F Multiplicity of Equilibria under Perfect Equilibrium

Considering refinements for our game, one natural candidate is Selten’s (1975) trembling-hand perfect equilibrium (PE). In this section we show that in our common-value SPA with asymmetric information, PE does not provide the natural unique prediction one would expect in the most basic setting with two agents: one informed agent with a binary signal, and one uninformed agent. Note that in this setting there is a unique TRE and it is a strong TRE in pure strategies. In this natural equilibrium, the informed bids his posterior value while the uninformed bids to match the lowest possible bid of the informed.

Formally, consider the setting with two agents, one informed agent with a binary signal, and one uninformed agent. Assume that the common value is 0 conditional on the informed low signal, and 1 conditional on his high signal. Each signal is realized with probability \(1/2\). Each agent’s action space (bid space) is the set \([0, 1]\) (an infinite set). In the unique TRE, the informed bids 0 on the low signal and 1 on the high signal, while the uninformed always bids 0.

We note that PE is usually defined for finite normal form games while our game is a game of incomplete information with infinite strategy spaces (finite type spaces but infinite action spaces). The adaptation of the solution concept to incomplete information is relatively straightforward. The move to infinite games is more delicate and we discuss two adaptations that were suggested in Simon and Stinchcombe (1995) (extending these adaptations to the incomplete information setting) and show that neither provide a unique prediction.

We start by presenting Simon and Stinchcombe’s (1995) reformulation Selten’s (1975) trembling-hand perfect equilibrium for finite (normal form) games with complete information. Let \(N\) be a finite set of agents. For agent \(i \in N\) let \(A_i\) be a finite set of pure actions, and let \(A = \times_{i \in N} A_i\). Let \(\Delta_i\) (resp. \(\Delta_i^{fs}\)) be the set of probability distributions (resp. full support probability distributions) on \(A_i\). Let \(\Delta = \times_{i \in N} \Delta_i\) and \(\Delta^{fs} = \times_{i \in N} \Delta_i^{fs}\). For \(\mu \in \Delta\), let \(Br_i(\mu - i)\) denote \(i\)’s set of mixed-strategy best-responses to the vector of strategies of the others \(\mu - i\).

**Definition 11** (Selten (1975)) Consider a finite game. Fix \(\epsilon > 0\). A vector \(\mu^\epsilon = (\mu^\epsilon_i)_{i \in N}\) in \(\Delta^{fs}\) is an \(\epsilon\)-Perfect Equilibrium if for each agent \(i \in N\) it holds that\(^{28}\)

\[
d_i(\mu^\epsilon_i, Br_i(\mu^\epsilon - i)) < \epsilon
\]

where \(d_i(\mu_i, \nu_i) = \sum_{a_i \in A_i} |\mu_i(a_i) - \nu_i(a_i)|\).

\(^{28}\)Informally, his strategy is at most \(\epsilon\) away from being a best response.
A vector $\mu = (\mu_i)_{i \in \mathbb{N}}$ in $\Delta$ is a Perfect Equilibrium if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 such that (1) for each $j$, $\mu^\epsilon_j$ is an $\epsilon_j$-Perfect Equilibrium and (2) for every $i \in \mathbb{N}$ it holds that $\mu^\epsilon_j$ converges in distribution to $\mu_i$ when $j$ goes to infinity.

Loosely speaking, for a finite (normal form) game a Perfect Equilibrium is a limit, as $\epsilon$ goes to 0, of a sequence of full support strategy vectors, each element of such a vector is $\epsilon$ close to being a best response to the other agent’s strategies in that element of the sequence of strategy vectors.

We next discuss two adaptations, suggested in [Simon and Stinchcombe, 1995], of PE to infinite games. The first is called “limit-of-finite” which considers the limit of a sequence of strategies in a sequence of finite games, in each game only a finite subset of actions is allowed and every player’s strategy has full support. The distance from every action to the set of allowed actions goes to zero and the sequence of strategies converges to the ”limit-of-finite”. The second is called strong perfect equilibrium which looks directly at the infinite game and requires positive mass to every nonempty open subset and the sequence of strategies converges to the strong perfect equilibrium.

Next, we adjust these concepts to games with incomplete information, finite types spaces but infinite action spaces, and show that neither predict a unique equilibrium in the simple setting discussed above.\footnote{We note that with tremble that is independent of the signal of the informed agent, such multiplicity of equilibria result cannot be proven. Yet, the unique equilibrium that is the result of any such tremble is \textit{not} the one we would expect. In the same setting of an item of a common value 0 or 1, with equal probability, and two agents, one perfectly informed and one uninformed, we observe the following. For any tremble of the informed that is independent of the informed agent’s signal, the best response of the uninformed agent is to bid the unconditional expectation (half) as this is the value of the item conditional on winning in the case the informed trembles (and if he does not, the uninformed agent just pays the exact value of the item if winning, as the price is set by the informed agent).}

\section*{F.1 Limit of Finite Games}

We next define the notion of limit-of-finite Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. The approach is to define perfect equilibrium as the limit of $\epsilon$-perfect equilibria for sequences of successively larger (more refined) finite games.

Let $N$ be a finite set of agents. For agent $i \in N$ let $T_i$ be a finite set of types for agent $i$. Assume that the agents have a common prior over types. Let $A_i$ be a compact (infinite) set of actions. Let $B_i$ be a nonempty finite subset of $A_i$, and let $B = \times_{i \in N} B_i$. For such a $B_i$, let $\Delta_i(B_i)$ (resp. $\Delta^f_i(B_i)$) be the set of probability distributions (resp. full support probability distributions) on $B_i$.

A $B_i$-supported mixed strategy $\mu_i(B_i)$ for agent $i$ is a mapping from his type $t_i$ to an element of $\Delta_i(B_i)$. For a profile of mixed strategies $\mu(B) = (\mu_i(B_i))_{i \in N}$, agent $i$ and type $t_i \in T_i$, let
$Br_i^\delta(B_i,\mu_{-i})$ denote $i$’s set of $B_i$-supported mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}(B_{-i})$ (with respect to the given prior and the utility functions) when his type is $t_i$.

**Definition 12** Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$ and $\delta > 0$. For each agent $i \in N$ let $B_i^\delta$ denote a finite subset of $A_i$ within (distance) $\delta$ of $A_i$. A vector $\mu^{(\epsilon,\delta)} = (\mu^{(\epsilon,\delta)}_i)_{i \in N}$ such that for each $i$ and $t_i \in T_i$ it holds that $\mu^{(\epsilon,\delta)}_i(t_i) \in \Delta^{fs}(B_i^\delta)$ is an $(\epsilon, \delta)$-Perfect Equilibrium if for each agent $i \in N$ and type $t_i \in T_i$ it holds that

$$d^\delta_i(\mu^{(\epsilon,\delta)}_i(t_i), Br_i^\delta(B_i^\delta)) < \epsilon$$

where $d^\delta_i(\mu_i, \nu_i) = \sum_{a_i \in B_i^\delta} |\mu_i(a_i) - \nu_i(a_i)|$.

A vector $\mu = (\mu_i)_{i \in N}$ is a limit-of-finite Perfect Equilibrium if there exists two infinite sequences of positive numbers $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ both converging to 0 such that (1) for each $j$, $\mu^{(\epsilon_j,\delta_j)}$ is an $(\epsilon_j, \delta_j)$-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu^{(\epsilon_j,\delta_j)}_i(t_i)$ converges in distribution to $\mu_i(t_i)$ when $j$ goes to infinity.

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent.

**Proposition 4** Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) limit-of-finite perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids $y$.

**Proof.**

Consider the following natural way to make our game finite by discretizing the bids: fix a large natural number $m$ and only allow bids of the form $k/m$ for $k \in \{0, 1, \ldots, m\}$. Note that as $m$ grows to infinity the distance between any bid $y$ and such a set of bids decreases to zero.

Fix $\epsilon > 0$ that is small enough. Fix $m$ that is large enough and fix $k_0 \in \{1, \ldots, m-1\}$ such that $(k_0 + 1)/m$ has minimal distance to $y$ out of all bids of form $k/m$. To prove the claim we present a profile of strategies with full support over the discrete set of bids that is close to the profile in which the informed bids according to his dominant strategy while the uninformed always bids $y$. The strategies that we build have an atom of size at least $1 - \epsilon$ on the specified bids. For the informed with low signal, the probability on every bid other than 0 is proportional to $\epsilon^2$, while for the informed with high signal the probability of every bid other than 1 is proportional to $\epsilon^3$, except for $k_0/m$ for which he assigns probability of about $\epsilon$. This motivates the uninformed to bid
right above this "gift" given by the informed bidder with high signal, and we show that such a bid is his best response. We next define the strategies formally.

The informed agent with low signal is bidding $0$ with probability $1 - \epsilon^2$, and for any $k \in \{1, \ldots, m\}$ he bids $k/m$ with probability $\epsilon^2/m$. The informed agent with high signal is bidding $1$ with probability $1 - \epsilon$. He bids $k_0/m$ with probability $\epsilon - \epsilon^3$, and for any $k \in \{0, \ldots, m-1\}$ such that $k \neq k_0$, he bids $k/m$ with probability $\epsilon^3/(m-1)$.

The uninformed agent is bidding $(k_0 + 1)/m$ with probability $1 - \epsilon$, and for any $k \in \{0, \ldots, m\}$ such that $k \neq k_0 + 1$ he bids $k/m$ with probability $\epsilon/m$.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly $\epsilon$ close to that strategy. It remains to show that the strategy of the uninformed is $\epsilon$ close to his best response (to the strategy of the informed). We claim that if $\epsilon$ is small enough the best response of the uninformed to the strategy of the informed is to bid $(k_0 + 1)/m$ with probability $1$. Indeed, consider any bid $j/m$:

- If $j = k_0 + 1$ then the informed has positive utility as when the value is high he has utility of at least $1/m$ with probability at least $(\epsilon - \epsilon^3)$. When the value is low his loss is at most $(k_0 + 1)/m$ and this happens only with probability at most $\epsilon^2$. For small enough $\epsilon$ the loss will be smaller than the gain.

- If $j = 0$ then the uninformed has utility $0$.

- If $0 < j < k_0$ then the uninformed wins item of value $1$ with probability at most $j\epsilon^3/(2 \cdot (m-1))$ (as the quality is high with probability $1/2$ and in such case he only wins if the informed is bidding below him), thus his expected value is at most $j\epsilon^3/(2 \cdot (m-1))$. On the other hand his expected payment is at least $(1/4) \cdot (\epsilon^2/m) \cdot (1/m)$ (in case it is low value he pays at least $1/m$ with probability $(1/2) \cdot (\epsilon^2/m)$ - the probability of the other bidding $1/m$ and tie is broken in favor of him). Thus his expected utility is at most $j\epsilon^3/(2 \cdot (m-1)) - \epsilon^2/4m^2$ which is negative for small enough $\epsilon > 0$.

- If $j = k_0$ then we claim that this bid is dominated by bidding $(k_0 + 1)/m$. Due to random tie breaking the bid of $k_0/m$ only wins half of the times when the value is high and the informed is also bidding $k_0/m$. By increasing his bid to $(k_0 + 1)/m$ the uninformed will always win in this case. The effect of this change is linear in $\epsilon$. The negative effect due to winning more when the informed gets the low signal is only of the order of $\epsilon^2$, thus for small enough $\epsilon$ it will be smaller.

- If $j > k_0 + 1$ then we claim that this bid is dominated by bidding $(j - 1)/m$. This follow since
the probability of winning high value items decreases by order of $\varepsilon^3$, while the probability of not paying for low value items decreases by order of $\varepsilon^2$.

Note that the proof of the proposition shows that PE does not provide a unique prediction even if we consider finite discrete action spaces. This seems to indicate that the problem with PE (with respect to our setting) is deeper than just its extension to games with infinite action spaces.

### F.2 Strong Perfect Equilibrium

We next define the notion of strong Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. Let $N$ be a finite set of agents. For agent $i \in N$ let $T_i$ be a finite set of types for agent $i$. Assume that the agents have a common prior over types. Let $A_i$ be a compact (infinite) set of actions. Let $\Delta_i$ be the set of probability measures on $A_i$, while $\Delta_i^{fs}$ be the set of probability measures on $A_i$ assigning positive mass to every nonempty open subset of $A_i$. We measure the distance between two measures $\mu, \nu$ on an infinite actions space using the following metric:

$$\rho(\mu, \nu) = \sup \{|\mu(B) - \nu(B)| : B \text{ measurable}\}$$

A mixed strategy $\mu_i$ for agent $i$ is a mapping from his type $t_i \in T_i$ to an element of $\Delta_i$. For a profile of mixed strategies $\mu = (\mu_i)_{i \in N}$ agent $i$ and type $t_i \in T_i$, let $Br_i^{t_i}(\mu_{-i})$ denote $i$'s set of mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}$ (with respect to the given prior and the utility functions) when his type is $t_i$.

**Definition 13** Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\varepsilon > 0$. A vector $\mu^{\varepsilon} = (\mu_i^{\varepsilon})_{i \in N}$ such that for each $i$ and $t_i \in T_i$ it holds that $\mu_i(t_i) \in \Delta_i^{fs}$ is a strong $\varepsilon$-Perfect Equilibrium if for each agent $i \in N$ and type $t_i \in T_i$ it holds that

$$\rho_i(\mu_i^{t_i}(t_i), Br_i^{t_i}(\mu_{-i}^{t_i})) < \varepsilon$$

A vector $\mu = (\mu_i)_{i \in N}$ is a strong Perfect Equilibrium if there exists an infinite sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ which converges to 0 such that (1) for each $j$, $\mu^{\varepsilon_j}$ is a strong $\varepsilon_j$-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{t_i}(t_i)$ converges in distribution to $\mu_i(t_i)$ when $j$ goes to infinity.

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent. The construction of the strategies in the next proposition is very similar to the one in Proposition 4.
Proposition 5  Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any \( y \in (0, 1) \), the following is a (pure strategy) strong perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids \( y \).

Proof.

Fix some \( y \in (0, 1) \). Consider the following tremble for a given \( \epsilon > 0 \) that is small enough.

The informed agent with low signal is bidding with CDF \( F_L(x) = 1 - \epsilon^2 + x\epsilon^2 \) for \( x \in [0, 1] \). (He bids 0 with probability \( 1 - \epsilon^2 \) or uniformly between 0 and 1 with probability \( \epsilon^2 \).)

The informed agent with high signal is bidding with CDF \( F_H \): For \( x \in [0, y - \epsilon] \) it holds that \( F_H(x) = xe^3 \). For \( x \in (y - \epsilon, y] \) it holds that \( F_H(x) = F_H(y - \epsilon) + (x - y + \epsilon)(1 - \epsilon^2) \). For \( x \in (y, 1) \) it holds that \( F_H(x) = F_H(y) + (x - y)e^3 \), and finally, \( F_H(1) = 1 \). (He bids 1 with probability \( 1 - \epsilon + \epsilon^4 \), uniformly between \( y - \epsilon \) and \( y \) with probability \( \epsilon - \epsilon^3 \), and uniformly over all other bids in \([0, 1]\) with the remaining probability \( \epsilon^3(1 - \epsilon) \).)

The uninformed agent is bidding with CDF \( G \): For \( x \in [0, y) \) it holds that \( G(x) = x\epsilon \). For \( x = y \) it holds that \( G(x) = G(y) = y\epsilon + 1 - \epsilon \). For \( x \in (y, 1] \) it holds that \( G(x) = G(y) + (x - y)\epsilon \). (He bids \( y \) with probability \( 1 - \epsilon \) or uniformly between 0 and 1 with probability \( \epsilon \).)

Clearly these strategies have full support and their limit as \( \epsilon \) goes to 0 is as required.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly \( \epsilon \) close to that strategy. It remains to show that the strategy of the uninformed is \( \epsilon \) close to his best response (to the strategy of the informed). We claim that if \( \epsilon \) is small enough the best response of the uninformed to the strategy of the informed is to bid \( y \) with probability 1. Indeed, consider any bid \( z \):

- If \( z = 0 \) then the agent has utility 0.

- If \( z = y \) then for small enough \( \epsilon > 0 \) the agent has positive utility. Indeed his expected gain from high value items is at least \( 1/2 \cdot F_H(y)(1 - y) = (\epsilon - \epsilon^3(1 - y + \epsilon))(1 - y)/2 \geq c\epsilon e^{3} \) for some constant \( c > 0 \) (for small enough \( \epsilon > 0 \)), while his expected loss from low value items is at most \( 1/2 \cdot (1 - F_L(0))y \leq (y/2)\epsilon^2 \leq \epsilon^2 \).

- If \( 0 < z < y \) then for small enough \( \epsilon > 0 \) it holds that \( 0 < z < y - \epsilon \). Moreover, for small enough \( \epsilon > 0 \) the agent has negative utility. Indeed his expected gain is at most \( 1/2 \cdot F_H(z) \cdot 1 \leq z\epsilon^3 \), while his expected loss is at least \( 1/2 \cdot (F_L(z) - F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4 \).

- If \( z > y \) then for small enough \( \epsilon > 0 \) the agent can increase his utility by bidding \( y \) instead of bidding \( z \). Indeed his expected loss of value by bidding \( y \) instead of \( z \) is at most \( 1/2 \cdot (F_L(z) - F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4 \).


\[(F_H(z) - F_H(y)) \cdot 1 = (y - z)e^3/2, \text{ while his expected reduction in payment is at least}
1/2 \cdot (F_L(z) - F_L(y)) \cdot y \geq (z - y)e^2/2.\]

\section{Relation to the work of Einy et al. (2002)}

Einy et al. (2002) study common-value second-price auction in domains that are \textit{connected}. For \textit{connected domains} Einy et al. (2002) consider the concept of \textit{sophisticated equilibrium}, which makes successive rounds of dominated strategy eliminations. This process might result in multiple equilibria and that paper points out a single sophisticated equilibrium that Pareto-dominates the rest in terms of bidders resulting utilities, and it is also the only sophisticated equilibrium that guarantees every bidder nonnegative utility. Moreover, this is the only sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain.

In this section we observe that Theorem 4 applies to any \textit{connected domain}, as any such domain satisfies the strong-high-signal property. Moreover, we observe that for \textit{connected domains} the TRE of Theorem 4 is exactly the one pointed out by Einy et al. (2002). Finally, we show that some domains that satisfy the strong-high-signal property are not connected. Some obvious such domains are monotonic domains in which the mapping from the state of the world to signals is not deterministic (yet they still satisfy the strong-high-signal property), but we also present examples of domains in which the mapping is deterministic yet they are not connected and for which Theorem 4 applies.

Before formally presenting \textit{connected domains} we present an example due to Einy et al. (2002) and the TRE we (as well as Einy et al. (2002)) pick for that domain.

\textbf{Example 2} Assume that there are two buyers and four states of the world \(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}\), with \(v(\omega_i) = i\) and states all are equally probable ( \(H(\omega_i) = 1/4 \) for all \(i \in \{1, 2, 3, 4\}\)). If the state is \(\omega_1\) then agent 1 gets the signal \(L_1\), otherwise he gets \(H_1\). If the state is \(\omega_4\) then agent 2 gets the signal \(H_2\), otherwise he gets \(L_2\). In \(\mu\), the TRE of Theorem 4 it holds that \(\mu_2(H_2) = v(H_1, H_2) = 4, \mu_1(H_1) = v(H_1, L_2) = 2.5\) and \(\mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 1.\)

We next define \textit{connected domains}.

\textbf{Definition 14} A domain is a \textit{connected domain} if the following hold. Each agent \(i\) has a partition \(\Pi_i\) of the state of nature and his signal is the element of the partition that include the realized state. The information partition \(\Pi_i\) of bidder \(i\) is connected (with respect to the common value \(v\))
if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in $\pi_i$. A common-value domain is connected (with respect to the common value) if for every agent $i$ his information partition $\Pi_i$ is connected.

Lemma 8 Every connected domain satisfies the strong-high-signal property.

Proof. Let $\Pi^*$ be the coarsest partition of $\Omega$ that refines the partition $\Pi_j$ for every agent $j$. Let $\sigma$ denote an element of $\Pi^*$. Let $v(\sigma)$ denote the expected value of the item conditional on $\sigma$. We prove the claim by induction on the number of elements in $\Pi^*$. If this number is 1 the claim trivially holds as the domain in which no agent gets any information satisfies the property by definition.

Assume that we have proven the claim for every $\Pi^*$ of size smaller than $k$, we prove the claim for $\Pi^*$ of size $k$. Consider that element $\sigma$ of $\Pi^*$ such that $v(\sigma)$ is maximal. There must exist an agent $i$ and signal $s_i$ such that $s_i$ implies $\sigma$, otherwise $\Pi^*$ is not the coarsest refinement. There is only one combination of signals that has value $v(\sigma)$, in that combination each agent gets the best signal (the one with the highest value conditional on the signal). Now, as the domain is connected it holds that $v(\sigma) > v(t)$ for every combination of signals $t$. This implies that $s_i$ has the required properties from the top signal at a domain that satisfies the strong-high-signal property. Removing this signal creates another connected domain, and its coarsest partition has only $k - 1$ elements, so by the induction hypothesis it satisfies the strong-high-signal property. We conclude that the original domain satisfies the strong-high-signal property as we need to show. ■

Proposition 6 For every connected domain the TRE of Theorem 4 is exactly the same as the unique sophisticated equilibrium picked by Einy et al. (2002) (the sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain).

Proof. Einy et al. (2002) show that unique sophisticated equilibrium that they pick can be computed as follows. One can look at $\Pi^*$, the coarsest partition of $\Omega$ that refines the partition $\Pi_j$ for every agent $j$. Let $\sigma$ denote an element of $\Pi^*$. Let $v(\sigma)$ denote the expected value of the item conditional on $\sigma$. An order over elements $\sigma_1, \sigma_2 \in \Pi^*$ is naturally defined by the order on the corresponding values $v(\sigma_1)$ and $v(\sigma_2)$. For agent $j$ with signal $\pi_j \in \Pi_j$ the bid is defined to $\min v(\sigma_1)$ and $v(\sigma_2)$. An equivalent definition is that agent $j$ with signal $\pi_j \in \Pi_j$ bids $\min \{v(\pi_j, \pi_{-j}) \mid \pi_{-j} \in S_{-j}$ and $(\pi_j, \pi_{-j})$ is feasible$\}$, which is exactly $\mu_j(s_j)$ as defined in Theorem 4. ■

We next show that there are domains that are not connected yet satisfy the strong-high-signal property. This implies that Theorem 4 applies to a strict superset of the domains that are handled by Einy et al. (2002). We start with a simple example with only one informed bidder.

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Example 3  Consider a domain with two buyers and three states of the world \( \Omega = \{\omega_1, \omega_2, \omega_3\} \), with \( v(\omega_1) = 0 \), \( v(\omega_2) = 4 \), \( v(\omega_3) = 10 \) and all states are equally probable \( H(\omega_i) = 1/3 \) for all \( i \in \{1, 2, 3\} \). If the state is \( \omega_1 \) or \( \omega_3 \) then agent 1 gets the signal \( H_1 \), otherwise he gets \( L_1 \). Agent 2 is not informed at all. This example is covered by Theorem 4 and moreover it is covered by Theorem 3. Yet, this domain is not connected, as signal \( H_1 \) of agent 1 indicates that the state is \( \omega_1 \) or \( \omega_3 \) and does not include \( \omega_2 \).

We also present an example with more than one informed bidder, in this example there are 2 agents and each has a binary signal.

Example 4  Assume that there are two buyers and four states of the world \( \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} \) with \( v(\omega_1) = 0 \), \( v(\omega_2) = 4 \), \( v(\omega_3) = 6 \), \( v(\omega_4) = 10 \), and all states are equally probable \( H(\omega_i) = 1/4 \) for all \( i \in \{1, 2, 3, 4\} \). If the state is \( \omega_4 \) then agent 1 gets the signal \( H_1 \), otherwise he gets \( L_1 \). If the state is \( \omega_1 \) or \( \omega_3 \) then agent 2 gets the signal \( L_2 \), otherwise he gets \( H_2 \). (note that this is not connected as \( \omega_2 \) does not belong to \( L_2 \)). In the TRE \( \mu \) of Theorem 4 it holds that \( \mu_1(H_1) = v(H_1, H_2) = 10 \), \( \mu_2(H_2) = v(L_1, H_2) = 4 \) and \( \mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 3 \).

While Example 3 presents a very simple domain that is not connected, it is clear that there exists a different representation of the states of the world for which a domain with exactly the same signal structure and posteriors, is indeed connected. In this new representation each state corresponds to one of the informed agent’s signals and the value corresponds to the posterior value given that signal. That is, we can define \( \Omega' = \{\omega'_1, \omega'_2\} \), with \( v(\omega'_1) = 5 \), \( v(\omega'_2) = 4 \), and the probabilities are \( H(\omega'_1) = 2/3 \) and \( H(\omega'_2) = 1/3 \). If the state is \( \omega'_1 \) then agent 1 gets the signal \( H_1 \), otherwise he gets \( L_1 \). Agent 2 is not informed at all. Clearly under the new representation the domain is connected, and the domain is equivalent to the original domain.

One might wonder if any domain that satisfies the strong-high-signal property can be transformed to an equivalent connected domain. We next show that this is not the case, presenting a domain that satisfies the property and cannot be represented by a connect domain. This shows that Theorem 4 applies to domains that do not have a representation as connected domains.

