Online Appendix for
“Proxy Advisory Firms: The Economics of Selling Information to Voters”
Andrey Malenko and Nadya Malenko

1. Supplementary analysis for the proof of Proposition 1: Proof that for any \( q \), the equilibrium \( w_s (0) = 0, w_s (1) = 1 \), and \( w_0 = \frac{1}{2} \) exists.

Consider the decision of shareholder \( i \) with signal \( s_i \) when other informed shareholders (i.e., shareholders that acquired private signals) vote according to strategy \( w_s (s_j) \), and uninformed shareholders (i.e., shareholders that did not acquire private signals) vote according to strategy \( w_0 = \frac{1}{2} \). Given \( q \), the probability that each shareholder votes “for” in state \( \theta \in \{0, 1\} \) equals

\[
\Pr [v_j = 1|\theta = 1] = q (w_s (1) p + w_s (0) (1 - p)) + (1 - q) \frac{1}{2} = qp + (1 - q) \frac{1}{2},
\]

\[
\Pr [v_j = 1|\theta = 0] = q (w_s (1) (1 - p) + w_s (0) p) + (1 - q) \frac{1}{2} = q (1 - p) + (1 - q) \frac{1}{2}.
\]

Shareholder \( i \)'s vote affects the decision if \( \frac{N - 1}{2} \) other shareholders vote “for” and \( \frac{N - 1}{2} \) vote “against.” The expected value of the proposal to shareholder \( i \) in this case is

\[
\tilde{u} (s_i) = \mathbb{E} [u (1, \theta) | s_i, PIV_i] = \Pr [\theta = 1|s_i, PIV_i] - \Pr [\theta = 0|s_i, PIV_i],
\]

where \( PIV_i \) denotes the event in which shareholder \( i \)'s vote determines the outcome (i.e., if \( \sum_{i \neq j} v_j = \frac{N - 1}{2} \)). Applying the Bayes’ rule,

\[
\tilde{u} (s_i) = \frac{\Pr [s_i|\theta = 1] \Pr [\sum_{j \neq i} v_j = \frac{N - 1}{2}|\theta = 1] - \Pr [s_i|\theta = 0] \Pr [\sum_{j \neq i} v_j = \frac{N - 1}{2}|\theta = 0]}{\Pr [s_i|\theta = 1] \Pr [\sum_{j \neq i} v_j = \frac{N - 1}{2}|\theta = 1] + \Pr [s_i|\theta = 0] \Pr [\sum_{j \neq i} v_j = \frac{N - 1}{2}|\theta = 0]},
\]

\[
= D (s_i) \times (\Pr [s_i|\theta = 1] - \Pr [s_i|\theta = 0]) \left( \frac{1}{2} + q (p - \frac{1}{2}) \right)^{-\frac{N - 1}{2} - \frac{1}{2}} \left( \frac{1}{2} - q (p - \frac{1}{2}) \right)^{\frac{N - 1}{2}},
\]

where \( D (s_i) > 0 \). The best response of shareholder \( i \) is to vote “for” \( (v_i = 1) \) if \( \tilde{u} (s_i) \geq 0 \) and vote “against” \( (v_i = 0) \) if \( \tilde{u} (s_i) \leq 0 \). When \( s_i = 1, \Pr [s_i|\theta = 1] - \Pr [s_i|\theta = 0] = 2p - 1 > 0 \). When \( s_i = 0, \Pr [s_i|\theta = 1] - \Pr [s_i|\theta = 0] = 1 - 2p < 0 \). Therefore, the optimal strategy of shareholder \( i \) is indeed \( v_i = s_i \). Hence, \( w_s (s) = s \) is an equilibrium.

