to verify global incentive compatibility.
Lemma 11 Assume that $\theta$ is a sum of the signal $s$ and a random variable $\varepsilon$ with smooth cumulative density $H(\varepsilon)$:

$$\theta = s + \varepsilon$$

$$\varepsilon \sim H(\varepsilon)$$

Then $F(\theta|s)$ are ordered by FOSD and if $G(s)$ satisfies the monotone hazard rate condition

$$\frac{\partial}{\partial s} \left( \frac{1-G(s)}{g(s)} \right) \leq 0,$$

then the relaxed solution $q^R(s,\theta)$ satisfies global incentive compatibility.

Proof. First, the FOSD ordering follows because $F(\theta|s) = H(\theta - s)$ is decreasing in $s$. Second, when $\theta$ is a function of $s$ and a random variable $\varepsilon \sim H(\varepsilon)$, and $\theta$ is strictly increasing in $\varepsilon$, it can be shown that $\frac{\partial}{\partial s} \lambda(s,\theta) = \frac{\theta_{ss}}{s_{ss}}$ and $\frac{\partial}{\partial s} \lambda(s,\theta) = \theta_{ss} - \frac{\theta_{s\varepsilon}}{s_{s\varepsilon}} \theta_s$. In this parameterization $\theta_{ss} = \theta_{s\varepsilon} = 0$ which implies $\frac{\partial}{\partial s} \lambda(s,\theta) = \frac{\partial}{\partial s} \lambda(s,\theta) = 0$ and therefore the conditions of Lemma 10 are met.  

No absolute conclusions can be drawn without a characterization of the solution when it differs from the relaxed solution. That being said, the preceding discussion suggests that if $S$ is ordered by FOSD, the common-prior model cannot explain observed cell phone service pricing.

It is possible, however, to work backwards to see what type distributions might explain observed pricing. Following the approach taken in this paper, now assume that consumers may freely dispose of minutes beyond their finite satiation points. As seen earlier in the paper, this essentially imposes the constraint that marginal price be non-negative. Further, when marginal costs are assumed to be zero, this constraint binds whenever $\frac{\partial}{\partial s} (1 - F(\theta|s))$ is weakly negative.

Observed menus of tariffs, with increasing levels of included minutes, might then correspond to the relaxed solution if the RSOSD ordered conditional distributions satisfy equation (39) for some cutoff $\theta^*(s)$ increasing in $s$. The relaxed solution would then involve zero marginal price up to $q^R(s,\theta^*(s))$ on each tariff, and positive marginal prices at higher quantities.

$$\frac{\partial}{\partial s} (1 - F(\theta|s)) \begin{cases} \leq 0 & \theta \leq \theta^*(s) \\ > 0 & \theta > \theta^*(s) \end{cases} \quad (39)$$

As the sufficient conditions for the relaxed solution to be optimal given in Lemma 10 are somewhat opaque, it is again useful to turn to a particular parameterization where global incentive compatibility can be verified.

Lemma 12 Assume that $\theta$ is a function of the signal $s$ and a mean zero ($E[\varepsilon] = 0$) random variable $\varepsilon$ with smooth cumulative distribution $H(\varepsilon)$ as given below:

$$\theta = z + \alpha s + s^{1+\gamma} \varepsilon$$
\[ \alpha, \gamma \geq 0 \]

Then \( F(\theta|s) \) are ordered by RSOSD and \( \frac{\partial}{\partial s} (1 - F(\theta|s)) \) where \( \leq 0 \quad \theta \leq \theta^*(s) \quad \text{for some } \theta^*(s) \) which is constant for \( \alpha \gamma = 0 \) and strictly increasing for \( \alpha \gamma > 0 \). Moreover, if \( G(s) \) satisfies the monotone hazard rate condition \( \frac{\partial}{\partial s} \left( \frac{1-G(s)}{g(s)} \right) \leq 0 \), and \( \bar{s} \) is not too small \( (\bar{s} \geq \frac{1+\gamma}{g(2)}) \), then the relaxed solution \( q^R(s, \theta) \) satisfies global incentive compatibility.