The domain we consider is the domain presented in Example 4 with \( v(\omega_2) \) assigned a value of 2 instead of 4. Clearly in a connected domain that is equivalent to that domain it must be the case that signals \( H_1 \) and \( H_2 \) are both received for some subset of states of the world such that for each such state the value is at least as high as the value if signal \( H_1 \) is not received. Now connectivity for \( H_2 \) implies that \( v(L_1) \geq v(L_2) \) which does not hold for the domain we are considering.
H  Details of the Proof of Theorem 1

As explained in Appendix B, Theorem 1 is implied by Theorem 4 in the special case $V_{HH} = \max\{V_{HL}, V_{LH}\}$. In this section, we present proofs of Lemmas 1-4 which imply that Theorem 1 holds for the remaining case $V_{HH} > \max\{V_{HL}, V_{LH}\}$. Throughout this appendix, we maintain Assumption 1, label bidders following equation (1), and assume that $V_{HH} > \max\{V_{HL}, V_{LH}\}$. We use $i$ to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use $j$ to denote that bidder, and assume that $j \neq i$. To simplify the notation we denote $v_1 = V_{HL}$, $v_2 = V_{LH}$, and $v_i = v(H_i, L_j)$, and (without loss of generality) normalize $V_{LL} = 0$ and $V_{HH} = 1$. In this notation, equation (1) becomes $\Pr[H_1, L_2](1 - v_1) \leq \Pr[L_1, H_2](1 - v_2)$ and our assumption $V_{HH} > \max\{V_{HL}, V_{LH}\}$ is $1 > \max\{v_1, v_2\}$. When equation (1) holds with equality, we label bidders such that $v_1 \geq v_2$ following equation (28). We define additional notation as it is first used throughout the appendix. For those reading nonlinearly, please refer to the notation summary in Table 1.

H.1  Proof of Lemma 1 (Necessary conditions part I)

Let $R$ be a standard distribution and fix some $\epsilon > 0$. Consider a NE $\mu^\epsilon$ of the $(\epsilon, R)$-tremble of the game $\lambda$ in which bidders never submit dominated bids. We first characterize bidding given a low signal $L_i$:

Lemma 9  At $\mu^\epsilon$ the following must hold. For each bidder $i \in \{1, 2\}$ it holds that: (1) Bidder $i$ with signal $L_i$ always bids $V_{LL} = 0$. (2) Bidder $i$ with signal $H_i$ always bids at least $v_i$.

Proof.  By assumption, bidders do not make weakly dominated bids. Therefore, bidder $i$ bids at least 0 given signal $L_i$ and at least $v_i = v(H_i, L_j)$ given signal $H_i$. Similarly, bidder $i$ bids no more than $v_j = v(L_i, H_j)$ given signal $L_i$ and no more than 1 given signal $H_i$. Bidder $i$ with signal $L_i$ cannot bid $b \in (0, v_j)$ because she would only win when bidder $j$ has a low signal and the value is zero but she would pay a positive amount due to the random bidder. Increasing the bid to $v_j$ incurs the same losses conditional on $L_j$ as bidding just below $v_j$ and earns zero conditional on $H_j$ because any wins are priced at their value $v_j$. Therefore bidder $i$ must bid 0 given a low signal, and the same is true for bidder $j$ by similar logic.

Given Lemma 9 we focus in the rest of the proof on the bidding of each bidder $i$ given his high signal $H_i$. Thus, if we say that some bid “is optimal for $i$”, we mean that it is a best response for $i$, conditional on signal $H_i$. We define $G_{H_i}$ to be the cumulative distribution function of bidder $i$’s bids conditional on $i$ having signal $H_i$, that is $G_{H_i} = \mu^\epsilon(H_i)$. Moreover, we define $G_{H_i}^-(b) = \sup_{x < b} G_{H_i}(x)$ to be the left-hand limit of $G_{H_i}$ evaluated at $b$. 

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### Table 1: Notation Summary for Section H

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(H_i)$</td>
<td>$v(H_i) = E[v</td>
<td>H_i] = \Pr[H_j</td>
</tr>
<tr>
<td>$v_1, v_2, v_i$</td>
<td>$v_1 = V_{HL}, v_2 = V_{LH}, v_i = v(H_i, L_j)$</td>
<td></td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>$\bar{v} = \max{v_1, v_2}$</td>
<td></td>
</tr>
<tr>
<td>$\mu^\epsilon$</td>
<td>NE in undominated bids of tremble $\lambda(\epsilon, R)$</td>
<td></td>
</tr>
<tr>
<td>$\bar{R}(b)$</td>
<td>$\bar{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$</td>
<td></td>
</tr>
<tr>
<td>$\bar{r}(b)$</td>
<td>$\bar{r}(x) = \epsilon \cdot r(x)$</td>
<td></td>
</tr>
<tr>
<td>$\Pi_i(b_i)$</td>
<td>i’s $E[\text{profit}]$ when bidding $b_i$ with signal $H_i$</td>
<td>Equation (39), page 67</td>
</tr>
<tr>
<td>$\Pi_i^{-1}(b_i), \Pi_i^+(b_i)$</td>
<td>Left and right limits of $\Pi_i(b_i)$</td>
<td></td>
</tr>
<tr>
<td>$G_{H_i}$</td>
<td>CDF of i’s bids conditional on $H_i$ ($G_{H_i} = \mu^\epsilon(H_i)$)</td>
<td></td>
</tr>
<tr>
<td>$G_{H_i}^{-}(b)$</td>
<td>$\sup_{x &lt; b} G_{H_i}(x)$ (left-hand limit of $G_{H_i}$ evaluated at $b$)</td>
<td></td>
</tr>
<tr>
<td>$b_i$</td>
<td>$\inf{b : G_{H_i}(b) &gt; 0}$ (infimum bid by $i \in {1, 2}$ with signal $H_i$)</td>
<td></td>
</tr>
<tr>
<td>$\bar{b}$</td>
<td>$\min{b_1, b_2}$ (infimum bid of any bidder with a high signal)</td>
<td></td>
</tr>
<tr>
<td>$b_{\min}$</td>
<td>$\max{b_1, b_2}$</td>
<td></td>
</tr>
<tr>
<td>$\bar{b}_i$</td>
<td>$\sup{b : G_{H_i}(b) &lt; 1}$ (supremum bid by $i \in {1, 2}$ with signal $H_i$)</td>
<td></td>
</tr>
<tr>
<td>$b_{\max}$</td>
<td>$\max{\bar{b}_1, \bar{b}<em>2} \geq b</em>{\min}$ (supremum bid of any bidder with a high signal)</td>
<td></td>
</tr>
<tr>
<td>$x_i(b)$</td>
<td>i’s supremum bid below $b$</td>
<td>Equation (45), page 72</td>
</tr>
<tr>
<td>$\beta_i(\Gamma)$</td>
<td>$\int_a^b \frac{1}{1-x^2} \cdot r(x)dx / \int^b_a \frac{x}{1-x^2} \cdot r(x)dx$</td>
<td>Equation (58), page 79</td>
</tr>
<tr>
<td>$\varphi(b)$</td>
<td>$\frac{\hat{R}(b_{\min})}{\hat{R}(b)}$</td>
<td></td>
</tr>
</tbody>
</table>
Let $\Pi_i(b_i)$ be the expected profit for bidder $i$ conditional on signal $H_i$ and bid $b_i$. Let $b_r$ be the random bidder’s bid if it enters and 0 otherwise. The distribution of $b_r$ is given by $\hat{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$, with density $\hat{r}(b) = \epsilon r(b)$ for $b > 0$. Then $\Pi_i(b_i)$ is:

$$\Pi_i(b_i) = \begin{cases} 0 & \text{if } b_i < 0 \\ \frac{1}{2} (\Pr[L_j | H_i] v_i + \Pr[H_j | H_i] G_{HJ}(0)) (1 - \epsilon) & \text{if } b_i = 0 \\ \Pr[L_j | H_i] \hat{R}(b_i) (v_i - E[b_r | b_r < b_i]) & \text{if } b_i > 0 \\ + \Pr[H_j | H_i] \hat{R}(b_i) G_{HJ}^{-}(b_i) (1 - E[\max\{b_r, b_j\} | \max\{b_r, b_j\} < b_i]) & \text{if } b_i > 0 \end{cases}$$

For $b_i > 0$, the first term handles the case that $j$ receives the signal $L_j$, in this case he bids $V_{LL} = 0$ and the price is set by the random bidder. The second and third terms handle the case that $j$ receives the signal $H_j$. The second term is for the case that $b_j < b_i$, while the third handles the case that $b_j = b_i$. Noting that

$$\hat{R}(b_i) G_{HJ}^{-}(b_i) E[\max\{b_r, b_j\} | \max\{b_r, b_j\} < b_i] = \hat{R}(b_i) G_{HJ}^{-}(b_i) \int_{b_i}^{b_i} \left( 1 - \frac{\hat{R}(x) G_{HJ}(x)}{\hat{R}(b_i) G_{HJ}(b_i)} \right) dx = b_i \hat{R}(b_i) G_{HJ}^{-}(b_i) - \int_{0}^{b_i} \hat{R}(x) G_{HJ}(x) dx,$$

profits for $b_i > 0$ may be written more explicitly as

$$\Pi_i(b_i > 0) = \Pr[L_j | H_i] \left( \hat{R}(b_i) v_i - \int_{0}^{b_i} x \hat{r}(x) dx \right)$$

$$+ \Pr[H_j | H_i] \left( \hat{R}(b_i) G_{HJ}^{-}(b_i) (1 - b_i) + \int_{0}^{b_i} \hat{R}(x) G_{HJ}(x) dx \right) + \Pr[H_j | H_i] \hat{R}(b_i) \frac{1}{2} \left( G_{HJ}(b_i) - G_{HJ}^{-}(b_i) \right) (1 - b_i). \quad (39)$$

Let $\Pi_i^-(b_i)$ be the left-hand limit of $\Pi_i(b_i)$ and $\Pi_i^+(b_i)$ be the right-hand limit of $\Pi_i(b_i)$. If $\Pi_i(b_i)$ is discontinuous at $b_i$, then $\Pi_i^-(b_i) < \Pi_i(b_i) < \Pi_i^+(b_i)$. In particular,

$$\Pi_i^+(b_i) - \Pi_i^-(b_i) = 2 \left( \Pi_i^+(b_i) - \Pi_i(b_i) \right) \geq 0, \quad (40)$$

and

$$\Pi_i^+(b_i) - \Pi_i(b_i) = \begin{cases} \frac{1}{2} (\Pr[L_j | H_i] v_i + \Pr[H_j | H_i] G_{HJ}(b_i)) (1 - \epsilon) & \text{for } b_i = 0 \\ \frac{1}{2} \Pr[H_j | H_i] \hat{R}(b_i) \left( G_{HJ}(b_i) - G_{HJ}^{-}(b_i) \right) (1 - b_i) & \text{for } b_i > 0. \end{cases} \quad (41)$$

In this expression, $\frac{1}{2} (G_{HJ}^{-}(b_i) + G_{HJ}(b_i))$ is the probability that $i$ wins with bid $b_i$ given $H_j$, accounting for the fact that there is a tie with probability $(G_{HJ}(b_i) - G_{HJ}^{-}(b_i))$ that is broken 50–50. There is a discontinuity in $\Pi_i(b_i)$ at $b_i = 0$ if $v_i > 0$ or $G_{HJ}(0) > 0$ and a discontinuity
at \( b_i < 1 \) if \( j \) has an atom at \( b_i \) so that \( G_{Hj}(b_i) > G_{Hj}^{-}(b_i) \). This leads to the following results in Lemmas 10 and 11.

**Lemma 10** At \( \mu^c \) the following must hold. If \( b \in [0, 1] \) is an optimal bid for \( i \), then \( \Pi_i \) is continuous at \( b \).

**Proof.** This follows from equations (40)–(41): if \( \Pi_i(b) \) is discontinuous at \( b \), then \( \Pi_i(b) < \Pi_i^+(b) \) and there exists \( \epsilon > 0 \) such that \( \Pi_i(b) < \Pi_i(b + \epsilon) \). Hence \( b \) is not an optimal bid for \( i \). ■

**Lemma 11** At \( \mu^c \) the following must hold. Assume that \( G_{Hj} \) is discontinuous at \( b < 1 \) (\( j \) has an atom at \( b \)), then \( \exists \delta > 0 \) such that bidding in the interval \((b - \delta, b] \) is not optimal for \( i \) as it is strictly dominated by bidding \( b + \delta \).

**Proof.** Consider the difference in \( i \)'s expected profit from bidding \( b + \delta \) instead of \( b \). In the limit as \( \delta \) goes to zero, the difference is given by equation (41). Next consider the difference in \( i \)'s expected profit from bidding \( b + \delta \) instead of \( b - \delta \). In the limit as \( \delta \) goes to zero the difference is double (equation (40)). In both cases, in the limit as \( \delta \) goes to zero, the difference is positive because we have assumed both that \( b < 1 \) and that \( j \) has an atom at \( b \) so that \( \left(G_{Hj}(b) - G_{Hj}^{-}(b)\right) > 0 \). This proves the result. ■

Define \( v_i^{\text{win}}(b) \) to be the expected value of the items \( i \) gets, conditional on winning with bid \( b \) and signal \( H_i \). Then,

\[
v_i^{\text{win}}(b) = \begin{cases} \frac{\Pr[H_i|H_i]G_{Hj}(0) + \Pr[L_j|H_i]v_i}{\Pr[H_i|H_i]G_{Hj}(0) + \Pr[L_j|H_i]} & \text{for } b = 0 \\ \frac{\Pr[H_i|H_i](G_{Hj}(b) + G_{Hj}^{-}(b)) / 2 + \Pr[L_j|H_i]v_i}{\Pr[H_i|H_i](G_{Hj}(b) + G_{Hj}^{-}(b)) / 2 + \Pr[L_j|H_i]} & \text{for } b > 0. \end{cases} \tag{42}
\]

**Lemma 12** At \( \mu^c \) the following must hold. If \( b \in [0, 1] \) is an optimal bid of bidder \( i \) then \( b \geq v_i^{\text{win}}(b) \).

**Proof.** By equation (42), \( v_i^{\text{win}}(b) \) is a convex combination of \( v_i \) and 1. Therefore, it holds for all \( b \) that \( 0 \leq v_i \leq v_i^{\text{win}}(b) \leq 1 \). If \( b = 1 \) then this means that \( v_i^{\text{win}}(b) \leq b \). Suppose that \( b < 1 \). If \( v_i^{\text{win}}(b) = v_i \) then the claim follows from \( b \geq v_i \) (Lemma 9). Therefore consider the remaining case in which \( v_i^{\text{win}}(b) > v_i \). Assume in contradiction that \( b < 1 \) is an optimal bid for \( i \) and \( b < v_i^{\text{win}}(b) \). It must hold that \( G_{Hj}(b) > 0 \), since \( G_{Hj}(b) = 0 \) implies \( v_i^{\text{win}}(b) = v_i \). As \( b \) is optimal for \( i \), Lemma 11 implies that \( j \) does not have an atom at \( b \). Moreover, it must be that \( b > 0 \), since all bids are nonnegative, \( j \) has no atom at \( b \), and \( G_{Hj}(b) > 0 \).

We show that for \( \delta \in (0, \min\{1 - b, v_i^{\text{win}}(b) - b\}) \), bidding \( b + \delta \) gives higher expected payoff than bidding \( b \). Consider the difference in expected payoff from such an increase in the bid. There are two events to consider that lead to different outcomes from the two bids: (1) First, \( b_j < b \) and

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For any such $b_r \in (b, b + \delta)$ such that $i$ was already bidding above $j$, but raising her bid causes her to bid above the random bidder and win the auction. Given that $j$ has no atom at $b$ and $b > 0$, this event occurs with positive probability, $\Pr[L_j|H_i] + \Pr[H_j|H_i]G_{H_j}(b)(\hat{R}(b + \delta) - \hat{R}(b)) > 0$, causing $i$ to win an item worth an average of $v_i^\text{win}(b) = E[v|b_j < b]$ but pay at most $b + \delta < v_i^\text{win}(b)$. (2) Second, $b_j \geq b$ and $i$ wins bidding at $b + \delta$ but not at $b$. This causes $i$ to win additional items worth 1 (as $b_j \geq b > 0$ implies signal $H_j$) but pay at most $b + \delta < 1$. Both events weakly increase expected payoffs from bidding $b + \delta$ relative to bidding $b$, and the first does so strictly, as it occurs with positive probability. Therefore bidding $b$ is not a best response, a contradiction that proves the result. ■

Where $G_{H_j}(b)$ is differentiable, the derivative of $\Pi_i(b_i)$ (equation (39)) with respect to $b_i$ is

$$
\frac{d\Pi_i(b_i)}{db_i} = \Pr[L_j|H_i] \hat{r}(b_i)(v_i - b_i) + \Pr[H_j|H_i]\left(\hat{r}(b_i)G_{H_j}(b_i) + \hat{R}(b_i)g_j(b_i)\right)(1 - b_i). \tag{43}
$$

The next result follows from equation (43) evaluated over an interval for which $g_j(b)$ is zero. To state the result, we first define $\beta_i(\Gamma)$ to be the expected value conditional on $i$ winning with signal $H_i$ given that $i$’s probability of winning is 1 for $(H_i, L_j)$ and $\Gamma$ for $(H_i, H_j)$:

$$
\beta_i(\Gamma) = \frac{\Pr[H_j|H_i] \Gamma + \Pr[L_j|H_i]v_i}{\Pr[H_j|H_i] \Gamma + \Pr[L_j|H_i]}. \tag{44}
$$

**Lemma 13** At $\mu^\epsilon$ the following must hold. For $1 \geq b^+ > b^- \geq 0$ suppose that $G_{H_j}(b^-) = G_{H_j}(b^+)$ ($j$ does not bid on $(b^-, b^+)$). Let $\Gamma = G_{H_j}(b^-)$.

1. If $\beta_i(\Gamma) \in (b^-, b^+)$ then $\beta_i(\Gamma)$ strictly dominates any other bid by $i$ in $[b^-, b^+]$.
2. If $\beta_i(\Gamma) \notin (b^-, b^+]$ then all bids $b \in (b^-, b^+]$ are strictly suboptimal for $i$. Moreover, if $\beta_i(\Gamma) \leq b^-$, then $i$’s payoff is decreasing in $b$ over $(b^-, b^+]$.
3. If $i$ has an optimal bid $b \in (b^-, b^+]$ it holds that $b = \beta_i(\Gamma) < 1$.

**Proof.** $G_{H_j}(b)$ is constant over $(b^-, b^+]$ and thus $g_j(b) = 0$ for every $b \in (b^-, b^+]$. Therefore $\Pi_i(b_i)$ is continuous and differentiable in $b_i$ for every $b_i \in (b^-, b^+]$. Moreover, since $g_j(b_i)$ is zero for any such $b_i$, the derivative with respect to $b_i$ is

$$
\frac{d\Pi_i(b_i)}{db_i} = \hat{r}(b_i)(\Pr[H_j|H_i] \Gamma(1 - b_i) + \Pr[L_j|H_i](v_i - b_i)).
$$

As we assume that $\Pr[L_j|H_i] > 0$, $\Pr[H_j|H_i] > 0$, and $\hat{r}(b_i) > 0$, this function of $b_i$ is not identically 0. The function has a unique 0 at $\beta_i(\Gamma)$, it is positive for $b_i < \beta_i(\Gamma)$, and it is negative for $b_i > \beta_i(\Gamma)$. The results then follow by the following augments:
(1) If (a) \( \beta_i(\Gamma) \in (b^-, b^+) \) then it follows from the derivative \( \frac{d\Pi_i(b_i)}{db_i} \) that \( b = \beta_i(\Gamma) \) uniquely maximizes \( \Pi_i(b) \) for bids \( b \) within the interval \((b^-, b^+)\). If (b) \( \beta_i(\Gamma) = b^+ \) then \( d\Pi_i(b_i)/db_i > 0 \) for \((b^-, b^+)\). Therefore \( \Pi_i(b_i) \) is higher at \( b^+ \) than at any \( b_i \in (b^-, b^+) \) because \( \Pi_i(b_i) \) is either continuous at \( b^+ \) or increases discretely at \( b^+ \) (depending on whether or not \( j \) has an atom at \( b^+ \)). In either case (a) or case (b), \( b = \beta_i(\Gamma) \) is also strictly better than bidding \( b^- \), either by continuity at \( b^- \) if \( j \) does not have an atom at \( b^- \) or by Lemma 11 if \( j \) does have an atom at \( b^- \). Therefore part (1) holds.

(2) (a) Suppose \( \beta_i(\Gamma) \leq b^- \): In this case, \( d\Pi_i(b_i)/db_i < 0 \) for \((b^-, b^+)\) and there is no optimal bid in \((b^-, b^+)\). If \( j \) bids an atom at \( b^+ \) and \( b^+ < 1 \) then \( b^+ \) is not an optimal bid for \( i \) by Lemma 11. If \( j \) does not bid an atom at \( b^+ \) or \( b^+ = 1 \), then \( \Pi_i(b_i) \) is continuous at \( b^+ \) (equations (40)–(41)). Therefore \( \Pi_i(b_i) \) is lower at \( b^+ \) than at any other \( b_i \in (b^-, b^+) \). In either case there is no optimal bid within \((b^-, b^+)\). (b) Suppose \( \beta_i(\Gamma) > b^- \): Inspection of equation (42) shows that \( v_i^{\text{win}}(b) \) is nondecreasing in \( b \). This fact and Lemma 12 imply that any optimal bid \( b_i > b^- \) must be at least \( \beta_i(\Gamma) \) because \( v_i^{\text{win}}(b_i) = \beta_i(\Gamma) \) for all \( b_i \in (b^-, b^+) \). Therefore there is no optimal bid within \((b^-, b^+)\). Thus part (2) holds.

(3) Note that \( \beta_i(\Gamma) < 1 \) given \( v_i < 1 \) and its definition in equation (44). Next, from parts (1) and (2) it follows that if \( b \in (b^-, b^+) \) is an optimal bid for \( i \), either \( b = \beta_i(\Gamma) \) or \( b = b^+ \). It therefore remains to show that if \( b = b^+ \) is an optimal bid that \( b^+ = \beta_i(\Gamma) \). Proof: Suppose that \( b^+ \) is an optimal bid. By part (2), \( \beta_i(\Gamma) \in (b^-, b^+) \). Suppose that \( \beta_i(\Gamma) \in (b^-, b^+) \). Then \( d\Pi_i(b_i)/db_i < 0 \) for all \( b \in (\beta_i(\Gamma), b^+) \). By continuity of \( \Pi_i(b) \) at \( b^+ \) (Lemma 10), bidding \( \beta_i(\Gamma) \) therefore strictly dominates bidding \( b^+ \), a contradiction. Thus \( \beta_i(\Gamma) = b^+ \). This completes the proof of part (3). ■

Corollary 6 At \( \mu' \) the following must hold. If bidder \( i \in \{1, 2\} \) bids an atom at \( b \in [0, 1) \), then \( b = \beta_i(G_{H_j}(b)) \).

Proof. For \( b \in (0, 1) \): Lemma 11 implies that \( j \) does not bid in the interval \((b - \delta, b)\) for some \( \delta > 0 \) and we can apply Lemma 13 for \( b^+ = b \) and \( b^- = b - (\delta/2) \). The result is then implied by part (3) of the lemma. For \( b = 0 \): If \( i \) bids 0, Lemma 12 implies that \( v_i^{\text{win}}(0) = 0 \). As \( v_i^{\text{win}}(0) = \beta_i(G_{H_j}(0)) \) (equations (42) and (44)), this implies \( b = \beta_i(G_{H_j}(b)) = 0 \). ■

Define \( b_i \) to be the infimum bid by \( i \in \{1, 2\} \), \( b_i = \inf \{b : G_{H_i}(b) > 0\} \). Let \( b = \min \{b_1, b_2\} \) be the infimum of all bids of any bidder with a high signal. Let \( b_{\text{min}} = \max \{b_1, b_2\} \). Note that undominated bidding requires \( b_i \geq v_i \) and hence \( b_{\text{min}} \geq \max \{v_1, v_2\} \).

Corollary 7 At \( \mu' \) the following must hold. Suppose that \( j \in \{1, 2\} \) has an optimal bid \( b \) satisfying \( b < 1 \) and \( b \leq b_i \). Then \( b = v_j \).
Proof. For \( b > 0 \): Note that \( G_{H_i}(b) = 0 \) because Lemma 11 implies that \( i \) does not have an atom at \( b \) if \( b = b_i \). Thus, as \( i \) does not bid less than \( b_i \) but \( j \) has an optimal bid \( b > 0 \) at or below \( b_i \), Lemma 13 part (3) implies \( b = \beta_j(0) = v_j \). For \( b = 0 \): If 0 is an undominated bid for \( j \), then \( v_j = 0 \) and hence \( b = v_j \). ■

Corollary 8 At \( \mu^c \) the following must hold. Assume that bids \( b^- \) and \( b^+ \) satisfy \( 0 \leq b^- < b^+ \leq 1 \) and are optimal bids for bidder \( i \in \{1, 2\} \). Then for bidder \( j \neq i \) it holds that \( G_{H_j}(b^+) > G_{H_j}(b^-) \).

Proof. Proof is by contradiction. Suppose that \( G_{H_j}(b^+) = G_{H_j}(b^-) \). By Lemma 14 part (3), \( b^+ = \beta_i(G_{H_j}(b^-)) \). By Lemma 13 part (1) the bid \( b^+ \) strictly dominates \( b^- \), a contradiction. ■

Lemma 14 At \( \mu^c \) the following must hold.

1. \( b_{\min} = \max\{b_i, b_j\} < 1 \).

2. Suppose both bidders have the same infimum bid: \( b_i = b_j = b = b_{\min} \). Then \( b = \max\{v_i, v_j\} \). If \( v_i = v_j \), then neither bidder bids an atom at \( b \) (that is, \( G_{H_j}(b) = G_{H_i}(b) = 0 \)). However, if \( v_i < v_j \) then \( j \) bids an atom at \( b = v_j \) and \( i \) does not bid at \( b \).

3. Suppose bidder \( i \) has a higher infimum bid: \( b_i > b_j \). Then \( b = b_j = v_j \) and \( j \) bids an atom with some positive weight \( \Gamma > 0 \) at \( v_j \) but nowhere else at or below \( b_j \):

\[
G_{H_j}(b) = \begin{cases} 
0 & b < v_j \\
\Gamma & b \in [v_j, b_i]
\end{cases}
\]

Moreover, \( b_{\min} = b_i > v_i \).

Proof. (1) Suppose not and \( b_j = 1 \) for some \( j \in \{1, 2\} \). Then from equation (39) we see that, for \( i \neq j \), \( \Pi_i^+(v_i) = \Pr[L_j|H_i]\left(\hat{R}(v_i)v_i - \int_0^{v_i} x \hat{r}(x)dx\right) \) and \( \Pi_i(b) = \Pi_i^+(v_i) + \int_{v_i}^{b_i}(v_i - x)\hat{r}(x)dx \) for all \( b > v_i \). This implies bidding \( b > v_i \) is not optimal for \( i \) because \( \Pi_i^+(v_i) > \Pi_i(b) \). As \( i \) must bid at least \( v_i \), this means that \( i \) bids \( v_i \) with probability 1. Applying Lemma 13 with \( b^- = v_i \) and \( b^+ = 1 \) implies that \( j \)'s bid of 1 must equal \( \beta_j(1) \). This is a contradiction as \( v_j < 1 \) and Assumption 1 imply \( \beta_j(1) < 1 \).

(2) It cannot be the case that both bidders have an atom at \( b \). As part (1) implies \( b < 1 \), this follows from Lemma 11. Therefore, suppose \( i \) does not have an atom at \( b \). This implies that \( \Pi_j(b) \) is continuous at \( b \). The assumption that \( b_j = b \) implies that \( j \) bids with positive probability either at \( b \) or in every neighborhood above \( b \). Therefore, \( b \) must be an optimal bid for \( j \) by continuity of \( \Pi_j(b) \) at \( b \). Because \( j \) has an optimal bid at \( b_j = 1 \), Corollary 7 implies that \( b_i = v_j \). Moreover, \( b_j > v_i \) by Lemma 9. Therefore \( v_i \leq v_j \) and \( b = \max\{v_i, v_j\} \).
Suppose that $v_i < v_j$ and $j$ does not bid an atom at $b$. Then $\Pi_i(b)$ is continuous at $b$ and hence $b < 1$ is an optimal bid for $i$ and Corollary \[7\] implies $b = v_i$, which is a contradiction. Thus $v_i < v_j$ implies $j$ has an atom at $b$. (Hence Lemma \[11\] implies that $i$ does not bid at $b < 1$.)

Suppose that $v_i = v_j$ and $j$ has an atom of weight $\Gamma > 0$ at $b$. In this case, $b = v_i = v_j$ as we showed above that $b = \max\{v_i, v_j\}$. As we have assumed $b_i = b$, Lemma \[12\] implies that bidder $i$’s infimum bid must be at least $b_i \geq v_i^{\mathrm{win}}(b)$. This is a contradiction because (i) equation \[42\] and $G_{H_j}(b) = \Gamma > 0$ imply $v_i^{\mathrm{win}}(b) > v_i$, and (ii) $v_i = b = b_i$. Thus $v_i = v_j$ implies neither bidder has an atom at $b$.

(3) The assumption $b_j < b_i$ implies that $j$ bids with some positive probability $\Gamma > 0$ below $b_j$. By Corollary \[7\], $j$ can only bid below $b_j$ at $v_j$. Therefore $j$ bids with atom $\Gamma$ at $b_j = v_j$ and nowhere else below $b_j$. Moreover, given $b_i < 1$ from part (1), Lemma \[11\] and Corollary \[8\] together imply that $j$ does not bid at $b_j$, and therefore $v_i^{\mathrm{win}}(b_j) = \beta_i(\Gamma)$. Finally, Lemma \[12\] implies that for all bids $b \geq b_j$, bidder $i$ must bid at least $v_i^{\mathrm{win}}(b_j) = \beta_i(\Gamma) > v_i$. ■

Lemma \[14\] begins to characterize bidder’s infimum bids. Our next goal is to prove that if $j$ has an atom at $b$ then $b$ is $j$’s infimum bid. Before proving this result in Lemma \[16\], we prove some helpful claims collected in Lemma \[15\]. To do so, we first define $x_i(b)$ for $i \in 1, 2$ to be the supremum bid placed below $b$ by bidder $i$:

$$x_i(b) = \begin{cases} \sup \{x : G_{H_i}(x) < G_{H_i}^-(b)\} & G_{H_i}^- (b) > 0 \\ -\infty & G_{H_i}^- (b) = 0 \end{cases} \quad (45)$$

For example, suppose that bidder $j$ has an atom at $b \in (0, 1)$. By Lemma \[11\] bidder $i$ does not bid in $(b - \delta, b]$ for some $\delta > 0$. In this case, $x_i(b) < b$ is the supremum point below $b$ at which bidder $i$ does place a bid.

**Lemma 15** At $\mu^\epsilon$, if $j$ has an atom at $b \in (0, 1)$ and $b$ is not $j$’s infimum bid ($0 \leq b_j < b$) then:

1. It holds that $v_j \leq x_j(b) < x_i(b) < b$.

2. In the interval $(x_j(b), b)$, $i$ bids an atom at $x_i(b) = \beta_i(G_{H_j}(x_i(b)))$ but nowhere else.

3. $j$ bids with an atom at $x_j(b) = \beta_j(G_{H_i}(x_j(b)))$.

4. $b = \beta_j(G_{H_i}(b))$.

**Proof.** We prove the claims:

1. We prove that $v_j \leq x_j(b) < x_i(b) < b$: First, by assumption ($b_j < b$) bidder $j$ bids with positive probability below $b$. Such bids must be at least $v_j$ and therefore $x_j(b) \geq v_j$. Second,
given $b < 1$, Lemma 11 implies that $x_i(b) < b$ and $G_{Hi}$ is continuous at $b$. Third, it only remains to show $x_j(b) < x_i(b)$. To do so we show that assuming either (a) $x_j(b) > x_i(b)$ or (b) $x_j(b) = x_i(b)$ leads to a contradiction and so can be ruled out. In both cases, deriving a contradiction relies on the fact that $G_{Hi}(x_i(b)) = G_{Hi}(b)$, which holds because $G_{Hi}(x)$ is everywhere right continuous, continuous at $b$, and constant on $(x_i(b), b)$.

(a) Suppose that $x_j(b) > x_i(b)$: Then there exists some bid $b^- \in [x_i(b), b)$ where $j$ bids. Then by Corollary 8 $G_{Hi}(b^-) < G_{Hi}(b)$ which contradicts $G_{Hi}(x_i(b)) = G_{Hi}(b)$ and $x_i(b) \leq b^- < b$.

(b) Suppose that $x_j(b) = x_i(b)$: Then for all $\delta > 0$, bidder $j$ has an optimal bid in the interval $(x_i(b) - \delta, x_i(b)]$. So, by Lemma 11 $i$ does not have an atom at $x_i(b) < 1$ and hence $\Pi_j(b_j)$ is continuous at $b_j = x_i(b)$. Since $j$ has an optimal bid at $x_i(b)$ or arbitrarily close to $x_i(b)$, continuity implies that $x_i(b)$ must be an optimal bid. Then by Corollary 8 $G_{Hi}(x_i(b)) < G_{Hi}(b)$ which contradicts $G_{Hi}(x_i(b)) = G_{Hi}(b)$.

2. By part (1) and the definition of $x_i(b)$, $j$ does not bid with positive probability in the interval $(x_j(b), b)$ but $i$ does. As a result, part (3) of Lemma 13 implies part (2).

3. There are two cases, either $b_i = x_i(b)$ or $b_i < x_i(b)$. (If $x_i(b) < b_i$ then $x_i(b) = -\infty$, which is ruled out by part (1).) Case (i) $b_i = x_i(b)$: By part (1), $j$ bids with positive probability below $x_i(b)$. Therefore, if bidder $i$’s infimum bid is at $b_i = x_i(b)$, Lemma 14 part 3 implies that $j$ bids with an atom at $x_j(b) = v_j$. Case (ii) Bidder $i$ bids with positive probability below $x_i(b)$ and $b_i < x_i(b)$: Parts (1) and (2) of this lemma can be applied to the atom at $x_i(b)$ and, noting that $x_j(x_i(b)) = x_j(b)$, these imply that $j$ bids with an atom at $x_j(b) = \beta_j(G_{Hi}(x_j(b))) > v_i$.

4. Part (4) follows from Corollary 6.

Lemma 16 At $\mu^e$, if $j \in \{1, 2\}$ has an atom at $b < 1$ then $b$ is $j$’s infimum bid: $b = b_j$.

Proof. Suppose not and $j$ bids an atom at $b < 1$ and with positive probability in a neighborhood of $b^- < b$. Then by Lemma 15, $j$ bids with an atom at $x_j(b) = \beta_j(G_{Hi}(x_j(b)))$, $i$ bids with an atom at $x_i(b) \in (x_j(b), b)$, $b = \beta_j(G_{Hi}(b))$, and there are no other bids in the interval $(x_j(b), b)$. We will show a contradiction by showing that $\Pi_j(b) > \Pi_j(x_j(b))$. Let $\Gamma_1 = G_{Hi}(x_j(b))$ and $\Gamma_2 = G_{Hi}(x_i(b)) = G_{Hi}(b)$.

Let $\Pi_j^-$ and $\Pi_j^+$ be the left and right hand limits of $\Pi_j$ respectively. We write down the difference in profit between bidding at $x_j(b)$ and $b$ for bidder $j$ in three parts corresponding to
The inequalities in equations (46) and (47) imply that $b > x$ which is positive since $Atb \beta$. The fact that $\int \hat{R}(x_i(b)) - \int \hat{R}(x_j(b))$ gives both $b$ and $x_i(b)$ is positive since $b = \beta_j(\Gamma_2)$ implies the following integral is positive:

$$\int_{x_i(b)}^{b} (\beta_j(\Gamma_1) - t) \hat{r}(t) dt = \int_{x_i(b)}^{b} (b - t) \hat{r}(t) dt > 0. \quad (46)$$

The inequalities in equations (46) and (47) imply that

$$\Pi_j^-(x_i(b)) - \Pi_j^+(x_i(b)) > (G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) \hat{R}(x_i(b)) (x_i(b) - x_j(b))$$

$$+ \Pr[H_i|H_j] (G_{Hi}(x_i(b)) - x_i(b)) = \Pi_j^-(x_i(b)) - \Pi_j^+(x_i(b)) \hat{R}(x_i(b)) (1 - x_i(b)) \quad (48)$$

Substituting $x_j(b) = \beta_j(G_{Hi}(x_j(b))) = \frac{G_{Hi}(x_j(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j] v_j}{G_{Hi}(x_j(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]}$ into the right-hand side of equation (48) and canceling and regrouping terms gives

$$\hat{R}(x_i(b)) (G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) \left( \frac{G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j] v_j}{G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]} - x_i(b) \right).$$

Finally, since $G_{Hi}(x_i(b)) = G_{Hi}(b) = x_j(b)$ we can substitute in $b = \frac{G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j] v_j}{G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]}$ yielding

$$\hat{R}(x_i(b)) (G_{Hi}(x_i(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]) (b - x_i(b)),$$

which is positive since $b > x_i(b)$. Thus $\Pi_j^-(b) - \Pi_j^+(x_j(b)) > 0$. ■

Recall the definition $b_{min} = \max\{b_1, b_2\}$. In addition, for $i \in \{1, 2\}$, define $b_i = \sup\{b : G_{Hi}(b) < 1\}$ and $b_{max} = \max\{b_1, b_2\}$. Notice that $b_{max} \geq b_{min}$.

**Lemma 17** At $\mu^*$ the following must hold.

1. If $b_{max} > b_{min}$ then both bidders have the same supremum bid: $\bar{b}_i = \bar{b}_j = b_{max}$.
2. Both $G_{H1}$ and $G_{H2}$ are continuous for all $b \in (b_{min}, 1)$. Moreover, both $G_{H1}$ and $G_{H2}$ are increasing over the interval $(b_{min}, b_{max})$. 

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3. Suppose that $b_i > b_j$ so that $b_{\min} = b_i > b = b_j$. Then $j$ bids an atom at $b = b_j = v_j$ with mass $\Gamma_j$, $i$ bids an atom at $b_{\min} = b_i = \beta_1(G_{H_j}(v_j))$ with mass $\Gamma_i$, and $G_{H_j}(v_j) = G_{H_j}(b_{\min})$. If $b_{\max} = b_{\min}$ then both atoms have mass $\Gamma_i = \Gamma_j = 1$. Otherwise, $i$’s atom at $b_{\min}$ has mass:

$$\Gamma_i = \frac{\Pr[L_i| H_j]}{\Pr[H_i| H_j]} \frac{b_{\min}^{b_{\min}} (x - v_j) \hat{r}(x) dx}{\Pr(b_{\min}) (1 - b_{\min})}. \quad (49)$$

**Proof.**

1. Suppose not and $b_{\max} > \bar{b}_i > \bar{b}_j$. Then $j$ does not bid over $(\bar{b}_j, \bar{b}_i)$ but $i$ bids with positive probability in $(\bar{b}_j, \bar{b}_i)$. By part (3) of Lemma 13 and the definition of $\bar{b}_i$, this positive probability must be concentrated at a single atom at $\bar{b}_i = \beta_i(1)$. As $\beta_i(1) < 1$, Lemma 16 implies that $\bar{b}_i$ is $i$’s infimum bid: $\bar{b}_i = b_i$. Thus $\bar{b}_i = b_i \leq b_{\min} \leq b_{\max} = \bar{b}_i$, so $b_{\min} = b_{\max}$, a contradiction.

2. By Lemma 16, $G_{H_i}$ and $G_{H_j}$ are continuous for all $b \in (b_{\min}, 1)$. To show that they must also be increasing over $(b_{\min}, b_{\max})$ we consider and rule out two types of flat spots. Throughout, we assume $b_{\max} > b_{\min}$ (the claim is trivially satisfied for $b_{\max} = b_{\min}$).

Suppose that at least one bidder, say $i$, does not bid in an interval $(b^-, b^+)$ such that $G_{H_i}(b^-) = G_{H_i}^{-}(b^+) = \Gamma$ where

$$b_{\min} \leq b^- = \inf \{b : G_{H_i}(b) = \Gamma\} < b^+ = \sup \{b : G_{H_i}(b) = \Gamma\} \leq b_{\max}.$$ 

Note that $b_{\max} > b_{\min}$ and part (1) imply $\bar{b}_i = b_{\max}$ so $\Gamma < 1$. By part (3) of Lemma 13, $j$ can place at most one bid over $(b^-, b^+]$ and it must be less than 1. Thus, by Lemma 16, it cannot be an atom. Thus $G_{H_j}(b^-) = G_{H_j}(b^+)$ and neither bidder bids with positive probability over $(b^-, b^+)$. Moreover, $G_{H_j}(b^-) = G_{H_j}(b^+)$ implies that $b^+ < b_{\max}$. (Otherwise $G_{H_j}(b^-) = G_{H_j}(b^+) = 1$ and $\bar{b}_j = b^-$, which contradicts part (1).) This means that $b^+ < 1$, and hence the contrapositive of Lemma 16 implies that there are no atoms on $(b_{\min}, b^+]$ and so $G_{H_i}(b^-) = G_{H_i}(b^+) = \Gamma$.

Now consider two cases. First suppose that $b^- > b_{\min}$. In this case, $b^- > b_{\min}$ implies $\Gamma > 0$ and there are no atoms on $[b^-, b^+]$. Thus $\Pi_i(b)$ is continuous at $b^-$ and $b^+$. Then, the definitions of $b^-$ and $b^+$ (and $\Gamma \in (0, 1)$) therefore imply that $b^-$ and $b^+$ are both optimal bids for $i$. By Corollary 8, $G_{H_j}(b^-) < G_{H_j}(b^+)$, a contradiction.

Second, suppose that $b^- = b_{\min}$. By the definition of $b^+$ and the fact that $j$ does not bid an atom at $b^+$, meaning $\Pi_i$ is continuous at $b^+$, $b^+$ must be an optimal bid for $i$. As (by definition) $b_{\min}$ must be the infimum bid of one or both bidders, the lack of bidding over $(b_{\min}, b^+]$ implies that one (but not both by Lemma 11) bidder has an atom at $b_{\min}$.

Suppose (i) $i$ has the atom at $b_{\min}$. Then $i$ has optimal bids at $b_{\min}$ and $b^+$ but $G_{H_j}(b_{\min}) = G_{H_j}(b^+)$, contradicting Corollary 8.

Suppose instead (ii) that $j$ has the atom at $b_{\min}$. By Corollary 8, $b^+$ is not an optimal bid for $j$ because $b_{\min}$ is optimal but $G_{H_i}(b_{\min}) = G_{H_i}(b^+)$. Because $i$ does not bid an atom at $b^+$, $\Pi_j(b)$
is continuous at \( b^+ \) and \( j \) does not have an optimal bid in a neighborhood \((b^+ - \delta, b^+ + \delta)\) for \( \delta > 0 \) sufficiently small. However, \( i \) must bid with positive probability in this interval by definition of \( b^+ \). By part \( \text{(3)} \) of Lemma \( \text{13} \), this probability must be concentrated at an atom, which (for any \( \delta < 1 - b^+ \)) contradicts Lemma \( \text{16} \)’s requirement that there be no atoms on \((b_{\min}, 1)\).

\( \text{(3)} \) Lemma \( \text{14} \) and \( b_j < b_i \) imply that \( j \) bids an atom at \( b_i = b_j = v_j \) and \( G_{H_j}(v_j) = G_{H_j}(b_{\min}) \).

The final step in the proof is to show that \( i \) bids an atom at \( b_{\min} \) and to calculate its size. Then Corollary \( \text{6} \) implies \( b_{\min} = \beta_i(G_{H_j}(b_{\min})) \), which means that \( b_{\min} = b_i = \beta_i(G_{H_j}(v_j)) \). We complete the final step for two cases:

(i) The assumption that \( b_j < b_i = b_{\min} = b_{\max} \) implies that \( i \) bids \( b_{\min} \) with probability 1 and \( G_{H_j}(b_{\min}) = 1 \). Therefore, as \( b_j = v_j \) and \( G_{H_j}(v_j) = G_{H_j}(b_{\min}) = 1 \), \( j \) bids \( v_j \) with probability 1.

(ii) \( b_{\max} > b_{\min} \): By part \( \text{(2)} \) of this Lemma, for any \( \delta > 0 \) bidder \( j \) has an optimal bid within the interval \((b_{\min}, b_{\min} + \delta)\). We show that this means that bidder \( i \) must have an atom at \( b_{\min} = b_i \): Suppose not and \( G_{H_i}(b_{\min}) = G_{H_i}(0) \). Then \( b_{\min} \) will be an optimal bid for \( j \) by continuity but \( b_j \) is also an optimal bid for \( j \). This contradicts Corollary \( \text{6} \) given \( G_{H_i}(b_{\min}) = G_{H_i}(0) \).

To compute \( \Gamma_i \) we observe that the utility of \( j \) is the same across all bids in the support, in particular at his atom at \( b_j = v_j \) and at any optimal bid \( b_j > b_{\min} \) that is arbitrarily close to \( b_{\min} \) (such a bid exists for any \( \delta > 0 \) in the interval \((b_{\min}, b_{\min} + \delta)\) since \( b_{\max} > b_{\min} \)). Thus the change in utility from increasing the bid from \( v_j \) to such \( b_j \) is zero. The next equation presents this utility change in the limit when \( b_j \) tends to \( b_{\min} \) from above.

\[
\Pi_j^+(b_{\min}) - \Pi_j(v_j) = \hat{R}(b_{\min}) \Pr[H_i|H_j] \Gamma_i(1 - b_{\min}) - \Pr[L_i|H_j] \int_{v_j}^{b_{\min}} (x - v_j) \hat{r}(x) \, dx = 0
\]

Solving for \( \Gamma_i \) then yields equation \( \text{49} \).

We are now ready to prove Lemma \( \text{1} \).

**Proof.** (of Lemma \( \text{1} \)) (1) Follows from Lemma \( \text{14} \) and Lemma \( \text{16} \). (2) Follows from the definition of \( b_{\max} \) and Lemma \( \text{17} \) parts \( \text{i} \) and \( \text{ii} \). (3) \( G_{H_i}(b) = 0 \) for every \( b \in (0, b_{\min}) \) and \( G_{H_j}(b) = 0 \) for every \( b \in (0, v_j) \) follow from the definitions of \( b_i \) and \( b_j \) and part \( \text{i} \) of this lemma. \( G_{H_j}(b) = G_{H_j}(v_j) \) for every \( b \in (v_j, b_{\min}] \) follows Lemma \( \text{14} \) part \( \text{iii} \). (4) Follows from Lemma \( \text{14} \) part \( \text{ii} \) and Lemma \( \text{16} \). (5) Follows almost entirely from Lemma \( \text{17} \) part \( \text{iii} \). The fact that \( b_{\min} \leq v(H_i) \) and \( b_{\min} = v(H_i) \) if and only if \( G_{H_j}(v_j) = 1 \) follows from inspection of equation \( \text{26} \), the fact that \( v(H_i) = \Pr[H_j|H_i] + v_i \Pr[L_j|H_i] \), and \( v_i < 1 \). The fact that \( b_{\min} > \max\{v_i, v_j\} \) follows because Lemma \( \text{17} \) part \( \text{iii} \) implies \( b_{\min} > v_j \) and \( G_{H_j}(v_j) > 0 \) while equation \( \text{26} \), \( G_{H_j}(v_j) > 0 \), and \( v_i < 1 \) imply that \( b_{\min} > v_i \). (6) Follows from undominated bids, \( \text{4} \) and \( \text{5} \), and the assumption \( \max\{v_1, v_2\} < 1 \).
H.2 Proof of Lemma 2 (Necessary Conditions Part II)

In this section we prove Lemma 2. Let $R$ be a standard distribution, $\epsilon > 0$, and $\mu^\epsilon$ be a Nash equilibrium in undominated bids of the game $\lambda(\epsilon, R)$.

H.2.1 Characterizations of the CDFs of $G_{H1}$ and $G_{H2}$ for $b > b_{min}$

**Lemma 18** At $\mu^\epsilon$, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid $b \in (b_{min}, b_{max})$, if $G_{Hj}(b)$ is differentiable at $b$ then it holds that

$$\frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b - v_i}{1-b} \cdot \frac{\hat{r}(b)}{R(b)} = g_j(b) + \frac{\hat{r}(b)}{R(b)} \cdot G_{Hj}(b)$$  \hspace{1cm} (50)

**Proof.** By Lemma 1 part (2), all bids $b \in (b_{min}, b_{max})$ are optimal for bidder $i$. $\Pi_i(b)$ is differentiable at $b$ because $G_{Hj}(b)$ is differentiable at $b$ by assumption. Therefore the first-order condition $d\Pi_i(b)/db = 0$ is necessary for optimality of $i$'s bid $b$. Equation (50) follows from setting equation (43) to zero and rearranging terms.]

Our next lemma follows by applying the following well known differential-equation result (Boyce and DiPrima 1986 Theorem 2.1) to the first-order condition derived in Lemma 18.

**Lemma 19** Assume that $q(x) = u'(x) + p(x) \cdot u(x)$ holds for every $x \in (b_{min}, b)$ but a set of measure zero, and $p(x)$ and $q(x)$ are continuous on the interval. Define $z(x) = e^{\int_{b_{min}}^{x} p(y)dy}$. Then every function $u(x)$ that satisfies the assumption is of the form

$$u(b) = \frac{1}{z(b)} \left( \int_{b_{min}}^{b} z(x)q(x)dx + u(b_{min}) + C \right)$$  \hspace{1cm} (51)

for some $C$.

**Lemma 20** At $\mu^\epsilon$, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid $b$ satisfying $b < 1$ and $b \in [b_{min}, b_{max}]$, it must hold that

$$G_{Hj}(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{R(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_i}{1-x} \cdot r(x)dx + G_{Hj}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{R(b)}$$  \hspace{1cm} (52)

**Proof.** At any bid $b \in (b_{min}, b_{max})$ for which $G_{Hj}$ is differentiable, equation (50) holds by Lemma 18. $G_{Hj}$ is differentiable almost everywhere because it is nondecreasing. \footnote{See, for example, Theorem 31.2 in Billingsley 1995.} $G_{Hj}$ and $G_{Hi}$ are continuous for all $b \in (b_{min}, 1)$ (Lemma 1 part 1).

This is a first-order ODE. We apply Lemma 19 with $u(b) = G_{Hj}(b)$, $u'(b) = g_j(b)$, $q(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b - v_i}{1-b} \cdot \frac{\hat{r}(b)}{R(b)}$ and $p(b) = \frac{\hat{r}(b)}{R(b)}$. We observe that $\int_{b_{min}}^{\hat{R}(x)} p(y)dy = \int_{b_{min}}^{\hat{R}(x)} \frac{\hat{r}(y)}{R(y)}dy = \ln(\hat{R}(x)) -$
\ln(\hat{R}(b_{\min})) \text{ and thus } z(x) = e^{\int_{b_{\min}}^{x} p(y) dy} = \hat{R}(x)/\hat{R}(b_{\min}). \text{ Therefore, for all } b \in (b_{\min}, b_{\max}),

\begin{equation}
G_{Hj}(b) = \frac{\hat{R}(b_{\min})}{\hat{R}(b)} \left( \int_{b_{\min}}^{b} \frac{\hat{R}(x)}{\hat{R}(b_{\min})} \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x-v_i}{1-x} \cdot \hat{r}(x) dx + G_{Hj}(b_{\min}) + C \right) \\
= \frac{1}{\hat{R}(b)} \int_{b_{\min}}^{b} \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x-v_i}{1-x} \cdot \hat{r}(x) dx + \frac{\hat{R}(b_{\min})}{\hat{R}(b)} (G_{Hj}(b_{\min}) + C) \tag{53}
\end{equation}

As \(G_{Hj}\) is right continuous everywhere and continuous for all \(b \in (b_{\min}, 1)\), the constant \(C\) is 0 and equation (52) then follows for all \(b\) that satisfy \(b < 1\) and \(b \in [b_{\min}, b_{\max}]\).