**Proof.** First, RSOSD follows as increasing \( s \) involves a mean preserving spread and a FOSD shift.

Second \( \theta^*(s) \) is found by setting \( \lambda(\theta, s) = \theta_s \) equal to zero and solving for \( \theta \). This yields \( \theta^*(s) = z + \alpha \left( \frac{\gamma}{1+\gamma} \right) s \) and \( \frac{\partial}{\partial s} \theta^*(\theta) = \alpha \left( \frac{\gamma}{1+\gamma} \right) \) which is strictly positive if \( \alpha \gamma > 0 \).

Third \( \frac{\partial}{\partial \theta} \lambda(\theta, s) = \frac{\theta_s}{s} = \frac{1+\gamma}{s} \). Thus condition (1) reduces to \( \frac{1-G(s)}{g(s)} (1+\gamma) \leq s \). Under the monotone hazard rate assumption, this inequality is hardest to satisfy for \( \bar{s} \). Thus the assumption \( \bar{s} \geq \frac{1+\gamma}{g(2)} \) is sufficient and period two global incentive compatibility is satisfied.

Fourth, it can be shown that:

\[
\frac{\partial}{\partial s} \lambda(s, \theta) = -\frac{1}{s} \cdot \lambda(s, \theta) - \frac{\gamma}{s} \alpha
\]

Therefore \( \lambda(s, \theta) \geq 0 \) implies that \( \frac{\partial}{\partial s} \lambda(s, \theta) \leq 0 \). So condition (2) of Lemma 10 is satisfied whenever \( \lambda(s, \theta) \geq 0 \). (This implies that \( q^R(s, \theta) \) is increasing in \( s \) whenever \( \lambda(s, \theta) \geq 0 \) and corresponds to the observed fall in overage rates on higher tariffs.)

Of course this condition will not always be met when \( \lambda(s, \theta) < 0 \). This does not matter, however, because for \( \lambda(s, \theta) < 0 \) zero marginal costs imply that the satiation (free disposal) constraint is binding and \( q^R(s, \theta) = q^S(\theta) \). This in turn means that \( \frac{\partial}{\partial \theta} q^R(s, \theta) = 0 \) and condition (2) of Lemma 9 is satisfied, which is what counts. Thus first period global incentive compatibility is guaranteed.  

As an example that meets the conditions of Lemma 12, take \( \theta = 5s + s^2 \varepsilon \) for \( s \sim U[2,3] \), and \( \varepsilon \sim N(0,1) \). In this example \( \theta^*(s) = \frac{5}{2} s \).

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\[ ^{40} \text{The critical points } \theta^*(s) \text{ will be within the support of } \theta \text{ given } s \text{ as long as the support of } \varepsilon \text{ is large enough: } \varepsilon \leq \frac{\alpha}{1+\gamma} s^{-\gamma}. \]

\[ ^{41} \text{Note that for } \gamma = 0, \text{ first period incentive compatibility holds without relying on the satiation constraint. In particular, for } \gamma = 0, \theta_{ss} = 0 \text{ and so } \lambda(s, \theta) \cdot \frac{\partial}{\partial \theta} \lambda(s, \theta) = \theta_s \left( \theta_{ss} - \frac{\theta_s^2}{s} \right) \leq 0 \text{ since } \theta_{ss}, \theta_s > 0. \text{ This means that condition (2) of Lemma 10 always holds. The fact that this works out so nicely is related to the fact that } \theta^*(s) \text{ is constant in } s \text{ and therefore all the distributions } F(\theta|s) \text{ cross once at this common point. (Essentially the case Courty and Li (2000) focus on). However, this would correspond to a menu of tariffs which all have the same number of included minutes. To explain observed tariffs } \gamma > 0 \text{ is required so that } \theta^*(s) \text{ is increasing.} \]