### H.2.2 Preliminary small \(\epsilon\) results

We next show that for sufficiently small \(\epsilon\) it holds that \(b_{\max} > b_{\min}\) (ruling out the case \(b_{\max} = b_{\min}\) allowed for in Lemma [1].

**Lemma 21** At \(\mu^\epsilon\) the following must hold. If \(\epsilon > 0\) is small enough then \(b_{\max} > b_{\min}\).

**Proof.** Assume that \(b_{\max} = b_{\min}\). It cannot be the case that \(b_{\min} = b\) as it means that both agents are bidding an atom (of size 1) at \(b\). By Lemma [14] \(b_{\min} < 1\). Lemma [11] then implies that both agents cannot have an atom at \(b_{\min}\), a contradiction. We conclude that \(b_{\min} > b\).

Given \(b_{\max} = b_{\min} > b\), Lemma [1] implies that one agent, say \(j\), is bidding an atom of size 1 at \(v_j\), while the other agent \(i\) is bidding an atom of size 1 at \(b_{\min} = \beta(1) = v(H_i) < 1\). \((v(H_i) < 1\) follows from \(v_i < 1\).\) When \(\epsilon\) is small enough agent \(j\) can deviate and get higher utility by bidding \(b^+ \in (b_{\min}, 1)\). This deviation has two effects. First it means that \(j\) has additional wins when \(i\) has a low signal and the random bidder bids between \(v_j\) and \(b^+\) causing \(j\) to pay more than the value \(v_j\). This costs bidder \(j\)

\[ \epsilon \Pr[L_i|H_j] \int_{v_j}^{b^+} (x-v_j) r(x) dx < \epsilon. \]

In addition, the deviation means that \(j\) has additional wins when \(i\) has a high signal and the random bidder bids below \(b^+\). All of these incremental wins are valued at 1 but cost no more than \(b^+\) so increase \(j\)'s payoff. Considering just those incremental wins for which the random bidder does not enter, this benefit is bounded below by \((1-\epsilon) \Pr[H_i|H_j](1-b_{\min}) = (1-\epsilon) \Pr[H_i|H_j](1-v(H_i)) > 0\). Thus \(\epsilon < (1-\epsilon) \Pr[H_i|H_j](1-v(H_i))\) is a sufficient condition for the deviation to be strictly profitable. This contradiction shows \(b_{\max} > b_{\min}\).

We further show that \(b_{\max} < 1\) but tends to 1 as \(\epsilon\) goes to 0.

**Lemma 22** Fix a small \(\delta > 0\). At \(\mu^\epsilon\) the following must hold. If \(\epsilon > 0\) is small enough then it holds that \(1 > b_{\max} > 1-\delta\). That is, \(b_{\max} < 1\) but approaches 1 as \(\epsilon\) goes to 0.
Proof. By Lemma 20 for each bidder $i \in \{1, 2\}$ and $j \neq i$, $b_{\max}$ must satisfy:

$$G_{Hi}(b_{\max}) = \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \cdot \frac{\epsilon}{R(b_{\max})} \int_{b_{\min}}^{b_{\max}} \frac{x - v_i}{1 - x} \cdot r(x)dx + G_{Hi}(b_{\min}) \cdot \frac{\hat{R}(b_{\min})}{R(b_{\max})}. \quad (54)$$

The integral $\int_{b_{\min}}^{b_{\max}} \frac{x - v_i}{1 - x} \cdot r(x)dx$ is finite for any $b_{\min} < b_{\max} < 1$ but approaches infinity in the limit as $b_{\max}$ goes to 1. This follows because $r(x)$ is bounded away from zero by $\mathcal{R} = \min_{x \in [0,1]} r(x) > 0$, Lemma 1 bounds $b_{\min} \leq k$ for some $k \in (0,1)$, and $v_i < 1$ by assumption. Thus:

$$\int_{b_{\min}}^{b_{\max}} \frac{x - v_i}{1 - x} \cdot r(x)dx \geq \int_{k}^{b_{\max}} \frac{x - v_i}{1 - x}dx = \hat{r} (1 - v_i)\ln \left( \frac{1 - k}{1 - b_{\max}} \right) + k - b_{\max}$$

and, for any fixed $\epsilon > 0$, it must hold that $b_{\max} < 1$ for the right-hand side of equation (54) to be finite. Therefore $G_{Hi}(b_{\max}) = 1$, as $b_{\max} \in (b_{\min}, 1)$ and Lemma 1 part (1) imply there is no atom at $b_{\max}$.

As $\epsilon$ approaches zero, Lemma 21 Lemma 1 and equation (27) imply that, for some bidder $i$, either $G_{Hi}(b_{\min}) = 0$ or $G_{Hi}(b_{\min})$ approaches zero (as in equation (27), $\hat{r}(x)$ approaches 0, $\hat{R}(b_{\min})$ approaches 1, and $b_{\min}$ is bounded away from 1). Let us fix this bidder to be named $i$ and consider equation (54). For the first term to approach $G_{Hi}(b_{\max}) = 1$ as $\epsilon$ approaches zero requires the integral to approach infinity. Thus $b_{\max}$ tends to 1 as $\epsilon$ goes to 0. $lacklozenge$

Next, for $0 \leq a < b < 1$, let

$$\chi(a, b) = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \int_a^b \frac{x - v_1}{1 - x}r(x)dx.$$  

(55)

Lemma 23 Fix a standard distribution $R$ and a sequence of positive $\epsilon$ converging to zero. For each $\epsilon$, let $\mu^{\epsilon}$ be a NE in undominated bids of the tremble $(\epsilon, R)$. For each $\epsilon > 0$ sufficiently small,

$$v_1 \geq v_2 \Leftrightarrow \chi(b_{\min}, b_{\max}) \leq \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2}. \quad (56)$$

Moreover, in the limit as $\epsilon$ converges to zero,

$$\lim_{\epsilon \to 0} \chi(b_{\min}, b_{\max}) = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2}. \quad (57)$$

Proof. For $0 \leq a < b < 1$, let

$$T(a, b) = \frac{\int_a^b \frac{x}{1-x} \cdot r(x)dx}{\int_a^b \frac{x - v_1}{1-x} \cdot r(x)dx}.$$  

(58)

We first prove some useful results about the function $T(a, b)$.

Claim 1 (1) $\chi(a, b) = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} T(a, b)$. (2) For all $0 \leq a < b < 1$, $T(a, b) > 1$. (3) For fixed $a \in [0,1)$, $\lim_{b \to 1} T(a, b) = 1$. 

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Proof. (1) Follows from substituting the definition of $T(a, b)$ into $\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2}$, multiplying through by $\int_a^b \frac{1}{1-x} \cdot r(x) dx$, and comparing to the definition of $\chi(a, b)$.

(2) Follows because $0 < a < b < 1$ implies $\frac{1}{1-x} > \frac{x}{1-x} \geq 0$ for all $x \in [a, b]$.

(3) Since $r$ is continuous and positive on the compact set $[0, 1]$, it has a positive minimum: $r(x) \geq r = \min_{x \in [0, 1]} r(x) > 0$. Then for $b > 1/2$, it holds that:

$$\int_a^b \frac{1}{1-x} r(x) dx \geq \int_a^b \frac{x}{1-x} r(x) dx \geq \int_{1/2}^b \frac{x}{1-x} dx \geq \frac{1}{2} \int_{1/2}^b \frac{1}{1-x} dx.$$ 

Now we observe that both the numerator and the denominator of $T(a, b)$ tend to infinity when $b$ tends to 1 as

$$\lim_{b \to 1} \int_{1/2}^b \frac{1}{1-x} dx = \lim_{b \to 1} \left( \ln \left( \frac{1}{2} \right) - \ln (1-b) \right) = \infty.$$ 

Thus by L'Hôpital's rule,

$$\lim_{b \to 1} \int_a^b \frac{1}{1-x} r(x) dx = \lim_{b \to 1} \int_a^b \frac{x}{1-x} r(x) dx = \lim_{b \to 1} \frac{1}{1-b^r(b)} = \lim_{b \to 1} \frac{1}{b} = 1.$$ 

Claim 2 $\lim_{\epsilon \to 0} T(b_{\min}, b_{\max}) = 1$.

Proof. First, note that for all $0 \leq a < b < 1$,

$$\frac{d}{da} T(a, b) = -\frac{r(a)}{1-a} \left( \int_a^b \frac{1}{1-x} r(x) dx \right)^2 \int_a^b \frac{x-a}{1-x} r(x) dx < 0.$$ 

Therefore, for all $\epsilon > 0$ sufficiently small, $0 \leq b_{\min} < b_{\max} < 1$ and $T(b_{\min}, b_{\max}) \in [1, T(0, b_{\max})]$. The lower bound $T(b_{\min}, b_{\max}) \geq 1$ follows from Claim 1. The upper bound $T(b_{\min}, b_{\max}) \leq T(0, b_{\max})$ follows from $dT(a, b)/da < 0$. Finally, Claim 1 and Lemma 22 imply $\lim_{\epsilon \to 0} T(0, b_{\max}) = 1$, which in turn implies that $\lim_{\epsilon \to 0} T(b_{\min}, b_{\max}) = 1$. ■

The fact that $\lim_{\epsilon \to 0} \chi(b_{\min}, b_{\max}) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2}$ follows immediately from Claims 1 and 2. The first result follows because, by assumption that $v_1, v_2 \in [0, 1)$ and Claim 2, for all $\epsilon > 0$ sufficiently small we have $0 \leq v_1 \leq v_1 T(b_{\min}, b_{\max}) < 1$ and $0 \leq v_2 \leq v_2 T(b_{\min}, b_{\max}) < 1$. Moreover, by Claim 1 $T(b_{\min}, b_{\max}) > 1$. Therefore, the following inequalities are equivalent:

$$\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} \leq \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2}$$

$$1 - v_1 T(b_{\min}, b_{\max}) (1 - v_2) \leq (1 - v_2 T(b_{\min}, b_{\max}) (1 - v_1)$$

$$v_1 (T(b_{\min}, b_{\max}) - 1) \geq v_2 (T(b_{\min}, b_{\max}) - 1)$$

$$v_1 \geq v_2$$

Given $\chi(a, b) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} T(a, b)$ from Claim 1, this proves the result. ■
H.2.3 Proof of Lemma 2

We next prove Lemma 2. For brevity, we define $\alpha_1 = \Pr[L_1|H_2]/\Pr[H_1|H_2]$, $\alpha_2 = \Pr[L_2|H_1]/\Pr[H_1|H_1]$, and observe that

$$\frac{\alpha_2}{\alpha_1} = \frac{\Pr[L_2|H_1]}{\Pr[L_1|H_2]} \cdot \frac{\Pr[H_1|H_2]}{\Pr[H_1|H_1]} = \frac{\Pr[L_2|H_1]}{\Pr[L_1|H_2]} \cdot \frac{\Pr[H_1]}{\Pr[H_1]} = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \quad (63)$$

**Proof.** (of Lemma 2) Recall that we label bidders following equation (1), or such that $v_1 \geq v_2$ when equation (1) holds with equality.

**Claim 3** Let $\phi(b) = \hat{R}(b_{\min})/\hat{R}(b)$. For sufficiently small $\epsilon > 0$ it holds that

$$\frac{1 - G_{H2}(b_{\min}) \cdot \phi(b_{\max})}{1 - G_{H1}(b_{\min}) \cdot \phi(b_{\max})} = \chi(b_{\min}, b_{\max}) \quad (64)$$

**Proof.** For sufficiently small $\epsilon > 0$, the inequality $b_{\min} < b_{\max} < 1$ follows from Lemmas 21-22.

Recall from Lemma 1 that $G_{H1}(b_{\max}) = G_{H2}(b_{\max}) = 1$. By $b_{\max} < 1$ and Lemma 20 for every bid $b \in [b_{\min}, b_{\max}]$ equation (52) holds. Therefore:

$$1 - G_{H2}(b_{\min}) \cdot \phi(b_{\max}) = \alpha_2 \cdot \frac{\epsilon}{\hat{R}(b_{\max})} \int_{b_{\min}}^{b_{\max}} \frac{x - v_1}{1 - x} \cdot r(x) dx,$$

$$1 - G_{H1}(b_{\min}) \cdot \phi(b_{\max}) = \alpha_1 \cdot \frac{\epsilon}{\hat{R}(b_{\max})} \int_{b_{\min}}^{b_{\max}} \frac{x - v_2}{1 - x} \cdot r(x) dx.$$

The claim follows from dividing the two equations (since for $0 \leq b_{\min} < b_{\max} < 1$ both sides of the two equations are not 0, thus such a division is well defined).

**Claim 4** For sufficiently small $\epsilon > 0$ it holds that: There are no atoms ($G_{H1}(b_{\min}) = G_{H2}(b_{\min}) = 0$) if and only if both bidders are symmetric: $v_1 = v_2$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$.

**Proof.** For sufficiently small $\epsilon > 0$, the inequality $b_{\min} < b_{\max} < 1$ follows from Lemmas 21-22.

By Lemma 1 part (4), if $G_{H1}(b_{\min}) = G_{H2}(b_{\min}) = 0$ then $\hat{b} = b_{\min} = v_1 = v_2$. In such a case equation (64) reduces to $\alpha_2 = \alpha_1$. Now, recall that $\frac{\alpha_2}{\alpha_1} = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]}$, thus $\Pr[H_1, L_2] = \Pr[L_1, H_2]$ and the two agents are completely symmetric.

Now, assume that both bidders are symmetric, that is, $v = v_1 = v_2$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$, we want to show that no bidder has an atom. We next show that it cannot be the case that $b_{\min} > \hat{b}$. This is sufficient as, by Lemma 1 $b_{\min} = \hat{b}$ and $v_1 = v_2$ imply that no bidder has an atom, that is $G_{H2}(b_{\min}) = G_{H1}(b_{\min}) = 0$.

We next show that symmetry and $b_{\min} > \hat{b}$ implies a contradiction. For symmetric bidders, equation (64) implies that $G_{H1}(b_{\min}) = G_{H2}(b_{\min})$. Using Lemma 1 we observe the following. One bidder, w.l.o.g. bidder 2, bids an atom at $\hat{b} = v_1 = v_2 = v$, the other bidder (bidder 1) bids
an atom at \( b_{\min} > b = v \). Moreover, \( G_{H2}(b_{\min}) = G_{H2}(v) \) so \( G_{H1}(b_{\min}) = G_{H2}(v) \). Denote \( \Gamma = G_{H1}(b_{\min}) = G_{H2}(v) \). By equation (26),

\[
b_{\min} = \beta_1(\Gamma) = \frac{\Pr[H_2|H_1] \Gamma + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1] \Gamma + \Pr[L_2|H_1]},
\]

or equivalently,

\[
\Gamma = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{\min} - v_1}{1 - b_{\min}}.
\]

By equation (27),

\[
\Gamma = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{\min}} (x - v_2) \hat{r}(x)dx}{\hat{R}(b_{\min})(1 - b_{\min})}.
\]

Thus,

\[
\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{\min}} (x - v_2) \hat{r}(x)dx}{\hat{R}(b_{\min})(1 - b_{\min})} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{\min} - v_1}{1 - b_{\min}},
\]

or due to symmetry in conditional probabilities \((\alpha_1 = \alpha_2)\) and values \((v_1 = v_2 = v)\),

\[
\int_{v}^{b_{\min}} (x - v) \hat{r}(x)dx = \hat{R}(b_{\min})(b_{\min} - v).
\]

Integration by parts implies that

\[
\int_{v}^{b_{\min}} (x - v) \hat{r}(x)dx = (b_{\min} - v) \hat{R}(b_{\min}) - \int_{v}^{b_{\min}} \hat{R}(x)dx,
\]

and this can only equal \( \hat{R}(b_{\min})(b_{\min} - v) \) when \( b_{\min} = v \), a contradiction.  

We next consider the case that \( \Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2) \) but the bidders are not symmetric \((v_1 > v_2 \text{ and } \Pr[H_1, L_2] < \Pr[L_1, H_2])\).

**Claim 5** Assume that \( \Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2) \) but the bidders are not symmetric, and it holds that \( v_1 > v_2 \) and \( \Pr[H_1, L_2] < \Pr[L_1, H_2] \). For sufficiently small \( \epsilon > 0 \), bidder 1 has an atom at \( b_{\min} = b_1 > v_1 \) and bidder 2 has an atom at \( v_2 = b_2 = b < b_{\min} \).

**Proof.** For sufficiently small \( \epsilon > 0 \), the inequality \( b_{\min} < b_{\max} < 1 \) follows from Lemmas 21, 22.

By Claim 4 as bidders are not symmetric it cannot be the case that both bidders have no atom. We next show that it cannot be the case that only one bidder has an atom. By Lemma 1 if only one bidder has an atom and \( v_1 > v_2 \) it must be the case that \( b = b_{\min} = v_1 > v_2 \) and bidder 1 has the atom at \( v_1 \). But in this case, as \( G_{H2}(b_{\min}) = 0 \), the LHS of equation (64) equals to \( \frac{1}{1 - G_{H1}(b_{\min} - \phi(b_{\max}))} > 1 \) (as \( 0 < \phi(b_{\max}) \leq 1 \) and \( G_{H1}(b_{\min}) > 0 \)), while the RHS of equation (64) is at most 1 by Lemma 23 and the assumptions \( \Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2) \) and \( v_1 > v_2 \), a contradiction.
We conclude that both bidders have an atom, each at his infimum bid. We next figure out which bidder has an atom at \( b \) and which has an atom at \( b_{\text{min}} \). We first show that it must be the case that, although they are positive away from the limit, both \( G_{H1}(b_{\text{min}}) \) and \( G_{H2}(b_{\text{min}}) \) tend to 0 as \( \epsilon \) goes to 0. By equation (27), for one bidder \( i \) it holds that \( G_{Hi}(b_{\text{min}}) \) must tend to 0 as \( \epsilon \) goes to 0. This follows because the numerator tends to 0 while the denominator does not as \( b_{\text{min}} \leq \max\{v(H1), v(H2)\} \) < 1 (Lemma 1). Now, as the RHS of equation (64) tends to 1 as \( \epsilon \) goes to 0 (by Lemma 23 and the assumption \( \Pr[H1, L2](1 - v1) = \Pr[L1, H2](1 - v2) \)), \( \epsilon > 0 \) implies that \( G_{H1}(b_{\text{min}}) - G_{H2}(b_{\text{min}}) \) must tend to 0. Now, as both \( G_{H1}(b_{\text{min}}) \) and \( G_{H2}(b_{\text{min}}) \) tend to 0 as \( \epsilon \) goes to 0, by equation (26) \( b_{\text{min}} \) must tend to \( v_i \), where \( i \) is the bidder who bids an atom at \( b_{\text{min}} \). Now recall that, in the case that both bidders have an atom, it holds that \( b_{\text{min}} > b_j = v_j \) (Lemma 1).

Thus, \( v_1 \geq v_j \), as otherwise \( b_{\text{min}} \) could not approach \( v_i \) without violating \( b_{\text{min}} > v_j \). Given \( v_1 > v_2 \), we conclude that \( b_{\text{min}} = b_1 > b = b_2 = v_2 \), that is, bidder 1 has an atom at \( b_{\text{min}} = b_1 > v_1 \) and bidder 2 has an atom at \( v_2 = b_2 = b < b_{\text{min}} \), as we need to show. ■

**Claim 6** Assume that \( \Pr[H1, L2](1 - v1) < \Pr[L1, H2](1 - v2) \). For sufficiently small \( \epsilon > 0 \), one of the following holds: Either bidder 1 has no atom and bidder 2 has an atom at \( v_2 = b_2 = b = b_{\text{min}} \).

Or, bidder 1 has an atom at \( b_{\text{min}} = b_1 > v_1 \) and bidder 2 has an atom at \( v_2 = b_2 = b < b_{\text{min}} \).

**Proof.** For sufficiently small \( \epsilon > 0 \), the inequality \( b_{\text{min}} < b_{\text{max}} < 1 \) follows from Lemmas 21 22.

By Claim 4 as bidders are not symmetric it cannot be the case that both bidders have no atom. We next consider the case that at least one bidder has an atom. By Lemma 23 the RHS of equation (64) tends to 1 as \( \epsilon \) goes to 0. Therefore, equation (64) implies that \( G_{H1}(b_{\text{min}}) \) tend to 0 as \( \epsilon \) goes to 0.

Now, if only one bidder has an atom it must be bidder 2, since \( G_{H2}(b_{\text{min}}) = 0 \) implies \( G_{H1}(b_{\text{min}}) < 0 \), a contradiction. Moreover, by Lemma 1 this atom must be at \( v_2 = b_2 = b = b_{\text{min}} \).

If on the other hand both bidders have an atom, we claim that bidder 1 has an atom at \( b_{\text{min}} = b_1 \) and bidder 2 has an atom at \( v_2 = b_2 = b < b_{\text{min}} \). Observe also that \( \phi(b_{\text{max}}) = \frac{R(b_{\text{min}})}{R(b_{\text{max}})} \) tends to 1 as \( \epsilon \) goes to 0. Now, if bidder 2 is the bidder with the atom at \( b_{\text{min}} \), by equation (27) \( G_{H2}(b_{\text{min}}) \) must tend to 0 as \( \epsilon \) goes to 0. This follows because the numerator tends to 0 while the denominator does not as \( b_{\text{min}} \leq \max\{v(H1), v(H2)\} \) < 1 (Lemma 1). Combining with \( G_{H1}(b_{\text{min}}) < G_{H2}(b_{\text{min}}) \) this will imply that \( G_{H1}(b_{\text{min}}) \) must also tend to 0 as \( \epsilon \) goes to zero. But then the LHS of equation (64) tends to 1 while the RHS tends to \( \frac{\Pr[H1, L2](1 - v1)}{\Pr[L1, H2](1 - v2)} \) < 1, a contradiction. We conclude that bidder 1 has an atom at \( b_{\text{min}} = b_1 \) and bidder 2 has an atom at \( v_2 = b_2 = b < b_{\text{min}} \). ■

We now complete the proof of Lemma 2.

1. The inequality \( b_{\text{min}} < b_{\text{max}} < 1 \) follows from Lemmas 21 22.

2. Claims 4 6 identify bidder
and \( j \) from Lemma 1 as \( i = 1 \) and \( j = 2 \) and provide the conditions for the three cases (no atom, one atom, and two atoms). (3) Applying Lemma 1 (part 5) given \( 1 > b_{\text{max}} > b_{\text{min}} > v_2, i = 1, \) and \( j = 2 \), yields equation (29). Applying Lemma 1 (part 4) given \( 1 > b_{\text{max}} > b_{\text{min}} = v_2, i = 1, \) and \( j = 2 \), yields \( G_{H1}(b_{\text{min}}) = 0 \), which is consistent with equation 29. Thus equation (29) always holds (as \( 1 > b_{\text{max}} > b_{\text{min}} \geq v_2 \)). (4) By Claim 3, equation (64), \( G_{H2}(b_{\text{min}}) \) must satisfy

\[
G_{H2}(b_{\text{min}}) = \frac{1}{\phi(b_{\text{max}})} - \left( \frac{1}{\phi(b_{\text{max}})} - G_{H1}(b_{\text{min}}) \right) \cdot \chi(b_{\text{min}}, b_{\text{max}})
\]

Equation (30) then follows from the definition of \( \phi(b_{\text{max}}) \) and \( \chi(b_{\text{min}}, b_{\text{max}}) \). (5) Equations 31–32 follow from Lemma 1, Lemma 20, \( 1 > b_{\text{max}} > b_{\text{min}} \geq v_2 \), \( i = 1, \) and \( j = 2 \). (6) Lemma 1 parts 4–5 show that equation 26 holds in the two-atom case, which becomes equation (33) when substituting \( i = 1 \) and \( j = 2 \). ■

H.3 Proof of Lemma 3 (Existence of NE in \( \lambda(\epsilon, R) \))

We next show that for any standard distribution \( R \), if \( \epsilon \) is small enough then there exists a mixed NE in the game \( \lambda(\epsilon, R) \). We prove existence of one of three types of equilibria depending on parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3) equilibrium depending on whether or not equation (70) in the proof is satisfied.

First, in Lemma 24, we use Lemmas 1, 2, 22, and 23 to find the limits of \( G_{H1}(b_{\text{min}}) \) and \( G_{H2}(v_2) \) as \( \epsilon \) goes to zero; they are useful later in proving Lemma 3 and convergence to the TRE. We introduce Lemma 24 here, as it leads to the following Observation 2 that indicates why equation (70) determines whether asymmetric equilibria involve one or two atoms.

**Lemma 24** Fix a standard distribution \( R \), a sequence of \( \epsilon \) converging to zero, and an associated sequence of NE \( \{\mu^\epsilon\} \) in the trembles \( \lambda(\epsilon, R) \). Then it holds that \( \lim_{\epsilon \to 0} G_{H1}(b_{\text{min}}) = 0 \) and

\[
\lim_{\epsilon \to 0} G_{H2}(v_2) = 1 - \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{1 - v_1}{1 - v_2}.
\]

**Proof.** By Lemma 2, \( G_{H1}(b_{\text{min}}) \) must satisfy equation (29) and \( G_{H2}(v_2) \) must satisfy equation (30) for sufficiently small \( \epsilon > 0 \). By inspection, as \( b_{\text{min}} \) is bounded away from 1 (Lemma 1), it is clear that \( \lim_{\epsilon \to 0} G_{H1}(b_{\text{min}}) = 0 \). Turning to \( G_{H2}(v_2) \), we note that: (1) \( \frac{R(b_{\text{max}})}{R(b_{\text{min}})} = \frac{1 - c + R(b_{\text{max}})}{1 - c + R(b_{\text{min}})} \) approaches 1. (2) Lemma 23 implies that \( \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \left( \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{1 - v_1}{1 - v_2} r(x) dx \right) \) approaches \( \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} \). Therefore equation (66) holds. ■
Observation 2 If ε is small enough and \( G_{H1}(b_{\min}) > 0 \) (bidder 1 has an atom, which implies that bidder 2 also has an atom) then it must hold that

\[
\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \leq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}
\]

(67)

Proof. If \( G_{H1}(b_{\min}) > 0 \) then equation (63) holds. In particular it must hold that

\[
\frac{G_{H2}(v_2)}{G_{H2}(v_2) + \alpha_2} = 1 - \frac{\alpha_2(1 - v_1)}{G_{H2}(v_2) + \alpha_2} > v_2
\]

(68)

Lemma 24 states that \( G_{H2}(v_2) \) tends to 1 when \( \Pr[H_1,L_2](1-v_1) \Pr[L_1,H_2](1-v_2) = 1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)} \) as \( \epsilon \) goes to zero. Thus it must hold that

\[
1 - \frac{\alpha_2(1 - v_1)}{\left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right) + \alpha_2} \geq v_2
\]

(69)

and the claim follows from reorganizing the last equation. ■

H.3.1 Proof of Lemma 3

Proof. Throughout the proof we index bidders 1 and 2 such that either 1) \( \alpha_1(1 - v_2) = \alpha_2(1 - v_1) \) and \( v_1 \geq v_2 \), or 2) \( \alpha_1(1 - v_2) > \alpha_2(1 - v_1) \). Moreover, we often distinguish between three cases:

1. No atom case. Bidders are symmetric: \( v = v_1 = v_2 \) and \( \Pr[H_1,L_2] = \Pr[L_1,H_2] \). In this case we show there exists an equilibrium in which \( b_{\min} = v \) and neither bidder has an atom: \( G_{H1}(b_{\min}) = G_{H2}(v_2) = 0 \).

2. One atom case. Bidders are asymmetric (\( v_1 \neq v_2 \) or \( \Pr[H_1,L_2] \neq \Pr[L_1,H_2] \)) and equation (70) holds:

\[
\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \geq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}
\]

(70)

Note that asymmetry and equation (70) imply that \( \alpha_1(1 - v_2) > \alpha_2(1 - v_1) \) and \( v_2 > v_1 \). This is so as by assumption the RHS of equation (70) is nonnegative, this implies that \( v_2 \geq v_1 \). If \( v_2 = v_1 \) then the equation implies that \( \alpha_1 = \alpha_2 \) which means the bidders are symmetric, a contradiction. Therefore \( v_2 > v_1 \) and thus \( \alpha_1(1 - v_2) > \alpha_2(1 - v_1) \) (since in the case that \( \alpha_1(1 - v_2) = \alpha_2(1 - v_1) \) we assume that \( v_1 \geq v_2 \)).

In this case we show that there exists an equilibrium in which \( b_{\min} = v_2 \) and only bidder 2 has an atom: \( G_{H2}(v_2) > 0 \) and \( G_{H1}(b_{\min}) = 0 \).

3. Two atom case. Bidders are asymmetric (\( v_1 \neq v_2 \) or \( \Pr[H_1,L_2] \neq \Pr[L_1,H_2] \)) and equation (70) is violated. Note that either 1) \( \alpha_1(1 - v_2) = \alpha_2(1 - v_1) \) and \( v_1 > v_2 \), or 2) \( \alpha_1(1 - v_2) > \alpha_2(1 - v_1) \) are both feasible. In this case we show that there exists an equilibrium in which \( b_{\min} > \max\{v_1, v_2\} \) and both bidders have atoms: \( G_{H2}(v_2) > 0 \) and \( G_{H1}(b_{\min}) > 0 \).
In all cases, bidder \( i \in \{1, 2\} \) with signal \( L_i \) is bidding \( V_{LL} = 0 \). We construct distributions \( G_{H1} \) and \( G_{H2} \) using the necessary conditions in Lemma 2 and show that they form a NE. Equations (31) and (32) define \( G_{H1} \) and \( G_{H2} \) as a function of the four parameters \( b_{min}, b_{max}, G_{H1}(b_{min}), \) and \( G_{H2}(v_2) \). As a preliminary step, we prove two useful claims. Then there are three remaining steps to the proof. First we show existence of parameters \( b_{min}, b_{max}, G_{H1}(b_{min}), \) and \( G_{H2}(v_2) \) that satisfy the necessary conditions in Lemma 2. Second, we show that, for the chosen parameters, \( G_{H1} \) and \( G_{H2} \) are well defined distributions (nondecreasing, and satisfying \( G_{H1}(0) = G_{H2}(0) = 0 \) and \( G_{H1}(1) = G_{H2}(1) = 1 \)). Third we show that the constructed bid distributions are best responses. By construction, bidder \( i \in \{1, 2\} \) is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives equal or lower utility.

We begin with two preliminary claims:

Claim 7 In all three cases (no atoms, one atom, two atoms) \( G_{H2}(b) \) as defined in Lemma 2 is increasing in \( b \) for every \( b \in (b_{min}, b_{max}) \).

Proof. We need to show that in all three cases \( G_{H2}(b) \) is increasing in \( b \) for every \( b \in (b_{min}, b_{max}) \). For any such \( b \), \( G_{H2}(b) \) satisfies equation (52), and its derivative with respect to \( b \) is

\[
g_2(b) = \frac{\hat{r}(b)}{R(b)} \left( \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_{H2}(b) \right).
\]

To prove the claim it is sufficient to show that for every \( b \in (b_{min}, b_{max}) \):

\[
g_2(b) \cdot \frac{\hat{R}(b)}{\hat{r}(b)} = \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_{H2}(b) > 0.
\]

(71)

If \( G_{H2}(b) \leq 0 \) the claim follows from \( 1 \geq b_{max} > b > b_{min} \geq \max\{v_1, v_2\} \). Next assume that \( G_{H2}(b) \geq 0 \). We observe that for small enough \( \epsilon \) this is an increasing function in \( b \) for \( b \in (b_{min}, b_{max}) \):

\[
\frac{d}{db} \left( \frac{\hat{R}(b)}{\hat{r}(b)} g_2(b) \right) = \alpha_2 \cdot \frac{1 - v_1}{(1 - b)^2} - g_2(b) = \alpha_2 \cdot \frac{1 - v_1}{(1 - b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)} \left( \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_{H2}(b) \right)
\]

\[
\geq \alpha_2 \cdot \frac{1}{(1 - b)^2} \left( (1 - v_1) - \frac{\hat{r}(b)}{\hat{R}(b)} (b - v_1) (1 - b) \right)
\]

\[
\geq \alpha_2 \cdot \frac{1}{(1 - b_{min})^2} \left( 1 - v_1 - \frac{r(b)}{1 - \epsilon} \right).
\]

As \( 1 > v_1 \) and \( r(b) \) is bounded from above (\( r \) is continuous on a compact interval), for small enough \( \epsilon \) this is positive.

Thus, as the function \( \frac{\hat{R}(b)}{\hat{r}(b)} g_2(b) \) is increasing, to prove that it is positive for any \( b > b_{min} \) it would be sufficient to show that it is at least 0 at \( b_{min} \), or equivalently, that the following holds:

\[
\alpha_2 \cdot \frac{b_{min} - v_1}{1 - b_{min}} \geq G_{H2}(b_{min}).
\]

(72)
We show that equation (72) is satisfied for each of the three cases.

In the first case (no atoms), \( G_{H2}(v_2) = 0 \), and equation (72) clearly holds because \( b_{\min} \geq v_1 \).

In the third case (two atoms), \( G_{H2}(v_2) \) satisfies equation (33), which is exactly equivalent to equation (72) holding with equality.

Finally we consider the second case (one atom) in which \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \), equation (70) holds and \( G_{H2}(b_{\min}) = G_{H2}(v_2) > 0 \) satisfies equation (30) with \( G_{H1}(b_{\min}) = 0 \), and additionally, \( b_{\min} = v_2 > v_1 \) (this corresponds to the case that only bidder 2 has an atom). These conditions imply that

\[
G_{H2}(v_2) = \frac{\hat{R}(b_{\max})}{\hat{R}(v_2)} \left( 1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x) dx} \right).
\]

Which means that we need to show that

\[
\frac{\alpha_2 v_2 - v_1}{1 - v_2} \geq \frac{\hat{R}(b_{\max})}{\hat{R}(v_2)} \left( 1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x) dx} \right) = G_{H2}(v_2).
\]

Equation (31) and continuity of \( G_{H1} \) at \( b_{\max} \) determines \( b_{\max} \) and implies that \( \hat{R}(b_{\max}) = \alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} \hat{r}(x) dx \), thus:

\[
\frac{\hat{R}(b_{\max})}{\hat{R}(v_2)} = \frac{\hat{R}(b_{\max})}{\hat{R}(b_{\max}) - \int_{v_2}^{b_{\max}} \hat{r}(x) dx} = \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x) dx}{\int_{v_2}^{b_{\max}} \left( \alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) dx}
\]

We can now express \( G_{H2}(v_2) \) as a function of \( b_{\max} \) as follows:

\[
G_{H2}(v_2) = \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x) dx}{\int_{v_2}^{b_{\max}} \left( \alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) dx} \left( 1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x) dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x) dx} \right)
\]

\[
= \frac{\int_{v_2}^{b_{\max}} (\alpha_1 (x - v_2) - \alpha_2 (x - v_1)) \frac{r(x)}{1-x} dx}{\int_{v_2}^{b_{\max}} \left( \alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) dx}
\]

Note that as \( \epsilon \) converges to 0, \( b_{\max} \) tends to 1 (Lemma 22) and \( G_{H2}(v_2) \) tends to \( 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} \) (Lemma 24). By equation (70) it is thus sufficient to prove that \( G_{H2}(v_2) \) is nondecreasing in \( b_{\max} \):

\[
\frac{d}{db_{\max}} G_{H2}(v_2) \geq 0.
\]

\[
\frac{dG_{H2}(v_2)}{db_{\max}} = \frac{1}{\left( \int_{v_2}^{b_{\max}} \left( \alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) dx \right)^2} \frac{r(b_{\max})}{1 - b_{\max}} \cdot \left( \frac{\alpha_1 \int_{v_2}^{b_{\max}} (\alpha_1 (x - v_2) - \alpha_2 (x - v_1)) \frac{r(x)}{1-x} dx}{\int_{v_2}^{b_{\max}} \left( \alpha_1 \frac{x-v_2}{1-x} - 1 \right) r(x) dx} \right)
\]
\[
\frac{dG_{H2}(v_2)}{db_{\text{max}}} = \frac{1}{r(b_{\text{max}})} \cdot \frac{\int_{v_2}^{b_{\text{max}}} (\alpha_1 x - v_2 - 1) d \alpha_2 (v_2 - v_1) - \alpha_1 (1 - v_2) + \alpha_2 (1 - v_1)}{1 - b_{\text{max}}} \cdot \frac{r(b_{\text{max}})}{1 - b_{\text{max}}} \]

\[
\cdot \int_{v_2}^{b_{\text{max}}} \frac{b_{\text{max}} - x}{1 - x} (\alpha_1 \alpha_2 (v_2 - v_1) - \alpha_1 (1 - v_2) + \alpha_2 (1 - v_1)) r(x) dx
\]

\[
= \alpha_1 (1 - v_2) \left( \frac{\alpha_2 (v_2 - v_1)}{1 - v_2} - \left( 1 - \frac{\alpha_2 (1 - v_1)}{\alpha_1 (1 - v_2)} \right) \right) \frac{r(b_{\text{max}})}{1 - b_{\text{max}}} \int_{v_2}^{b_{\text{max}}} \frac{b_{\text{max}} - x}{1 - x} r(x) dx
\]

By equation (70), \( \alpha_2 \frac{v_2 - v_1}{1 - v_2} \geq \left( 1 - \frac{\alpha_2 (1 - v_1)}{\alpha_1 (1 - v_2)} \right) \), thus \( \frac{dG_{H2}(v_2)}{db_{\text{max}}} \geq 0 \) holds. (Moreover, when \( \alpha_2 \frac{v_2 - v_1}{1 - v_2} = \left( 1 - \frac{\alpha_2 (1 - v_1)}{\alpha_1 (1 - v_2)} \right) \), \( \frac{dG_{H2}(v_2)}{db_{\text{max}}} = 0 \) and \( G_{H2}(v_2) \) attains its limit for any \( b_{\text{max}} < 1 \). \( \blacksquare \)

**Claim 8** In all three cases (no atoms, one atom, two atoms) the expression,

\[
\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{R(b)} \cdot \int_{b_{\text{min}}}^{b} \frac{x - v_2}{1 - x} r(x) dx + G_{H1}(b_{\text{min}}) \cdot \frac{\hat{R}(b_{\text{min}})}{R(b)}
\]

which defines \( G_{H1}(b) \) for \( b \in [b_{\text{min}}, b_{\text{max}}] \), is increasing in \( b \) for every \( b > b_{\text{min}} \). Hence \( G_{H1}(b) \) as defined above is increasing in \( b \) for every \( b \in (b_{\text{min}}, b_{\text{max}}) \).

**Proof.** The same arguments as the ones presented in the proof of Claim 7 show that it is sufficient to prove that

\[
\alpha_1 \cdot \frac{b_{\text{min}} - v_2}{1 - b_{\text{min}}} \geq G_{H1}(b_{\text{min}}).
\]

When bidder 1 does not have an atom (when no bidder has an atom, or only bidder 2 has an atom), this trivially holds since \( b_{\text{min}} \geq v_2 \). We are left to prove the claim when both bidders have an atom and \( G_{H1}(b_{\text{min}}) > 0 \) satisfies equation (29). We need to show that

\[
\alpha_1 \cdot \frac{b_{\text{min}} - v_2}{1 - b_{\text{min}}} \geq \alpha_1 \cdot \frac{\int_{v_2}^{b_{\text{min}}} (x - v_2) \hat{r}(x) dx}{\hat{R}(b_{\text{min}}) (1 - b_{\text{min}})}
\]

which holds as \( \int_{v_2}^{b_{\text{min}}} (x - v_2) \frac{\hat{r}(x)}{\hat{R}(b_{\text{min}})} dx \leq (b_{\text{min}} - v_2) \frac{\hat{R}(b_{\text{min}}) - R(v_2)}{\hat{R}(b_{\text{min}})} \leq (b_{\text{min}} - v_2) \).

Having proven Claims 7,8, we now proceed with the remaining three steps of the proof.

**Step 1.** Existence of parameters \( b_{\text{min}}, b_{\text{max}}, G_{H1}(b_{\text{min}}), \) and \( G_{H2}(v_2) \):

**Case 1 (no atoms):** First consider the case that the bidders are symmetric. We define \( b_{\text{min}} = v_2 \) and \( G_{H1}(b_{\text{min}}) = G_{H2}(v_2) = 0 \). By the necessary conditions at \( b_{\text{max}} \) it must hold that

\[
1 = G_{H1}(b_{\text{max}}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{R(b_{\text{max}})} \cdot \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - x} r(x) dx
\]

The RHS increases continuously from zero towards infinity as \( b_{\text{max}} \) increases from \( v_2 \) towards 1 (Claim 8), so there exists a unique value of \( b_{\text{max}} \in (v_2, 1) \) that solves this equation. It is clear that
\( b_{\text{max}} \) must tend to 1 as \( \epsilon \) goes to 0. Note that all the necessary conditions presented in Lemma 2 for the symmetric case are now satisfied.

**Case 2 (one atom):** Next consider the case that bidders are asymmetric and equation (70) holds (implying \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \) and \( v_1 < v_2 \)). We define \( b_{\text{min}} = v_2 \) and \( G_{H1}(b_{\text{min}}) = 0 \). As \( G_{H1}(b_{\text{min}}) = 0 \), \( b_{\text{max}} \in (v_2, 1) \) can be determined exactly as in the symmetric case. Finally, we set \( G_{H2}(v_2) \) using equation (30). Observe that \( G_{H2}(v_2) \) as defined tends to 1 as \( \epsilon \) tends to 0 (Lemma 23), thus for sufficiently small \( \epsilon \) it is positive.

**Case 3 (two atoms):** Finally, consider the case that bidders are asymmetric and equation (70) is violated. We define \( G_{H1}(b_{\text{min}}) \) as a function of \( b_{\text{min}} \) by equation (29). We define \( G_{H2}(v_2) \) as a function of \( b_{\text{min}} \) by equation (33), or equivalently by:

\[
G_{H2}(v_2) = \frac{\Pr[L_{2|H1}] \cdot b_{\text{min}} - v_1}{\Pr[H_{2|H1}] \cdot 1 - b_{\text{min}}}.
\] (76)

The arguments below show that \( b_{\text{min}} > \max\{v_1, v_2\} \), which ensures that \( G_{H1}(b_{\text{min}}) > 0 \) and \( G_{H2}(v_2) > 0 \). By substituting \( G_{H1}(b_{\text{min}}) \) and \( G_{H2}(v_2) \) into equations (31) and (32), which determine \( G_{H1}(b) \) and \( G_{H2}(b) \), and evaluating these equations at \( b_{\text{max}} \), for which it must hold that \( G_{H1}(b_{\text{max}}) = G_{H2}(b_{\text{max}}) = 1 \), we derive that we need to find \( b_{\text{min}} \) and \( b_{\text{max}} \) that satisfy the following pair of equations:

\[
1 = \alpha_1 \cdot \frac{1}{R(b_{\text{max}})} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) dx + \alpha_1 \cdot \frac{1}{R(b_{\text{max}})} \int_{v_2}^{b_{\text{max}}} \frac{x - v_2}{1 - b_{\text{min}}} \cdot \hat{r}(x) dx
\] (77)

\[
1 = \alpha_2 \cdot \frac{1}{R(b_{\text{max}})} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) dx + \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \cdot \frac{\hat{R}(b_{\text{max}})}{R(b_{\text{max}})}
\] (78)

Let \( \bar{v} = \max\{v_1, v_2\} \). We first show that when \( \epsilon \) is small enough, for any \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) we can find a unique \( b_{\text{max}} \in (b_{\text{min}}, 1) \) that solves equation (77). We denote such a solution by \( b_{\text{max}}(b_{\text{min}}) \). When \( b_{\text{max}} = b_{\text{min}} \), the RHS of equation (77) equals \( \epsilon \cdot h(b_{\text{min}}) \) for \( h(b_{\text{min}}) = \frac{\alpha_1}{R(b_{\text{min}})} \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{x - v_2}{1 - b_{\text{min}}} \cdot r(x) dx \). As \( h \) is a continuous function on a compact set it is bounded, thus \( \epsilon \cdot h(b_{\text{min}}) < 1 \) for any \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) as long as \( \epsilon \) is small enough. Now, for every fixed \( b_{\text{min}} \in [\bar{v}, v(H_1)] \), the RHS of equation (77) is continuously increasing in \( b_{\text{max}} \) (by Claim 8 above) and goes to infinity when \( b_{\text{max}} \) tends to 1. Therefore there exists a unique \( b_{\text{max}} \in (b_{\text{min}}, 1) \) that solves the equation. Note that \( b_{\text{max}}(b_{\text{min}}) \) is a continuous function of \( b_{\text{min}} \) and, for any fixed \( b_{\text{min}} \), \( b_{\text{max}}(b_{\text{min}}) \) tends to 1 as \( \epsilon \) tends to 0.

Now we substitute \( b_{\text{max}}(b_{\text{min}}) \) into equation (78) and get the following equation in \( b_{\text{min}} \)

\[
1 = \alpha_2 \cdot \frac{1}{R(b_{\text{max}}(b_{\text{min}}))} \int_{b_{\text{min}}}^{b_{\text{max}}(b_{\text{min}})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) dx + \frac{b_{\text{min}} - v_1}{1 - b_{\text{min}}} \cdot \frac{\hat{R}(b_{\text{min}})}{R(b_{\text{max}}(b_{\text{min}}))}
\] (79)

To complete the proof we need to show that there exists \( b_{\text{min}} \in [\bar{v}, v(H_1)] \) that satisfies equation (79). The RHS of this equation is a continuous function of \( b_{\text{min}} \) on the compact set \([\bar{v}, v(H_1)]\). It will
therefore be sufficient to show that for \( b_{\text{min}} = v(H_1) \) the RHS is larger than 1, while for \( b_{\text{min}} = \bar{v} \) the RHS is smaller than 1. Once this is shown (below) we conclude that there exists \( b_{\text{min}} > \bar{v} \) such that the RHS is exactly 1. This \( b_{\text{min}} \) together with \( b_{\max} = b_{\max}(b_{\text{min}}) \) solve both equations (77) and (78) and satisfy \( 1 > b_{\max} > b_{\text{min}} > \bar{v} \).

To prove the remaining two inequalities, define:

\[
z(b_{\text{min}}) = \alpha_1 \cdot \frac{1}{R(b_{\max}(b_{\text{min}}))} \int_{b_{\text{min}}}^{b_{\max}(b_{\text{min}})} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx.
\]

Now, the RHS of equation (79) can be written as

\[
z(b_{\text{min}}) \cdot \chi(b_{\text{min}}, b_{\max}(b_{\text{min}})) + \frac{b_{\min} - v_1}{1-b_{\min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{\text{min}})}{R(b_{\max}(b_{\text{min}}))}.
\]

Fix \( b_{\text{min}} \). Note that equation (77) implies that \( z(b_{\text{min}}) \leq 1 \) and \( \lim_{\epsilon \to 0} z(b_{\text{min}}) = 1 \). This follows because \( b_{\text{min}} \) bounded away from 1 (Lemma 1) implies that the second term of the RHS of equation (77) is positive and tends to 0. Thus, by Lemma 23 as \( \epsilon \) tends to 0, the RHS of equation (79) tends to

\[
\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1-v_1}{1-v_2} + \frac{b_{\min} - v_1}{1-b_{\min}} \cdot \alpha_2 = \frac{\alpha_2(1-v_1)}{1-v_2} + \frac{b_{\min} - v_1}{1-b_{\min}} \cdot \alpha_2.
\]

For \( b_{\text{min}} = v(H_1) \), equation (81) exceeds 1 since by equation (26) it holds that \( b_{\text{min}} = v(H_1) \) if and only if \( G_{H_2}(v_2) = \frac{b_{\min} - v_1}{1-b_{\min}} \cdot \alpha_2 = 1 \), and the first term is positive by assumption. Thus, for sufficiently small \( \epsilon \), the RHS of equation (79) also exceeds 1 for \( b_{\text{min}} = v(H_1) \).

If \( b_{\text{min}} = \bar{v} \) we show that the RHS of equation (79) is less than 1 for sufficiently small \( \epsilon \). We consider two cases separately. First, if \( b_{\text{min}} = \bar{v} = v_2 \geq v_1 \), equation (81) is less than 1 as equation (70) is violated. Thus, for sufficiently small \( \epsilon \), the RHS of equation (79) is also less than 1. Second, if \( b_{\text{min}} = \bar{v} = v_1 > v_2 \), equation (81) is less than or equal to 1. However, we show that equation (80) (and hence the RHS of equation (79)) is less than equation (81) for all \( \epsilon > 0 \). This follows because: (1) \( b_{\text{min}} > v_2 \) implies that the second term on the RHS of equation (77) is positive so that the first term, which is \( z(b_{\text{min}}) \), is less than 1; and (2) \( v_1 > v_2 \) implies (by Lemma 23) that \( \chi(b_{\text{min}}, b_{\max}) \leq \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1-v_1}{1-v_2} \leq 1 \).

**Step 2.** \( G_{H_1} \) and \( G_{H_2} \) are well defined: We next argue that \( G_{H_1} \) and \( G_{H_2} \), as defined above by Step 1 and equations (31) and (32), are well defined distributions. The way we have chosen the parameters in Step 1 ensures that \( \max\{v_1, v_2\} < b_{\text{min}} < b_{\max} < 1 \), \( G_{H_1}(b_{\text{min}}), G_{H_2}(v_2) \geq 0 \), and \( G_{H_1}(b_{\text{max}}) = G_{H_2}(b_{\text{max}}) = 1 \). The two distributions are continuous from the right at \( b_{\text{min}} \), and by Claims 7,8 are increasing on \((b_{\text{min}}, b_{\max})\). Thus both are monotonically nondecreasing on \([0, \infty)\) with \( G_{H_1}(0) = G_{H_2}(0) = 0 \) and \( G_{H_1}(b_{\text{max}}) = G_{H_2}(b_{\text{max}}) = 1 \).

**Step 3.** Constructed bid distributions are best responses: To see that \( \mu^e \) is indeed a mixed NE we show that each bidder is best responding to the other. Observe that, by construction,
\( G_{H_1} \) and \( G_{H_2} \) ensure that each bidder is indifferent between all the bids in the support of her bid distribution. It only remains to show that all other bids earn equal or lower payoffs.

First consider bids above \( b_{\text{max}} \). Assumption 1 and \( \max\{v_1, v_2\} < 1 \) implies that \( \max\{v(H_1), v(H_2)\} < 1 \). Therefore, as \( b_{\text{max}} \) tends to 1 when \( \epsilon \) tends to 0, for small enough \( \epsilon \) it holds that \( b_{\text{max}} > \max\{v(H_1), v(H_2)\} \). Noticing that \( \beta_i(1) = v(H_i) \), this means that \( b_{\text{max}} \) exceeds both \( \beta_1(1) \) and \( \beta_2(1) \). Therefore, for small enough \( \epsilon \), part (2) of Lemma 13 implies that \( b_{\text{max}} \) strictly dominates any higher bid \( b > b_{\text{max}} \).

Second note that for bidder \( i \), bidding \( v_i \) weakly dominates any lower bid \( b < v_i \).

Third, we consider bids \( b \in [v_i, b_{\text{min}}] \) by bidder \( i \in \{1, 2\} \) outside the support of bidder \( i \)’s bid distribution for each of the three cases.

Consider case 1 (no atoms) in which \( b_{\text{min}} = v_1 = v_2 \). In this case, the utility from bidding \( b_{\text{min}} = v_2 \) equals the utility of any bid in \([v_2, b_{\text{max}}]\) by continuity.

Consider case 2 (one atom) in which \( b_{\text{min}} = v_2 \), \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \), and \( v_2 > v_1 \). Bidder 2 bids an atom at \( v_2 \), which strictly dominates any bid \( b < b_{\text{min}} \). Moreover, for bidder 2, bidding \( b_{\text{min}} \) is dominated by bids in the support by Lemma 11. Now turn to bidder 1. Lemma 13 part (1) and Lemma 11 imply that \( i \)’s atom at \( b_{\text{min}} \) dominates any bid in \([v_2, b_{\text{min}}]\) because \( b_{\text{min}} \) is defined by equation (33). For \( v_1 \geq v_2 \), \([v_2, b_{\text{min}}]\) includes all bids \([v_1, b_{\text{min}}]\) and we are done. For \( v_1 < v_2 \), we must also consider bids \([v_1, v_2]\), of which \( v_1 \) gives the highest payoff to bidder 1 by Lemma 13 part (1). As \( v_1 < v_2 \) implies \( \alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2) \), \( b_{\text{min}} \) must dominate \( v_1 \) for sufficiently small \( \epsilon \) by the same argument applied above in the one-atom case.

H.4 Proof of Lemma 4 (Convergence)

We first we provide a bound on \( G_{Hj} \) in Lemma 25. Then we apply this bound with necessary conditions in Lemma 2 to prove the convergence result in Lemma 4. Finally we note that the Theorem follows from Lemmas 3-4.

**Lemma 25** If \( \epsilon \) is small enough then the following holds. For every bidder \( i \in \{1, 2\} \) and \( j \neq i \)
and every \( b \in (b_{\min}, b_{\max}) \) it holds that:

\[
G_{H_j}(b) - G_{H_j}(b_{\min}) \leq \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{1 - \epsilon} \cdot r_{\max} \cdot (-b - \ln(1 - b))
\] (82)

where \( r_{\max} = \sup_{x \in [0,1]} r(x) \) is finite.

**Proof.** By Lemma 20

\[
G_{H_j}(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{1 - \epsilon} \cdot \int_{b_{\min}}^{b} \frac{x - v_i}{1 - x} r(x) dx + G_{H_j}(b_{\min}) \cdot \frac{\hat{R}(b_{\min})}{\hat{R}(b)}
\] (83)

For a standard distribution \( R \), \( r_{\max} \) is a finite upper bound for \( r(x) \). As \( v_i \geq 0 \) and \( r(b) \leq r_{\max} \) for all \( b \),

\[
\int_{b_{\min}}^{b} \frac{x - v_i}{1 - x} r(x) dx \leq \int_{b_{\min}}^{b} \frac{x}{1 - x} r(x) dx \leq r_{\max} \int_{0}^{b} \frac{x}{1 - x} dx = r_{\max} (-b - \log(1 - b))
\] (84)

As \( \hat{R}(b) \geq \hat{R}(b_{\min}) \geq 1 - \epsilon \), equation (82) follows. ■

**Corollary 9** Fix any \( b \in (\max\{v(H_1), v(H_2)\}, 1) \) and any \( \delta > 0 \). For small enough \( \epsilon > 0 \), for every bidder \( j \in \{1, 2\} \) it holds that \( G_{H_j}(b) - G_{H_j}(b_{\min}) < \delta \).

**Proof.** Lemma 1 implies \( b_{\min} \leq \max\{v(H_1), v(H_2)\} < b \). For \( b \in (\max\{v(H_1), v(H_2)\}, 1) \) and sufficiently small \( \epsilon \), equation (82) holds by Lemma 25. As for any fixed positive \( b < 1 \) the RHS tends to 0 when \( \epsilon \) tends to 0, the claim follows. ■

**H.4.1 Proof of Lemma 4**

**Proof.** Fix a standard distribution \( R \). We make three claims: (i) First, \( G_{H_2}(b) = 0 \) for \( b \in [0, v_2) \) and \( G_{H_2}(b_{\min}) = G_{H_2}(v_2) \) for all \( \epsilon \) sufficiently small. (ii) Second, \( \lim_{\epsilon \to 0} G_{H_1}(b_{\min}) = 0 \) and \( \lim_{\epsilon \to 0} G_{H_2}(v_2) = 1 - \frac{Pr[H_1|L_1, H_2]}{Pr[H_1, L_1, H_2]} \). (iii) Third, for any \( b \in (\max\{v(H_1), v(H_2)\}, 1) \), \( \lim_{\epsilon \to 0} (G_{H_1}(b) - G_{H_1}(b_{\min})) = 0 \) for both \( i \in 1, 2 \). It then follows that in the limit as \( \epsilon \) approaches zero, bidder 1 bids 1 with probability 1 while bidder 2 bids \( v_2 \) with probability \( 1 - \frac{Pr[H_1|L_1, H_2]}{Pr[H_1, L_1, H_2]} \) and 1 with complementary probability. Claims (i) and (ii) follow from Lemmas 2 and 24. Claim (iii) follows from Corollary 9. ■

**I Proof of Theorems 2 and 5**

Theorem 2 is an abbreviated statement of the complete Theorem 5. The proof of Theorem 5 proceeds in three parts. In Appendix 1.1 we show that the conditions in the theorem are necessary for a Nash equilibrium in monotone bidding strategies. In Appendix 1.2 we show that the same
In any NE, supremum bids are equal and do not exceed $V_{HH}$: $b_1 = b_2 = b \leq V_{HH}$.

**Proof.** For $j$, any bid $b > b_j$ earns strictly less than $(b + b_j)/2$ because the latter bid wins with the same probability but pays less. Therefore $b_j \leq b_j$, and by symmetric argument $b_i = b_j$. If $b > V_{HH}$, then the highest bids (at or in a neighborhood of $b$) earn negative expected profits so cannot be optimal, a contradiction, so $b \leq V_{HH}$. ■
Proof. The result follows by inspection of the difference:

\[
\Pi_i(b|H_j) - \Pi_i(b|L_i) = (\Pr[L_j|L_i](b - V_{LL}) - \Pr[L_j|H_i](b - v(H_i, L_j))) G_{L_j}(b) \\
+ (\Pr[H_j|H_i](V_{HH} - b) - \Pr[H_j|L_i](v(L_i, H_j) - b)) G_{H_j}(b)
\]

given the assumed relationship \( V_{LL} \leq V_{LH}, V_{HL} \leq V_{HH} \) (Assumption 1) and affiliation: \( \Pr[L_j|L_i] \geq \Pr[L_j|H_i] \) and \( \Pr[H_j|H_i] \geq \Pr[H_j|L_i] \) (Assumption 2). 

**Lemma 27** In any NE, for all \( i \in \{1, 2\} \), it holds that payoffs conditional on bidding \( b \) are increasing in type, \( \Pi_i(b|H_i) \geq \Pi_i(b|L_i) \), and this inequality is strict if \( V_{LL} < v(H_i, L_j) \) and \( G_{L_j}(b) > 0 \) or if \( v(L_i, H_j) < V_{HH} \) and \( G_{H_j}(b) > 0 \). \(^{31} \)

**Proof.** The result follows by inspection of the difference:

\[
\Pi_i(b|H_i) - \Pi_i(b|L_i) = (\Pr[L_j|L_i](b - V_{LL}) - \Pr[L_j|H_i](b - v(H_i, L_j))) G_{L_j}(b) \\
+ (\Pr[H_j|H_i](V_{HH} - b) - \Pr[H_j|L_i](v(L_i, H_j) - b)) G_{H_j}(b)
\]

given the assumed relationship \( V_{LL} \leq V_{LH}, V_{HL} \leq V_{HH} \) (Assumption 1) and affiliation: \( \Pr[L_j|L_i] \geq \Pr[L_j|H_i] \) and \( \Pr[H_j|H_i] \geq \Pr[H_j|L_i] \) (Assumption 2). 

**Lemma 28** In any NE, if \( i \) receives signal \( H_i \) then, with probability 1, \( i \) places a bid that wins with positive probability. Therefore \( G_{Hi}(b_j) = 0 \). This holds for all \( i \in \{1, 2\} \).

\(^{31}\)Recall that the pair \( \{v(H_i, L_i), v(L_i, H_j)\} \) corresponds to \( \{V_{HL}, V_{LH}\} \) if \( i = 1 \) or \( \{V_{LH}, V_{HL}\} \) if \( i = 2 \). We use this notation when we do not specify whether \( i \) is bidder 1 or 2.
**Proof.** Suppose not, and, given $H_i$, with probability $x > 0$, $i$ places a bid that never wins. Then $i$ must earn zero payoff given $H_i$. Conditional on signal $L_j$, at any bid $b > v(H_i, L_j)$, $j$ must earn a negative payoff with probability at least $\Pr[H_i|L_j]x$. Thus $j$ with signal $L_j$ must not bid higher than $v(H_i, L_j)$ and $b_j \leq v(H_i, L_j)$. I show a contradiction for each of three exhaustive cases: (1) $b_j < v(H_i, L_j)$; (2) $b_j = v(H_i, L_j) > V_{LL}$; (3) $b_j = v(H_i, L_j) = V_{LL}$. Note that both cases (2) and (3) suppose that $b_j = v(H_i, L_j)$. Because $j$ does not bid higher than $v(H_i, L_j)$ conditional on signal $L_j$, this implies that $j$ bids $b_j$ with probability 1 given $L_j$ so that $G_{L_j}(b_j) = 1$.

(1) Suppose that $b_j < v(H_i, L_j)$. In that case, $i$ with signal $H_i$ can deviate and bid $b_j + \epsilon$. For sufficiently small $\epsilon > 0$, $i$ wins with positive probability, and earns at least $v(H_i, L_j) - b_j - \epsilon > 0$, a contradiction.

(2) Suppose that $b_j = v(H_i, L_j) > V_{LL}$. Then it must hold that $G_{Li}(b_j) = 0$ so that $i$ bids more than $b_j$ with probability 1. Otherwise, $j$ with signal $L_j$ would earn negative payoff with probability at least $\Pr[H_i|L_j]x$ from its bid $b_j$ because the expected value of objects won would be a weighted average of $v(H_i, L_j)$ and $V_{LL}$ and hence fall strictly below $b_j$. Therefore bidder $i$ makes a bid $b \geq b_j$ given $L_i$ with nonnegative payoff. Moreover, $G_{L_j}(b_j) = 1$, as explained in the first paragraph of the proof. Thus, Lemma 27 implies that the same bid $b \geq b_j$ yields $i$ a positive payoff conditional on $H_i$—a contradiction.

(3) Suppose that $b_j = v(H_i, L_j) = V_{LL}$. In this case, given $L_j$, $j$ will make zero or negative payoff at any bid $b \geq V_{LL}$. As noted above, $b_j = v(H_i, L_j)$ implies that $j$ bids $b_j$ with probability 1 given $L_j$. Thus if with probability $x > 0$, conditional on $H_i$, $i$ places a bid that never wins, it must be the case that $b_{H_i} < b_j$. However, given $b_{H_i} < b_j$, a bid of $(b_{H_i} + b_j)/2 < V_{LL}$ would make a positive payoff for bidder $j$ with signal $L_j$—a contradiction.

**Lemma 29** In any NE, if $G_{S_i}()$ and $G_{S_j}()$ both have an atom at $b$, then

$$b = E[v \mid S_i \text{ and } j \text{ bids } b] = E[v \mid S_j \text{ and } i \text{ bids } b]$$

and both $\Pi_i[b \mid S_i]$ and $\Pi_j[b \mid S_j]$ are continuous at $b$.

**Proof.** To see why, note that were $b > E[v \mid S_i \text{ and } j \text{ bids } b]$, then it would be strictly better for $S_i$ to bid $b - \epsilon$ for sufficiently small $\epsilon > 0$ rather than to bid $b$. Similarly, were $b < E[v \mid S_i \text{ and } j \text{ bids } b]$, then it would be strictly better for $S_i$ to bid $b + \epsilon$ for sufficiently small $\epsilon > 0$ rather than to bid $b$. Thus $b = E[v \mid S_i \text{ and } j \text{ bids } b]$ and the second equality follows by symmetric argument. This directly implies continuity of $\Pi_i[b \mid S_i]$ and $\Pi_j[b \mid S_j]$ at $b$ as $\Pi_i^+ [b \mid S_i] - \Pi_i^- [b \mid S_i] = \Pr[j \text{ bids } b \mid S_i] (E[v \mid S_i \text{ and } j \text{ bids } b] - b)$. ■

**Lemma 30** It is not the case that $G_{H1}()$, $G_{L1}()$, $G_{H2}()$, and $G_{L2}()$ all have an atom at $b$. 95
Proof. Proof is by contradiction. Suppose that $G_{H1}(\cdot), G_{L1}(\cdot), G_{H2}(\cdot)$, and $G_{L2}(\cdot)$ all have atoms at $b$, of sizes $\Gamma_{H1}, \Gamma_{L1}, \Gamma_{H2}, \Gamma_{L2} > 0$. Lemma 29 implies that $b = E[v \mid L_1]$ and $j$ bids $b$. Therefore $V_{LL} = V_{HL}$ and $V_{LH} = V_{HH}$, as otherwise positive probabilities for all states (Assumption 1), affiliation (Assumption 2), and $\Gamma_{L2}, \Gamma_{H2} > 0$ would imply $E[v \mid L_1]$ and $j$ bids $b$. By symmetric argument, Lemma 29 also implies $V_{LL} = V_{HL}$ and $V_{LH} = V_{HH}$, so that $V_{LL} = V_{LH} = V_{HL} = V_{HH}$, a contradiction of $V_{LL} < V_{HH}$ (Assumption 1).

Lemma 31 In any NE with monotone bidding strategies, $b_1 = b_2 = b \geq V_{LL}$ and a bidder with a low signal earns an expected payoff of zero.

Proof. The proof is developed in a sequence of claims:

Claim 9 In any NE, $\max\{b_1, b_2\} \geq V_{LL}$.

Proof. Label bidders $i$ and $j$ such that $b_j \geq b_i$. Suppose $V_{LL} > b_j$. Consider two cases: (i) $b_j > b_i$; and (ii) $b_j = b_i$.

Case (i) $b_j > b_i$: Lemma 28 implies that $G_{H1}(b_j) = 0$. Therefore $b_j > b_i$ implies that bidder $i$ bids below $b_j$ given signal $L_i$ and hence must earn zero payoff. However, $V_{LL} > b_j$ implies that bidder $i$ with signal $L_i$ earns strictly positive payoff by bidding $b \in (b_j, V_{LL})$, a contradiction.

Case (ii) $b_j = b_i < V_{LL}$ implies that any bid $b \in (b_i, V_{LL})$ yields strictly positive payoff. As a result, all equilibrium bids must win with positive probability bounded away from zero, implying that both bidders must bid with an atom at $b_i$. However, in this case deviating to bid $b_i + \epsilon$ for sufficiently small $\epsilon > 0$ strictly increases payoffs, a contradiction.

Contradictions in both case (i) and (ii) imply $b_j = \max\{b_1, b_2\} \geq V_{LL}$.

Claim 10 In any NE with monotone bidding strategies, bidder $L_j$ makes zero payoff for $j \in \{1, 2\}$.

Proof. Suppose not and $L_j$ makes a positive payoff. Then, by Lemma 27 $H_j$ does as well. Then $L_j$ and $H_j$ must both win with positive probability. Thus $b_j \geq b_i$. By Claim 9 $b_j \geq V_{LL}$. In this case we must have $G_{H1}(b_{Lj}) > 0$, or $L_j$ would earn nonpositive payoff. Monotone bidding implies that $b_{Lj} = b_j \leq b_{Hj}$. By Lemma 28 we must have $G_{H1}(b_j) = 0$ and, given $G_{H1}(b_{Lj}) > 0$, then $H_i$ bids an atom at $b_j$: $G_{H1}(b_j) - G_{H1}(b_j) > 0$. Moreover, Lemma 28 implies that if $H_i$ bids at $b_j$, $j$ must have an atom at $b_j$ as well: $G_j(b_j) - G_j(b_j) > 0$. Therefore Lemma 29 applies and $b_j = E[v \mid H_i \text{ and } j \text{ bids } b_j] \in [V(L_j, H_i), V_{HH}]$. This is a contradiction because $b_{Lj} < V(L_j, H_i)$ is required for $L_j$ to have positive profit.

Claim 11 In any NE with monotone bidding strategies, $b_1, b_2 \geq V_{LL}$.
Proof. Suppose $b_j < V_{LL}$. Then $L_j$ can earn strictly positive payoff by bidding $b \in (b_j, V_{LL})$, which contradicts Claim 10.

Claim 12 In any NE with monotone bidding strategies, $b_1 = b_2 = b$.

Proof. Label bidders $i$ and $j$ such that $b_j \geq b_i$. Suppose that $b_j > b_i$. By Claim 11 and monotone bidding, $V_{LL} \leq b_i < b_j = b_{L_j}$. We consider and rule out three exhaustive cases:

(a) $G_j(b_j) = 0$: Lemma 28 implies that $G_{H_i}(b_j) = 0$, which in turn implies $G_{L_i}(b_j) > 0$ to satisfy $b_i < b_j$. Thus $\Pi_j(b_j; L_j) = \Pr[L_i | L_j]\frac{1}{2} (G_{L_i}(b_j) + G_{L_i}(b_j)) (V_{LL} - b_j) < 0$ and $\Pi_j^+(b_j; L_j) = \Pr[L_i | L_j] G_{L_i}(b_j)(V_{LL} - b_j) < 0$. As a result, bids by $j$ with signal $L_j$ at $b_j$ or in a neighborhood above $b_j$ yield negative payoffs, a contradiction.

(b) $G_j(b_j) > 0$ and $G_{H_i}(b_j) = 0$: The same contradiction as in (a) results.

(c) $G_j(b_j) > 0$ and $G_{H_i}(b_j) > 0$: Lemma 28 implies that $G_{H_i}(b_j) = 0$. Monotone bidding means that $G_j(b_j) > 0$ implies $G_{L_j}(b_j) > 0$. Thus, both $L_j$ and $H_j$ have atoms at $b_j$. Then, by Lemma 29 $\Pi_j(b; L_j)$ is continuous at $b_j$, and hence must be strictly negative at $b_j$ because $G_{H_i}(b_j) = 0$ and $b_j > b_i \geq V_{LL}$ imply that $\Pi_j^+(b_j; L_j) = \Pr[L_i | L_j] G_{L_i}(b_j)(V_{LL} - b_j) < 0$. Thus $j$ is not bidding optimally given $L_j$, a contradiction.

Claims 10, 11, and 12 complete the proof of Lemma 31.

Lemma 32 There are no gaps in bidding: In any NE with monotone bidding strategies, for $i \in \{1, 2\}$, $G_i(b)$ is increasing on $[b, b]$.

Proof. First we prove a claim.

Claim 13 Bidder $j$ does not bid in a gap in $i$’s bidding: Suppose there is a NE in which $G_i(x) = G_i^-(y) > 0$ for some $x < y$. Then $G_j(x) = G_j^-(y)$.

Proof. Note that $G_i(x) = G_i^-(y) > 0$ implies $G_{H_i}(x) = G_{H_i}^-(y)$, $G_{L_i}(x) = G_{L_i}^-(y)$, and $\max \{G_{L_i}(x), G_{H_i}(x)\} > 0$. Thus, for all $b \in (x, y)$, $\Pi_j(b|S_j) = \Pr[L_i|S_j] G_{L_i}(x) (v(L_i,S_j) - b) + \Pr[H_i|S_j] G_{H_i}(x) (v(H_i,S_j) - b)$, and this is decreasing in $b$. Thus bidder $j$ does not bid in the interval $(x, y)$, and Claim 13 follows.

Next we prove the lemma by contradiction. Suppose that the lemma does not hold, and for some $x$ and $y$ satisfying $b \leq x < y \leq b$, we have $G_i(x) = G_i(y)$. By the definitions of $b$ and $b$ and Lemmas 26 and 31, $G_i(x) = G_i(y) \in (0, 1)$. Let $y' = \sup \{b : G_i(b) = G_i(x)\}$ and $x' = \inf \{b : G_i(b) = G_i(x)\}$. Note that $b \leq x' < y' < b$. Then $G_i(x') = G_i^-(y') > 0$ and by Claim 13, $G_j(x') = G_j^-(y')$. Moreover, $G_j(x') = G_j^-(y') \in (0, 1)$ follows from Lemmas 26 and 31. $x' < b$, and $y' > b$. Then we have $y' = \sup \{b : G_j(b) = G_j(x)\}$ and $x' = \inf \{b : G_j(b) = G_j(x)\}$, otherwise Claim 13 would yield a contradiction.
Given the above, there is some \( S_i \in \{L_i, H_i\} \) and \( S_j \in \{L_j, H_j\} \) such that \( S_i \) and \( S_j \) are bidding at \( y' \) or in a neighborhood above \( y' \). Moreover, for all \( b \in (x', y') \),

\[
\Pi_i(b|S_i) = \Pr[L_j|S_i] G_{L_j}(x)(v(L_j, S_i) - b) + \Pr[H_j|S_i] G_{H_j}(x)(v(H_j, S_i) - b)
\]

\[
\Pi_j(b|S_j) = \Pr[L_i|S_j] G_{L_i}(x)(v(L_i, S_j) - b) + \Pr[H_i|S_j] G_{H_i}(x)(v(H_i, S_j) - b)
\]

are both decreasing in \( b \). Thus, for bidding \( y' \) or in a neighborhood above \( y' \) to be optimal, \( \Pi_i(b|S_i) \) and \( \Pi_j(b|S_j) \) must both increase discontinuously at \( y' \). (Otherwise bidding \( x' \) would be strictly more profitable.) This requires that \( G_i \) and \( G_j \) must both have atoms at \( y' \) for some \( S_i' \in \{L_i, H_i\} \) and \( S_j' \in \{L_j, H_j\} \). However, Lemma 29 then implies that both \( \Pi_i(b|S_i') \) and \( \Pi_j(b|S_j') \) are continuous at \( y' \). As \( \Pi_i(b|S_i') \) and \( \Pi_j(b|S_j') \) are both decreasing on \((x', y')\), continuity at \( y' \) implies that bidding \( x' \) is strictly more profitable than bidding \( y' \) for \( S_i' \) and \( S_j' \), a contradiction. \( \blacksquare \)

**Lemma 33** In any NE with monotone bidding strategies, for some \( i \in \{1, 2\} \), \( j \neq i \), \( b \in (V_{LL}, V_{HH}) \), and \( b^* \in [V_{LL}, \bar{b}] \) such that \( b^* \leq v(H_i, L_j) \), the following hold: \( G_i(b) \) and \( G_j(b) \) are continuous and increasing on the interval \([V_{LL}, \bar{b}]\), with \( G_i(\bar{b}) = G_j(\bar{b}) = 1 \). Moreover, \( L_i \) bids \( V_{LL} \) with probability 1 and \( G_{H_i}(b) \) is continuous and increasing from \( G_{H_i}(V_{LL}) \) to 1 on the interval \([V_{LL}, \bar{b}]\). \( G_{L_j}(b) \) is continuous and increasing from \( G_{L_j}(V_{LL}) \) to 1 on the interval \([V_{LL}, b^*] \) and \( G_{H_j}(b) \) is continuous and increasing from 0 to 1 on the interval \([b^*, \bar{b}] \). Also \( G_{H_i}(V_{LL}) = 0 \) if \( b^* > V_{LL} \).

**Proof.** Lemmas 26, 31 and 32 imply that there are no gaps in either bidder’s bidding strategy on \([b, \bar{b}]\), that is that \( G_i(b) \) is increasing from \( G_i(\bar{b}) \) to 1 on \([b, \bar{b}] \subset [V_{LL}, V_{HH}]\). The assumption of monotone bidding strategies \((G_{L_i}(b) < 1 \implies G_{H_i}(b) = 0)\) implies that for \( i \in \{1, 2\} \) and some \( b_i^* \in [b, \bar{b}] \), \( G_{L_i}(b) \) is increasing from \( G_{L_i}(\bar{b}) \) to 1 on the interval \([b, b_i^*] \) and \( G_{H_i}(b) \) is zero for all \( b < b_i^* \) and is increasing from \( G_{H_i}(b_i^*) \) to 1 on the interval \([b_i^*, \bar{b}] \).

For the rest of the proof, label bidders such that \( b_i^* \geq b_j^* \).

Claim: \( b_i^* = \bar{b} \) and \( b_j^* \leq v(H_i, L_j) \). Proof: Given \( b_j^* \geq b_i^* \), notice that any bid \( b \in (b, b_i^*) \) by \( j \) with signal \( L_j \) wins with positive probability but only wins items of value \( V_{LL} \) at a cost of \( b > \bar{b} \geq V_{LL} \) (the latter inequality following from Lemma 31). Thus there can be no such bids. As there are no gaps (Lemma 32), however, this means the interval must be empty and \( \bar{b} = b_i^* \). Moreover, note that \( b_j^* \leq v(H_i, L_j) \) as \( L_j \) would earn negative profits bidding above \( v(H_i, L_j) \).

The preceding claim implies \( L_i \) bids \( \bar{b} \) with probability 1 and we can define \( b^* = b_j^* \leq v(H_i, L_j) \).

Claim: \( \bar{b} = V_{LL} \). Proof: Suppose that \( \bar{b} > V_{LL} \). Then \( G_{H_i}(\bar{b}) > 0 \), as otherwise bidder \( L_j \) would earn negative profit from bidding at or in a neighborhood above \( \bar{b} \). Then \( G_{L_j}(\bar{b}) > 0 \) must hold to satisfy Lemma 28. Then, symmetrically, it must be that \( G_{H_j}(\bar{b}) > 0 \) (and hence \( b^* = \bar{b} \)) or \( L_i \) would earn negative profit from bidding \( \bar{b} \). Thus, all four types bid atoms at \( \bar{b} \). This contradicts Lemma 30, proving the claim.
Claim: $\bar{b} \in (V_{LL}, V_{HH})$. Proof: Notice that if $v(L_i, H_j) > V_{LL}$ then bidding $V_{LL}$ earns $H_j$ positive payoffs, so $\bar{b} < V_{HH}$ as $H_j$ earns at most 0 by bidding $V_{HH}$. However, if $v(L_i, H_j) = V_{LL}$ then $\bar{b} < V_{HH}$ because bidding $V_{HH}$ yields $H_j$ negative payoffs. Thus $\bar{b} < V_{HH}$. Moreover, we must have $\bar{b} > V_{LL}$, as $L_1$, $H_1$, $L_2$, and $H_2$ all bidding $V_{LL}$ with probability 1 would contradict Lemma 30.

Claim: $b^* > V_{LL}$ implies that $b^* < v(H_i, L_j)$. Proof: Given preceding claims, bidder $j$ with signal $L_j$ wins with positive probability when bidding $b \in (V_{LL}, b^*)$. Therefore, all such bids must equal $E[v \mid L_j$ and $j$ wins] for $L_j$ to earn zero expected payoff (Lemma 31). Monotonicity implies that $E[v \mid L_j$ and $j$ wins] $\leq E[v \mid L_j]$. The inequalities $v(H_i, L_j) \geq b^*$ and $b^* > V_{LL}$ imply that $v(H, L_j) > V_{LL}$ and therefore $E[v \mid L_j] < v(H_i, L_j)$. We conclude that $b^* < v(H_i, L_j)$.

Next we show that there are no atoms above $V_{LL}$.

Claim: There are no atoms on the interval $(b^*, \bar{b}]$. Proof: Suppose that $H_j$ bids an atom at $\hat{b} \in (b^*, \bar{b}]$. Then, as $\bar{b} < V_{HH}$ implies $\hat{b} < V_{HH}$, bidder $H_i$ will find profits increase discontinuously at $\hat{b}$:

$$\Pi_i (\hat{b} \mid H_i) - \Pi_i^- (\hat{b} \mid H_i) = \frac{1}{2} \Pr[H_j \mid H_i] \left(G_{H_j}(\hat{b}) - G_{H_j}^- (\hat{b})\right) (V_{HH} - \hat{b}) > 0.$$ 

Therefore, $H_i$ cannot bid in a neighborhood $(\hat{b} - \epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 32). Thus $H_j$ has no atom on $(b^*, \bar{b}]$. By symmetric argument, $H_i$ has no atom on $(b^*, \bar{b}]$.

Claim: If $V_{LL} < b^*$ then $H_i$ does not bid an atom on the interval $(V_{LL}, b^*)$. Proof: Suppose that $H_i$ bids an atom at $\hat{b} \in (V_{LL}, b^*)$. Then, as $b^* < v(H_i, L_j)$, bidder $L_j$ will find profits increase discontinuously at $\hat{b}$:

$$\Pi_j (\hat{b} \mid L_j) - \Pi_j^- (\hat{b} \mid L_j) = \frac{1}{2} \Pr[H_i \mid L_j] \left(G_{H_i}(\hat{b}) - G_{H_i}^- (\hat{b})\right) (v(H_i, L_j) - \hat{b}) > 0.$$ 

Therefore, $L_j$ cannot bid in a neighborhood $(\hat{b} - \epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 32). Thus $H_i$ has no atom on $(V_{LL}, b^*)$.

Claim: If $V_{LL} < b^*$ then bidder $j$ does not bid an atom on the interval $(V_{LL}, b^*)$. Proof: Suppose that $j$ bids an atom at $\hat{b} \in (V_{LL}, b^*)$. Then, as $b^* < v(H_i, L_j)$, bidder $H_i$ will find profits increase discontinuously at $\hat{b} < v(H_i, L_j) \leq V_{HH}$:

$$\Pi_i (\hat{b} \mid H_i) - \Pi_i^- (\hat{b} \mid H_i) = \Pr[L_j \mid H_i] \frac{1}{2} \left(G_{L_j}(\hat{b}) - G_{L_j}^- (\hat{b})\right) (v(H_i, L_j) - \hat{b}) + \Pr[H_j \mid H_i] \frac{1}{2} \left(G_{H_j}(\hat{b}) - G_{H_j}^- (\hat{b})\right) (V_{HH} - \hat{b}) > 0.$$ 

Therefore, $H_i$ cannot bid in a neighborhood $(\hat{b} - \epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 32). Thus $j$ has no atom on $(V_{LL}, b^*)$.
Having shown that there are no atoms above $V_{LL}$ (which implies $b^* < \bar{b}$ given that $\bar{b} > V_{LL}$) only two final claims remain to be shown.

Claim: $G_{Hi}(V_{LL}) = 0$ if $b^* > V_{LL}$. Proof: Notice that $b^* > V_{LL}$ implies $V_{LL} < v(H_i, L_j)$, as shown above. So if $G_{Hi}(V_{LL}) > 0$ then $L_j$ would find it optimal to bid above $V_{LL}$ such that $G_{Lj}(V_{LL}) = 0$. However, in this case $H_i$ wins with zero probability at bid $V_{LL}$, in contradiction to Lemma 33. Thus $G_{Hi}(V_{LL}) = 0$ if $b^* > V_{LL}$.

Lastly, note that the lemma also implies that $G_{Hj}(V_{LL}) = 0$ if $b^* = V_{LL}$. However, in this case the labeling of the bidders is arbitrary, and hence we only need to show that $G_{Hj}(V_{LL}) = 0$ or $G_{Hi}(V_{LL}) = 0$ which must hold by Lemma 30.

Now we are ready to move to first-order conditions. For each labeling of bidders, there are two cases, $b^* = V_{LL}$ and $b^* > V_{LL}$. So there are a total of four possible cases. For each of the four cases, we can derive necessary conditions, and show that only one of the four possibilities is feasible, according to the conditions in the Theorem.

Lemma 34 In any NE satisfying the conditions in Lemma 33 and $b^* > V_{LL}$, case (1) of Theorem 5 must hold.

Proof. Denote $v(H_i, L_j) = v_i$. Given the equilibrium structure specified by Lemma 33 and $b^* > V_{LL}$, the expected payoff functions for $b > V_{LL}$ are:

$$
\Pi_j(b|L_j) = \Pr[L_j|L_j](V_{LL} - b) + \Pr[H_i|L_j](v_i - b)G_{Hi}(b)
$$

$$
\Pi_j(b|H_j) = \Pr[L_i|H_j](v_j - b) + \Pr[H_i|H_j](V_{HH} - b)G_{Hi}(b)
$$

$$
\Pi_i(b|H_i) = \begin{cases}
\Pr[L_j|H_i](v_i - b)G_{Lj}(b) & b \leq b^* \\
\Pr[L_j|H_i](v_i - b) + \Pr[H_j|H_i](V_{HH} - b)G_{Hj}(b) & b > b^*
\end{cases}
$$

These yield the following necessary first-order conditions:

$$
0 = \frac{d}{db}\Pi_j(b|L_j) = \Pr[H_i|L_j](v_i - b)g_{Hi}(b) - \Pr[L_i|L_j] - \Pr[H_i|H_j]G_{Hi}(b) 
$$

$$
0 = \frac{d}{db}\Pi_j(b|H_j) = \Pr[H_i|H_j](V_{HH} - b)g_{Hi}(b) - \Pr[L_i|H_j] - \Pr[H_i|H_j]G_{Hi}(b) 
$$

$$
0 = \frac{d}{db}\Pi_i(b|H_i) = \Pr[L_j|H_i](v_i - b)g_{Lj}(b) - \Pr[L_j|H_i] - \Pr[H_j|H_i]G_{Hj}(b) 
$$

Which may be usefully re-written in the following form:

$$
g_{Hi}(b) - \frac{1}{v_i - b}G_{Hi}(b) = \frac{\Pr[L_i|L_j]}{\Pr[H_i|H_j]} - \frac{1}{v_i - b}, 
$$

$$
g_{Hi}(b) - \frac{1}{v_i - b}G_{Hi}(b) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} - \frac{1}{v_i - b}, 
$$

$$
g_{Lj}(b) - \frac{1}{v_i - b}G_{Lj}(b) = 0, 
$$

$$
g_{Hj}(b) - \frac{1}{v_i - b}G_{Hj}(b) = \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]} - \frac{1}{V_{HH} - b}, 
$$

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Applying the well known differential-equation result stated in Lemma 19 to the preceding first-order conditions (along with the fact that the bid distributions are everywhere right continuous and continuous for all \( b > V_{LL} \)) yields:

\[
G_{Hi}(b) = \frac{v_i - V_{LL}}{v_i} G_{Hi}(V_{LL}) + \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} \frac{b - V_{LL}}{v_i - b}, \quad b \in [V_{LL}, b^*]
\]

\[
G_{Hi}(b) = \frac{V_{HH} - b}{V_{HH} - b} G_{Hi}(b^*) + \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - b^*}{V_{HH} - b}, \quad b \in [b^*, \bar{b}]
\]

\[
G_{Lj}(b) = \frac{v_i - V_{LL}}{v_i - b} G_{Lj}(V_{LL}), \quad b \in [V_{LL}, b^*]
\]

\[
G_{Hj}(b) = \frac{V_{HH} - b}{V_{HH} - b} G_{Hj}(b^*) + \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - b^*}{V_{HH} - b}, \quad b \in [b^*, \bar{b}]
\]

These expressions are simplified as follows. The boundary condition \( G_{Lj}(b^*) = 1 \) implies that \( G_{Lj}(V_{LL}) = \frac{v_i - b^*}{v_i - V_{LL}} \). Additional boundary conditions from Lemma 33 are that \( G_{Hi}(V_{LL}) = G_{Hj}(b^*) = 0 \). Finally, evaluating \( G_{Hi}(b) \) at \( b^* \) using the equation for \( b \in [V_{LL}, b^*] \) implies \( G_{Hi}(b^*) = \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} \frac{b^* - V_{LL}}{v_i - b^*} \). Substitution therefore yields:

\[
G_{Hi}(b) = \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} \frac{b - V_{LL}}{v_i - b}, \quad b \in [V_{LL}, b^*]
\]

\[
G_{Hi}(b) = \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} \frac{V_{HH} - b - V_{LL}}{v_i - b} + \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - b^*}{V_{HH} - b}, \quad b \in [b^*, \bar{b}]
\]

\[
G_{Lj}(b) = \frac{v_i - b^*}{v_i - b}, \quad b \in [V_{LL}, b^*]
\]

\[
G_{Hj}(b) = \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - b^*}{V_{HH} - b}, \quad b \in [b^*, \bar{b}]
\]

The remaining boundary conditions are \( G_{Hi}(\bar{b}) = G_{Hj}(\bar{b}) = 1 \). The boundary condition \( G_{Hj}(\bar{b}) = 1 \) may be solved for \( \bar{b} \), yielding

\[
\bar{b} = (1 - Pr[H_j|H_i]) b^* + Pr[H_j|H_i] V_{HH}.
\]

Substituting \( \bar{b} = (1 - Pr[H_j|H_i]) b^* + Pr[H_j|H_i] V_{HH} \) into the final boundary condition \( G_{Hi}(\bar{b}) = 1 \) and solving for \( b^* \) yields

\[
b^* = V_{LL} + (v_i - V_{LL}) \left( \frac{1 - Pr[L_i|H_j]}{Pr[H_i|L_j]} \right) \left( \frac{Pr[L_j|H_i] Pr[H_j|H_i]}{Pr[H_j|L_j]} \right) \left( 1 - \frac{Pr[L_i|H_j]}{Pr[H_i|L_j]} \right),
\]

which is equivalent to

\[
b^* = \left( \frac{v_i Pr[L_j, H_i]}{Pr[L_j, H_i]} \right) \left( Pr[L_j, H_i] - Pr[H_j, L_i] \right) + Pr[L_j, L_i] Pr[H_i]
\]

\[
\left( Pr[L_j, H_i] - Pr[H_j, L_i] \right) + Pr[L_j, L_i] Pr[H_i].
\]

Note that the denominator in the second term of equation (88) is always positive. This follows because it can be re-written as

\[
1 + \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} + \frac{Pr[H_j|H_i]}{Pr[L_i|H_i]} \left( \frac{Pr[L_i|L_j]}{Pr[H_i|L_j]} - \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \right),
\]

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and affiliation (Assumption 2) implies that \( \frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]} - \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \geq 0 \). Therefore, equation (88) implies that \( b^* \) can only be higher than \( V_{LL} \), as assumed, if \( v_i > V_{LL} \) and if

\[
\frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \frac{\Pr[H_j|H_i]}{\Pr[L_j|H_i]} < 1.
\]

This is equivalent to the condition \( \Pr[H_i|H_j] > \Pr[H_i|H_j] \), which given the labeling of the bidders can only hold if \( i = 2, j = 1 \), and \( \Pr[H_2|H_1] > \Pr[H_1|H_2] \). Then the condition \( v_i > V_{LL} \) is \( V_{LH} > V_{LL} \). Making the substitution \( i = 2, j = 1, \) and \( v_i = V_{LH} \) into the preceding expressions for the bidding distributions, \( \bar{b} \), and \( b^* \) yields expressions coinciding with those in equations (18)–(22) in case (1) of Theorem 5.

**Lemma 35** In any NE satisfying the conditions in Lemma 33 and \( b^* = V_{LL} \), case (2) of Theorem 3 must hold.

**Proof.** Given the equilibrium structure specified by Lemma 33 and \( b^* = V_{LL} \), the expected payoff functions for \( b > V_{LL} \) are:

\[
\Pi_j (b|H_j) = \Pr[L_i|H_j] (v_j - b) + \Pr[H_i|H_j] (V_{HH} - b) G_{Hi}(b)
\]

\[
\Pi_i (b|H_i) = \Pr[L_j|H_i] (v_i - b) + \Pr[H_j|H_i] (V_{HH} - b) G_{Hj}(b)
\]

These yield the following necessary first-order conditions for all \( b \in (V_{LL}, \bar{b}) \):

\[
0 = \frac{d}{db} \Pi_j (b|H_j) = \Pr[H_i|H_j] (V_{HH} - b) g_{Hi}(b) - \Pr[L_i|H_j] - \Pr[H_i|H_j] G_{Hi}(b)
\]

\[
0 = \frac{d}{db} \Pi_i (b|H_i) = \Pr[H_j|H_i] (V_{HH} - b) g_{Hj}(b) - \Pr[L_j|H_i] - \Pr[H_j|H_i] G_{Hj}(b)
\]

Which may be usefully re-written in the following form:

\[
g_{Hi}(b) - \frac{1}{V_{HH} - b} G_{Hi}(b) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} V_{HH} - b
\]

\[
g_{Hj}(b) - \frac{1}{V_{HH} - b} G_{Hj}(b) = \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]} V_{HH} - b
\]

Applying the well-known differential-equation result stated in Lemma 19 to the preceding first-order conditions (along with the fact that the bid distributions are everywhere right continuous and continuous for all \( b > V_{LL} \)) yields:

\[
G_{Hi}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b} G_{Hi}(V_{LL}) + \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} b - V_{LL},
\]

\[
G_{Hj}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b} G_{Hj}(V_{LL}) + \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]} b - V_{LL}.
\]

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A condition of Lemma 33 is that $G_{H_j}(b^*) = 0$, which requires $G_{H_j}(V_{LL}) = 0$ given $b^* = V_{LL}$. Thus $G_{H_j}(b) = \frac{Pr[L_i|H_i]}{Pr[H_i|H_j]} \frac{b - V_{LL}}{V_{HH} - b}$. The boundary condition $G_{H_j} (\bar{b}) = 1$ may be written as

$$\bar{b} - V_{LL} \frac{V_{HH} - \bar{b} - b}{V_{HH} - b} = \frac{Pr[H_j|H_i]}{Pr[H_i|H_j]}.$$

and, solving for $\bar{b}$:

$$\bar{b} = (1 - Pr[H_j|H_i]) V_{LL} + Pr[H_j|H_i] V_{HH}.$$

Substituting the expression for $\bar{b}$ into the final boundary condition $G_{H_i} (\bar{b}) = 1$, yields

$$G_{H_i} (V_{LL}) = Pr[L_j|H_i] - \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} Pr[H_j|H_i] = \frac{Pr[H_i|H_j] - Pr[H_j|H_i]}{Pr[H_i|H_j]}.$$

Thus, for $b \in [V_{LL}, \bar{b}]$,

$$G_{H_i}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b} \frac{Pr[H_i|H_j] - Pr[H_j|H_i]}{Pr[H_i|H_j]} + \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - V_{LL}}{V_{HH} - b},$$

which is equivalent\(^{32}\) to

$$G_{H_i}(b) = \frac{V_{HH} - \bar{b}}{V_{HH} - b} \frac{Pr[H_i|H_j] - Pr[H_j|H_i]}{Pr[H_i|H_j]} + \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \frac{b - \bar{b}}{V_{HH} - b}.$$

Notice that $G_{H_i} (V_{LL}) = (Pr[H_i|H_j] - Pr[H_j|H_i]) / Pr[H_i|H_j]$ requires that $Pr[H_i|H_j] \geq Pr[H_j|H_i]$, implying that we must have $i = 2$ and $j = 1$. Thus the preceding expressions for bidding distributions and $\bar{b}$ coincide with those in equations (23)–(25) in case (2) of Theorem 5. Finally, Suppose $V_{LH} > V_{LL}$ and $Pr[H_2|H_1] > Pr[H_1|H_2]$. Then

$$\Pi_1(b = V_{LL}|L_1) = \frac{1}{2} Pr[H_2|L_1] (V_{HH} - V_{LL}) \frac{Pr[H_2|H_1] - Pr[H_1|H_2]}{Pr[H_2|H_1]} > 0,$$

which contradicts Lemma 31. Therefore it must hold that $V_{LH} = V_{LL}$ or $Pr[H_2|H_1] = Pr[H_1|H_2]$.

**Lemma 36** The conditions in Theorem 5 are necessary for any Nash equilibrium in monotone bidding strategies.

**Proof.** A direct implication of Lemmas 33, 34, and 35. \(\blacksquare\)

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\(^{32}\)This can be verified by substituting the expression for $\bar{b}$ into the second expression for $G_{H_i}$ and grouping terms by $(V_{HH} - V_{LL})$ and $(b - V_{LL})$. 103
I.2 Necessary conditions are sufficient

In this subsection, we show that the conditions in Theorem 5 characterize a Nash equilibrium. That is they are sufficient as well as necessary as shown in the previous subsection. We begin with one preliminary lemma (Lemma 37) and then proceed to the main result in Lemma 38.

**Lemma 37** The definitions of \(b^*\) and \(\bar{b}\) given in Theorem 5 imply the following bounds. Case 1: \(b^* \in (V_{LL}, V_{LH})\) and \(\bar{b} \in (b^*, V_{HH})\). Case 2: \(\bar{b} \in (V_{LL}, V_{HH})\).

**Proof.** In Case 1, the inequalities follow from \(V_{LH} > V_{LL}\), \(\Pr[L_1, H_2] > \Pr[H_1, L_2]\), Assumption 1 and equations (18) – (19). In Case 2, the inequalities follow from Assumption 1 and equation (23). The logic is that the equations define \(b^*\) and \(\bar{b}\) as the weighted average of two other values, and Assumption 1 (and \(V_{LH} > V_{LL}\), \(\Pr[L_1, H_2] > \Pr[H_1, L_2]\) in Case 1) guarantees that the weights are probabilities in \((0, 1)\).

**Lemma 38** The conditions in Theorem 5 characterize a Nash equilibrium.

**Proof.** First, notice that the described strategy profiles in Theorem 5 are valid. The bid distributions \(G_{S_i}(b)\) are nondecreasing by inspection. Moreover, evaluating \(G_{S_i}\) at \(b = V_{LL}\) and again at \(b = \bar{b}\), while substituting relevant expressions for \(b^*\) and \(\bar{b}\) yields \(G_{S_i}(V_{LL}) \geq 0\) and \(G_{S_i}(\bar{b}) = 1\) in all cases. It thus only remains to show that they are best responses. This entails two steps, first showing that bidders are indifferent over the support of their mixed strategies, and second showing that there are no profitable deviations outside the support of their mixed strategies. Indifference over the support of the mixed strategies is ensured by the fact that the bid distribution of bidder \(j\) was constructed to satisfy bidder \(i\)'s first-order condition \(d\Pi_i(b|S_i)/db = 0\) (and vice-versa) in the proofs of Lemmas 34 and 35.

It now remains to verify that there are no profitable deviations for a bidder outside the support of its mixed strategy. We need not consider bids outside \([V_{LL}, \bar{b}]\) because (i) bids below \(V_{LL}\) earn at most 0 while bids at \(V_{LL}\) earn at least 0, and (ii) bidding above \(\bar{b}\) increases payments without increasing the probability of winning, and therefore yields lower payoffs than at \(\bar{b}\). To show that there are no strictly profitable deviations within \([V_{LL}, \bar{b}]\), it is sufficient to show that bidder \(i\)'s expected profit function has increasing differences in \(b\) and \(S_i\) given \(j\)'s strategy: \(\frac{d}{db} (\Pi_i(b|H_j) - \Pi_i(b|L_j)) \geq 0\). Increasing differences implies that if bidder \(L_i\) is indifferent over bids on an interval, then bidder \(H_i\) weakly prefers to bid above that interval. Similarly, if bidder \(H_i\) is indifferent over bids on an interval, then bidder \(L_i\) weakly prefers to bid below that interval. This rules out all other possible deviations.
It straightforward to see that (in both case 1 and case 2) \( \frac{d}{db} (\Pi_i (b|H_i) - \Pi_i (b|L_i)) \geq 0 \) for all \( b \in [V_{LL}, \bar{b}] \) by examining equation (90) term-by-term.

\[
\begin{align*}
\frac{d}{db} (\Pi_i (b|H_i) - \Pi_i (b|L_i)) &= (\Pr [L_j|H_i] (v(H_i, L_j) - b) + \Pr [L_j|L_i] (b - V_{LL})) g_{Lj}(b) \\
&+ (\Pr [H_j|H_i] (V_{HH} - b) - \Pr [H_j|L_i] (v(L_i, H_j) - b)) g_{Hj}(b) \\
&+ (\Pr [L_j|L_i] G_{Lj}(b) + \Pr [H_j|L_i] G_{Hj}(b)) \\
&\geq (\Pr [L_j|H_i] G_{Lj}(b) + \Pr [H_j|H_i] G_{Hj}(b))
\end{align*}
\]

First,

\[
\Pr [L_j|H_i] (v(H_i, L_j) - b) + \Pr [L_j|L_i] (b - V_{LL}) = \\
\Pr [L_j|H_i] (v(H_i, L_j) - V_{LL}) + (\Pr [L_j|L_i] - \Pr [L_j|H_i])(b - V_{LL}) \geq 0,
\]

holds for all \( b \geq V_{LL} \) because \( v(H_i, L_j) \geq V_{LL} \) (Assumption 1) and \( \Pr [L_j|L_i] \geq \Pr [L_j|H_i] \) (affiliation). Thus the first term is nonnegative. Second,

\[
\Pr [H_j|H_i] (V_{HH} - b) \geq \Pr [H_j|L_i] (v(L_i, H_j) - b),
\]

holds for all \( b < V_{HH} \) because, \( \Pr [H_j|H_i] \geq \Pr [H_j|L_i] \) (affiliation) and \( V_{HH} \geq v(L_i, H_j) \) (Assumption 1). Thus the second term is nonnegative. Third,

\[
\Pr [L_j|L_i] G_{Lj}(b) + \Pr [H_j|L_i] G_{Hj}(b) \geq \Pr [L_j|H_i] G_{Lj}(b) + \Pr [H_j|H_i] G_{Hj}(b).
\]

This follows because both left and right hand sides are weighted averages of \( G_{Lj}(b) \) and \( G_{Hj}(b) \). As \( \Pr [H_j|H_i] \geq \Pr [H_j|L_i] \) (affiliation), the right-hand side places more weight on \( G_{Hj}(b) \). Given the strategies, \( G_{Hj}(b) \leq G_{Lj}(b) \), and the inequality follows. Thus the sum of the last two terms is nonnegative.

I.3 The conditions in Theorem 5 characterize a TRE

In this section, we show that the conditions in Theorem 5 characterize a TRE. As before, let \( \hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x), \hat{r}(x) = \epsilon \cdot r(x), \underline{r} = \min_{b \in [V_{LL}, V_{HH}]} r(b), \) and \( \bar{r} = \max_{b \in [V_{LL}, V_{HH}]} r(b) \).

Given a tremble \( \lambda (\epsilon, R) \), consider the following bidding strategy profile \( \mu^\epsilon \):

**Case (1)** \( V_{LL} > V_{LL} \) and \( \Pr [L_1, H_2] > \Pr [H_1, L_2] \): Bidder 2 bids \( V_{LL} \) given signal \( L_2 \) and bids over the interval \([V_{LL}, \bar{b}]\) with distribution \( G_{H2}^\epsilon(b) \) given signal \( H_2 \). Bidder 1 bids over the interval \([V_{LL}, \bar{b}]\) with distribution \( G_{L1}^\epsilon(b) \) given signal \( L_1 \) and bids over the interval \([\bar{b}, \bar{b}]\) with distribution \( G_{H1}^\epsilon(b) \) given signal \( H_1 \). These bidding distributions are described by equations...
For $b \in [V_{LL}, b^{*\epsilon}]$:

$$G'_{L1}(b) = \frac{\hat{R}(b^{*\epsilon}) V_{LH} - b^{*\epsilon}}{\hat{R}(b) \cdot V_{LH} - b} \quad (91)$$

$$G'_{H2}(b) = \frac{\Pr[L_2|L_1] b - V_{LL}}{Pr[H_2|L_1] \cdot V_{LH} - b} \quad (92)$$

and for $b \in [b^{*\epsilon}, \bar{b}]$:

$$G'_{H1}(b) = \frac{\Pr[L_1|H_2]}{Pr[H_1|H_2]} \frac{1}{V_{HH} - b} \left( b - b^{*\epsilon} - \left( 1 - \frac{\hat{R}(b^{*\epsilon})}{\hat{R}(b)} \right) (V_{LH} - b^{*\epsilon}) \right) \quad (93)$$

$$G'_{H2}(b) = \frac{\hat{R}(b^{*\epsilon}) \Pr[L_2|L_1]}{\hat{R}(b) \cdot Pr[H_2|L_1]} \frac{V_{HH} - b}{V_{LH} - b} \frac{V^{*\epsilon} - V_{LL}}{ \bar{b} - V_{HH} - b} \left( b - b^{*\epsilon} - \left( 1 - \frac{\hat{R}(b^{*\epsilon})}{\hat{R}(b)} \right) (V_{LH} - b^{*\epsilon}) \right) \quad (94)$$

where $\bar{b}$ and $b^{*\epsilon}$ satisfy $b^{*\epsilon} \in (V_{LL}, V_{LH})$, $\bar{b} \in (b^{*\epsilon}, V_{HH})$, and $G'_{H2}(\bar{b}) = G'_{H1}(\bar{b}) = 1$.

**Case (2)** $V_{LH} = V_{LL}$ or $\Pr[L_1, H_2] = \Pr[H_1, L_2]$: Bidder $i \in \{1, 2\}$ bids $V_{LL}$ given signal $L_i$ and bids over the interval $[V_{LL}, \bar{b}]$, given signal $H_i$, where $\bar{b}$ is given by equation (23). The bid distribution given $H_1$ is $G'_{H1}(b) = G_{H1}(b)$ (equation (24)) and given signal $H_2$ is $G'_{H2}(b)$ (equation (25)).

$$G'_{H2}(b) = \frac{\hat{R}(\bar{b}) V_{HH} - \bar{b}}{\hat{R}(b) V_{HH} - b} \frac{1}{Pr[L_2|H_1]} \frac{1}{Pr[H_2|H_1]} \frac{V_{HH} - b}{V_{LH} - b} \left( b - \bar{b} - \left( 1 - \frac{\hat{R}(\bar{b})}{\hat{R}(b)} \right) (V_{LH} - \bar{b}) \right) \quad (95)$$

We begin with $\bar{b}$ and $b^{*\epsilon}$ in case (1):

**Lemma 39** Given case (1) above ($V_{LH} > V_{LL}$ and $Pr[L_1, H_2] > Pr[H_1, L_2]$), for sufficiently small $\epsilon > 0$, functions $\bar{b}'$ and $b^{*\epsilon}$ exist that solve $G'_{H1}(\bar{b}') = G'_{H2}(\bar{b}') = 1$, vary continuously in $\epsilon$, and equal $b^{*\epsilon}$ and $\bar{b}$ (that are characterized by equations (18)–(19)) at $\epsilon = 0$. Moreover they satisfy $b^{*\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}' \in (b^{*\epsilon}, V_{HH})$.

**Proof.** We can rewrite $G'_{H1}(\bar{b}') = G'_{H2}(\bar{b}') = 1$ as $Y_1(b^{*\epsilon}, \bar{b}', \epsilon) = Y_2(b^{*\epsilon}, \bar{b}', \epsilon) = 0$ for

$$Y_1(b^{*\epsilon}, \bar{b}', \epsilon) = -b^{*\epsilon} + b^{*\epsilon} \left( 1 - \frac{1 - \epsilon + \epsilon R(b^{*\epsilon})}{1 - \epsilon + \epsilon R(\bar{b}')} \right) \Pr[L_1, H_2] \Pr[H_2] \left( \frac{\Pr[L_1, H_2]}{Pr[H_1, H_2]} \frac{(V_{LH} - b^{*\epsilon})^2}{V_{HH} - b^{*\epsilon}} - \frac{Pr[H_1, L_2]}{Pr[H_1, H_2]} \frac{(V_{LH} - b^{*\epsilon})(V_{HLL} - b^{*\epsilon})}{V_{HH} - b^{*\epsilon}} - \frac{Pr[L_1, L_2]}{Pr[H_1, H_2]} \frac{b^{*\epsilon} - V_{LL}}{Pr[L_1, H_2]} \left( \Pr[L_1, H_2] \Pr[L_1, H_2] \Pr[H_2] \right) \right) \quad (96)$$

$$Y_2(b^{*\epsilon}, \bar{b}', \epsilon) = -\bar{b}' + (1 - \Pr[H_1|H_2]) b^{*\epsilon} + \Pr[H_1|H_2] V_{HH} \left( 1 - \frac{1 - \epsilon + \epsilon R(b^{*\epsilon})}{1 - \epsilon + \epsilon R(\bar{b}')} \right) \Pr[H_1|H_2] \left( V_{LH} - b^{*\epsilon} \right) \quad (97)$$
Given that $R$ is continuously differentiable, it is apparent by inspection of equations (96)--(97) that $Y_1(b^{*\epsilon}, \bar{b}^{\epsilon}, \epsilon)$ and $Y_2(b^{*\epsilon}, \bar{b}^{\epsilon}, \epsilon)$ are continuously differentiable in $b^{*\epsilon}$, $\bar{b}^{\epsilon}$, and $\epsilon$ for all $b^{*\epsilon} < V_{HH}$ and $\epsilon < 1$. Moreover, at $\epsilon = 0$, equations (96)--(97) reduce to $Y_1(b^{*\epsilon}, \bar{b}^{\epsilon}, 0) = -b^{*\epsilon} + b^*$ and $Y_2(b^{*\epsilon}, \bar{b}^{\epsilon}, 0) = -\bar{b}^{\epsilon} + (1 - \Pr[H_1|H_2]) b^{*\epsilon} + \Pr[H_1|H_2] V_{HH}$, which are equivalent to equations (18)–(19). Therefore $b^{*\epsilon}$ and $\bar{b}^{\epsilon}$ that solve equations (96)--(97) at $\epsilon = 0$ exist (namely $b^*$ and $\bar{b}$). Further, differentiating yields

$$
\begin{pmatrix}
\frac{\partial}{\partial b^{*\epsilon}} Y_1(b^*, \bar{b}, 0) & \frac{\partial}{\partial \bar{b}^{\epsilon}} Y_1(b^*, \bar{b}, 0) \\
\frac{\partial}{\partial b^{*\epsilon}} Y_2(b^*, \bar{b}, 0) & \frac{\partial}{\partial \bar{b}^{\epsilon}} Y_2(b^*, \bar{b}, 0)
\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ (1 - \Pr[H_1|H_2]) & -1 \end{pmatrix},
$$

which has full rank. Therefore, the assumptions of the implicit function theorem hold. It implies that for sufficiently small $\epsilon$, solutions $b^{*\epsilon}$ and $\bar{b}^{\epsilon}$ that solve $Y_1(b^{*\epsilon}, \bar{b}^{\epsilon}, \epsilon) = Y_2(b^{*\epsilon}, \bar{b}^{\epsilon}, \epsilon) = 0$ exist that vary continuously in $\epsilon$ and equal $b^*$ and $\bar{b}$ at $\epsilon = 0$.

Finally, as $b^* \in (V_{LL}, V_{LH})$ and $\bar{b} \in (b^*, V_{HH})$ by Lemma 37, continuity implies that $b^{*\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}^{\epsilon} \in (b^{*\epsilon}, V_{HH})$, for sufficiently small $\epsilon$. ■

We proceed by proving three lemmas, which together prove that the conditions in Theorem 5 characterize a TRE. First, we show that a valid profile $\mu^{\epsilon}$ exists as described above for sufficiently small $\epsilon$. 

**Lemma 40** For sufficiently small $\epsilon > 0$, the strategy profile $\mu^{\epsilon}$ described at the beginning of Section L3 exists and is valid.\(^3\)

**Proof.** Case (1) ($V_{LH} > V_{LL}$ and $\Pr[L_1, H_2] > \Pr[H_1, L_2]$): For sufficiently small $\epsilon > 0$, Lemma 39 shows that $\bar{b}^{\epsilon}$ and $b^{*\epsilon}$ exist that solve $G_{H1}^{\epsilon}(\bar{b}^{\epsilon}) = G_{H2}^{\epsilon}(\bar{b}^{\epsilon}) = 1$, and satisfy $b^{*\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}^{\epsilon} \in (b^{*\epsilon}, V_{HH})$. Evaluating equations (91) and (93) at $b = b^{*\epsilon}$ yields $G_{L1}^{\epsilon}(b^{*\epsilon}) = 1$ and $G_{H1}^{\epsilon}(b^{*\epsilon}) = 0$. Evaluating equations (91)--(92) at $b = V_{LL}$ yields $G_{H2}^{\epsilon}(V_{LL}) = 0$ and

$$
G_{L1}^{\epsilon}(V_{LL}) = \frac{\hat{R}(b^{*\epsilon})}{\hat{R}(V_{LL})} \frac{V_{LH} - b^{*\epsilon}}{V_{LH} - V_{LL}} \geq 0,
$$

where the inequality follows from $V_{LH} > V_{LL}$ and $b^{*\epsilon} \in (V_{LL}, V_{LH})$.

Finally it remains to show that $G_{L1}^{\epsilon}(b)$, $G_{H1}^{\epsilon}(b)$, and $G_{H2}^{\epsilon}(b)$ are nondecreasing. For $b \in [V_{LL}, b^{*\epsilon}]$, derivatives of equations (91)--(92) are:

$$
g_{L1}^{\epsilon}(b) = -\frac{\hat{r}(b)}{\hat{R}(b)} \frac{\hat{R}(b^{*\epsilon})}{\hat{R}(b)} \frac{V_{LH} - b^{*\epsilon}}{V_{LH} - b} + \frac{\hat{R}(b^{*\epsilon})}{\hat{R}(b)} \frac{V_{LH} - b^{*\epsilon}}{(V_{LH} - b)^2},
$$

$$
g_{H2}^{\epsilon}(b) = \frac{\Pr[L_2|L_1]}{\Pr[H_2|L_1]} \frac{V_{LH} - V_{LL}}{(V_{LH} - b)^2}.
$$

\(^3\)Namely the described bidding distributions are valid cumulative distribution functions.
By inspection, \( g'_{H2}(b) > 0 \) as \( V_{LL} > V_{LL} \) in case (1), and thus \( G'_{H2}(b) \) is increasing on \([V_{LL}, b^*, \epsilon]\).

Note that for all \( b, b' \in [V_{LL}, V_{HH}] \):

\[
\frac{\hat{r}(b)}{R(b)} \in [\epsilon, \frac{\epsilon r}{1-\epsilon}] \quad \text{and} \quad \frac{\hat{R}(b)}{R(b')} \in \left[ 1 - \epsilon, \frac{1}{1 - \epsilon} \right].
\]

Therefore (as \( V_{LL} \leq b \leq b^*, \epsilon < V_{LL} \)), we can bound \( g'_{L1}(b) \) from below independently of \( b \):

\[
g'_{L1}(b) \geq g'_{L1} = (1 - \epsilon) \frac{V_{LL} - b^*, \epsilon}{(V_{LL} - V_{LL})^2 - (1 - \epsilon)^2}
\]

Moreover, \( \lim_{\epsilon \to 0} g'_{L1} = \frac{V_{LL} - b^*, \epsilon}{(V_{LL} - V_{LL})^2} > 0 \). Therefore \( G'_{L1}(b) \) is increasing for all \( b \in [V_{LL}, b^*, \epsilon] \) for sufficiently small \( \epsilon > 0 \).

For \( b \in [b^*, \bar{b}] \), derivatives of equations (93)-94 are:

\[
g'_{H1}(b) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{1}{V_{HH} - b} \left( b - b^*, \epsilon - \frac{\hat{R}(b^*, \epsilon)}{R(b)} (V_{LL} - b^*, \epsilon) \right) + \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{1}{V_{HH} - b} \left( 1 - \frac{\hat{R}(b^*, \epsilon)}{R(b)} \right) (V_{LL} - b^*, \epsilon)
\]

\[
g'_{H2}(b) = \frac{\hat{R}(b^*, \epsilon)}{R(b)} \left( \frac{V_{HH} - b^*, \epsilon}{V_{HH} - b} - \frac{\hat{R}(b)}{R(b)} \frac{V_{HH} - b^*, \epsilon}{V_{HH} - b} \right) \frac{\Pr[L_2|L_1]}{\Pr[H_2|H_1]} \frac{1}{V_{HH} - b} \left( b - b^*, \epsilon - \frac{\hat{R}(b^*, \epsilon)}{R(b)} (V_{LL} - b^*, \epsilon) \right) + \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{1}{V_{HH} - b} \left( 1 + \frac{\hat{R}(b^*, \epsilon)}{R(b)} \right) (V_{LL} - b^*, \epsilon)
\]

Given \( b^*, \epsilon \leq b \leq \bar{b} \), \( g'_{H1}(b) \) can be bounded below independent of \( b \) as

\[
g'_{H1}(b) \geq g'_{H1} = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{1}{V_{HH} - b^*, \epsilon} - \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{V_{LL} - b^*, \epsilon}{V_{HH} - b} \left( \frac{\hat{r}}{1 - \epsilon} + \frac{1}{V_{HH} - b^*, \epsilon} \right)
\]

Moreover, \( \lim_{\epsilon \to 0} g'_{H1} = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{1}{V_{HH} - b^*, \epsilon} > 0 \). Therefore \( G'_{H1}(b) \) is increasing for all \( b \in [b^*, \bar{b}] \) for sufficiently small \( \epsilon > 0 \). Similarly,

\[
g'_{H2}(b) \geq g'_{H2} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{1}{V_{HH} - b^*, \epsilon} - \epsilon \frac{\Pr[L_2|L_1]}{\Pr[H_2|L_1]} \frac{\hat{r}}{1 - \epsilon} \frac{V_{HH} - b^*, \epsilon}{V_{HH} - b^*, \epsilon - V_{LL}} \frac{V_{HH} - b^*, \epsilon}{V_{HH} - b} \left( \frac{\hat{r}}{1 - \epsilon} + \frac{1}{V_{HH} - b^*, \epsilon} \right)
\]

\[
\text{and} \lim_{\epsilon \to 0} g'_{H2} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{1}{V_{HH} - b^*, \epsilon} > 0 \). Therefore \( G'_{H2}(b) \) is increasing for all \( b \in [b^*, \bar{b}] \) for sufficiently small \( \epsilon > 0 \). Finally, evaluating equations (92) and (94) at \( b = b^*, \epsilon \) shows \( G'_{H2}(b^*, \epsilon) \) is continuous at \( b^*, \epsilon \). Therefore \( G'_{H2}(b) \) is nondecreasing on all of \([V_{LL}, \bar{b}]\) for sufficiently small \( \epsilon \).
Given bidders are indifferent over the support of their mixed strategies, and second showing that there are within the closure of the set of undominated bids.

First, we note that the set of undominated bids for bidder \( b \) is valid. Evaluating equation (95) at \( b \in \{ V_{LL}, \bar{b} \} \) yields \( G^{\epsilon}_{H2}(\bar{b}) = 1 \) and

\[
G^{\epsilon}_{H2}(V_{LL}) = \frac{\hat{R}(\bar{b})}{R(V_{LL})} \frac{V_{HH} - \bar{b}}{V_{HH} - V_{LL}} - \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \frac{1}{V_{HH} - V_{LL}} \left( \bar{b} - V_{LL} - \left( 1 - \frac{\hat{R}(\bar{b})}{R(V_{LL})} \right) (\bar{b} - V_{HL}) \right).
\]

Rearranging terms and substituting the definition of \( \bar{b} \) from equation (23), this becomes

\[
G^{\epsilon}_{H2}(V_{LL}) = \frac{1}{Pr[H_2|H_1]} \frac{1}{(V_{HH} - V_{LL})} \left( (V_{HL} - V_{LL}) \left( \frac{\hat{R}(\bar{b})}{R(V_{LL})} Pr[L_1|H_2] - Pr[L_2|H_1] \right) + \frac{\hat{R}(\bar{b})}{R(V_{LL})} Pr[H_2|H_1] - Pr[H_1|H_2] \right) (V_{HH} - V_{HL})
\]

Written in this way, it is clear that \( G^{\epsilon}_{H2}(b) \geq 0 \), which follows from the fact that \( V_{HH} \geq V_{HL} \geq V_{LL} \), \( Pr[H_2|H_1] \geq Pr[H_1|H_2] \) (and hence also \( Pr[L_1|H_2] \geq Pr[L_2|H_1] \)), and \( \frac{\hat{R}(\bar{b})}{R(V_{LL})} \geq 1 \). It only remains to show that \( G^{\epsilon}_{H2}(b) \) is nondecreasing from \( V_{LL} \) to \( \bar{b} \). Taking its derivative yields

\[
g^{\epsilon}_{H2}(b) = \frac{\hat{R}(\bar{b})}{R(b)} \left( \frac{V_{HH} - \bar{b}}{(V_{HH} - b)^2} - \frac{\hat{R}(\bar{b})}{R(b)} \frac{V_{HH} - \bar{b}}{V_{HH} - b} \right)
\]

\[
+ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \frac{1}{(V_{HH} - b)^2} \left( b - \frac{\hat{R}(\bar{b})}{R(b)} \frac{\hat{R}(\bar{b})}{R(b)} \frac{V_{HH} - \bar{b}}{V_{HH} - b} \right)
\]

\[
+ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \frac{1}{V_{HH} - b} \left( 1 + \frac{\hat{R}(\bar{b})}{R(b)} \frac{\hat{R}(\bar{b})}{R(b)} (\bar{b} - V_{HL}) \right),
\]

Given \( V_{LL} \leq b \leq \bar{b} < V_{HH} \), \( g^{\epsilon}_{H2}(b) \) can be bounded below independent of \( b \) as

\[
g^{\epsilon}_{H2}(b) \geq g^{\epsilon}_{H2} = \frac{V_{HH} - \bar{b}}{(V_{HH} - V_{LL})^2} - \frac{\epsilon \overline{e}}{(1 - \epsilon)^2} \left( 1 + \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \frac{V_{HL}}{V_{HH} - b} \right),
\]

Moreover, \( \lim_{\epsilon \to 0} g^{\epsilon}_{H2} = \frac{V_{HH} - \bar{b}}{(V_{HH} - V_{LL})^2} > 0 \). Therefore \( G^{\epsilon}_{H2}(b) \) is increasing for all \( b \in \{ V_{LL}, \bar{b} \} \) for sufficiently small \( \epsilon > 0 \). \( \blacksquare \)

Next we show that strategies are best responses.

**Lemma 41** For sufficiently small \( \epsilon > 0 \), the strategies in the profile \( \mu^{\epsilon} \) described above are best responses and constitute a NE of the tremble \( \lambda(\epsilon, R) \). Bidders bid within the closure of the set of undominated bids.

**Proof.** First, we note that the set of undominated bids for bidder \( i \) with signal \( S_i \) is \( (-\infty, V(S_i, H_j)) \) and its closure is \( (-\infty, V(S_i, H_j)) \). Therefore, \( \mu^{\epsilon} \) described above specifies that all bidders only bid within the closure of the set of undominated bids.

Next we show that the strategies are best responses. This entails two steps, first showing that bidders are indifferent over the support of their mixed strategies, and second showing that there are
no profitable deviations outside the support of their mixed strategies. Expected profits for bidder 
i may be written as:

\[ \Pi^*_i (b|S_i) = \hat{R}(b) (\Pr [L_j|S_i] (v (S_i, L_j) - b) G^e_{Lj} (b) + \Pr [H_j|S_i] (v (S_i, H_j) - b) G^e_{Hj} (b)) . \]

Differentiating yields

\[
\frac{d \Pi^*_i (b|S_i)}{db} = \Pr [L_j|S_i] (v (S_i, L_j) - b) (\hat{R}(b) g^e_{Lj} (b) + \hat{r}(b) G^e_{Lj} (b)) \\
+ \Pr [H_j|S_i] (v (S_i, H_j) - b) (\hat{R}(b) g^e_{Hj} (b) + \hat{r}(b) G^e_{Hj} (b)) \\
- \hat{R}(b) (\Pr [L_j|S_i] G^e_{Lj} (b) + \Pr [H_j|S_i] G^e_{Hj} (b)), \quad (100)
\]

and the first-order condition for bidder \( i \) to be locally indifferent over bids is \( d \Pi^*_i (b|S_i) / db = 0 \).

Case (2) requires verifying that \( d \Pi_1 (b|H_1) / db = d \Pi_2 (b|H_2) / db = 0 \) for \( b \in [V_{LL}, \bar{b}] \). Case (1) requires verifying that \( d \Pi'_1 (b|L_1) / db = 0 \) for \( b \in [V_{LL}, b^*] \), \( d \Pi'_1 (b|H_1) / db = 0 \) for \( b \in [b^*, \bar{b}'] \), and \( d \Pi'_2 (b|H_2) / db = 0 \) for \( b \in [V_{LL}, \bar{b}'] \). In each case, differentiation of cumulative bidding distributions and substitution into equation \([100]\) verifies the relevant first-order condition holds.

It now remains to verify that there are no profitable deviations for a bidder outside the support of its mixed strategy. We need not consider bids outside \([V_{LL}, \bar{b}]\) because (i) bids below \( V_{LL} \) earn at most 0 while bids at \( V_{LL} \) earn at least 0, and (ii) bids above \( b \) earn less than bids at \( b \).

To show that there are no strictly profitable deviations within \([V_{LL}, \bar{b}]\), it is sufficient to show that bidder \( i \)'s expected profit function has increasing differences in \( b \) and \( S_i \) given \( j \)'s strategy:

\[
\frac{d}{db} (\Pi^*_i (b|H_i) - \Pi^*_i (b|L_i)) \geq 0.
\]

Increasing differences implies that if bidder \( L_i \) is indifferent over bids on an interval, then bidder \( H_i \) weakly prefers to bid above that interval. Similarly, if bidder \( H_i \) is indifferent over bids on an interval, then bidder \( L_i \) weakly prefers to bid below that interval.

This rules out all other possible deviations.

We use equation \([100]\) to take the difference \( d \frac{d}{db} (\Pi^*_i (b|H_i) - \Pi^*_i (b|L_i)) \):

\[
\frac{d}{db} (\Pi^*_i (b|H_i) - \Pi^*_i (b|L_i)) = \\
(\Pr [L_j|H_i] (v (H_i, L_j) - b) + \Pr [L_j|L_i] (V_{LL} - b) - \hat{R}(b) g^e_{Lj} (b) + \hat{r}(b) G^e_{Lj} (b)) \\
+ (\Pr [H_j|H_i] (V_{HH} - b) - \Pr [H_j|L_i] (v (L_i, H_j) - b) - \hat{R}(b) g^e_{Hj} (b) + \hat{r}(b) G^e_{Hj} (b)) \\
+ \hat{R}(b) (\Pr [L_j|L_i] G^e_{Lj} (b) + \Pr [H_j|L_i] G^e_{Hj} (b)) \\
- \hat{R}(b) (\Pr [L_j|H_i] G^e_{Lj} (b) + \Pr [H_j|H_i] G^e_{Hj} (b)) \quad (101)
\]

It is then straightforward to see that (in both case 1 and case 2) \( d \frac{d}{db} (\Pi^*_i (b|H_i) - \Pi^*_i (b|L_i)) \geq 0 \) for all \( b \in [V_{LL}, \bar{b}] \) by examining equation \([101]\) term-by-term, using the same logic applied to equation \([90]\) on page 105.

Next we show convergence as \( \epsilon \) goes to zero.
Lemma 42 For both case (1) and case (2), there exists a sequence of positive \( \{\epsilon\} \) converging to 0 and an associated sequence of strategy profiles \( \{\mu^\epsilon\} \), which is as described above for each \( \epsilon \), that converges to the strategy profile described in the Theorem 5.

Proof.

Case 1 \((V_{LH} > V_{LL} \text{ and } \Pr[L_1, H_2] > \Pr[H_1, L_2])\): Lemma 39 shows that there exist a sequence of \( \{b^\epsilon, \tilde{b}^\epsilon\} \) that converge to \( \{b^*, \tilde{b}\} \). Note that \( \lim_{\epsilon \to 0} \frac{\hat{R}(b^\epsilon, \tilde{b}^\epsilon)}{\hat{R}(b)} = 1 \). Substituting these into equations (91)–(94) and comparing to equations (20)–(22) shows that \( \lim_{\epsilon \to 0} G_{S_i}^\epsilon = G_{S_i} \) for each \( S_i \in \{L_1, L_2, H_1, H_2\} \).

Case 2 \((V_{LH} = V_{LL} \text{ or } \Pr[L_1, H_2] = \Pr[H_1, L_2])\): Substituting \( \lim_{\epsilon \to 0} \frac{\hat{R}(\tilde{b})}{\hat{R}(b)} = 1 \) (which holds for all \( b \in [V_{LL}, V_{HH}] \)) into equation (95) and comparing to equation (25) shows that \( \lim_{\epsilon \to 0} G_{H_2}^\epsilon = G_{H_2} \). Bid distributions for \( S_i \in \{L_1, L_2, H_1\} \) coincide with the \( \epsilon = 0 \) case for all \( \epsilon \).

Finally, we apply the preceding lemmas to prove the result.

Lemma 43 The conditions in Theorem 5 characterize a TRE.

Proof. First, we note that the set of undominated bids for bidder \( i \) with signal \( S_i \) is \( (-\infty, V(S_i, H_j)) \) and its closure is \( (-\infty, V(S_i, H_j)) \). Therefore, \( \mu \) in Theorem 5 specifies that all bidders only bid within the closure of the set of undominated bids. This condition of a TRE is therefore satisfied. Lemmas 40, 41, and 42 therefore imply the NE in Theorem 5 is a TRE. Because the NE is unique in monotone bidding strategies, the TRE is as well.