Similarly, for an uninformed shareholder, the expected value of the proposal conditional on being pivotal is

\[
\tilde{u}_0 = D_0 \times \left( \frac{(qp + (1 - q) \frac{1}{2})^{\frac{N - 1}{2}} (1 - qp - (1 - q) \frac{1}{2})^{\frac{N - 1}{2}}}{(q (1 - p) + (1 - q) \frac{1}{2})^{\frac{N - 1}{2}} (1 - q (1 - p) - (1 - q) \frac{1}{2})^{\frac{N - 1}{2}}} \right) = 0,
\]

for some \( D_0 \), and hence it is indeed optimal to mix between voting “for” and “against.”

2. Value of signals. We derive the value of the private signal \( V_s (q_r, q_s) \) and the value of the advisor’s recommendation \( V_r (q_r, q_s) \) to shareholder \( i \) for given \( q_r, q_s \).

2.1. Value of a private signal. Shareholder \( i \)'s vote only makes a difference only if \( \sum_{j \neq i} v_j =
if \( q_i = 1 \) and on being pivotal, his utility from being informed is \( \frac{1}{2} \mathbb{E} [u(1, \theta) | s_i = 1, PIV_i] \). Similarly, conditional on being pivotal and his private signal being \( s_i = 0 \), the shareholder’s utility from being informed is \( -\frac{1}{2} \mathbb{E} [u(1, \theta) | s_i = 0, PIV_i] \). Overall, the shareholder’s value of acquiring a private signal is

\[
V_s(q_r, q_s) = Pr(s_i = 1) Pr(PIV_i | s_i = 1) \frac{1}{2} \mathbb{E} [u(1, \theta) | s_i = 1, PIV_i] - Pr(s_i = 0) Pr(PIV_i | s_i = 0) \frac{1}{2} \mathbb{E} [u(1, \theta) | s_i = 0, PIV_i].
\]

By the symmetry of the model, \( \mathbb{E} [u(1, \theta) | s_i = 1, PIV_i] = -\mathbb{E} [u(1, \theta) | s_i = 0, PIV_i] \) and \( Pr(PIV_i | s_i = 1) = Pr(PIV_i | s_i = 0) \), so we get

\[
V_s(q_r, q_s) = \frac{1}{2} Pr(PIV_i | s_i = 1) \mathbb{E} [u(1, \theta) | s_i = 1, PIV_i] - Pr(s_i = 0) Pr(PIV_i | s_i = 0) \frac{1}{2} \mathbb{E} [u(1, \theta) | s_i = 0, PIV_i].
\]

where

\[
Pr(PIV_i) = Pr(PIV_i | \theta = 1) = \pi Pr(PIV_i | r = 1, \theta = 1) + (1 - \pi) Pr(PIV_i | r = 0, \theta = 1)
\]

\[
= \pi P \left( \frac{1}{2} q_n + q_r + q_s p, N - 1, \frac{N-1}{2} \right) + (1 - \pi) P \left( \frac{1}{2} q_n - q_r + q_s p, N - 1, \frac{N-1}{2} \right).
\]

Hence, \( V_s(q_r, q_s) \) is given by (9).

### 2.2. Value of the advisor’s signal

As before, shareholder \( i \)’s vote makes a difference only if \( \sum_{j \neq i} v_j = \frac{N-1}{2} \). Conditional on \( r = 1 \) and on being pivotal, his utility from being informed is \( \frac{1}{2} \mathbb{E} [u(1, \theta) | r = 1, PIV_i] \). Similarly, conditional on \( r = 0 \) and on being pivotal, shareholder \( i \)’s utility from being informed is \( -\frac{1}{2} \mathbb{E} [u(1, \theta) | r = 0, PIV_i] \). Overall, the shareholder’s value of acquiring the advisor’s signal is

\[
V_r(q_r, q_s) = Pr(r = 1) Pr(PIV_i | r = 1) \frac{1}{2} \mathbb{E} [u(1, \theta) | r = 1, PIV_i] - Pr(r = 0) Pr(PIV_i | r = 0) \frac{1}{2} \mathbb{E} [u(1, \theta) | r = 0, PIV_i].
\]

By the symmetry of the model, \( \mathbb{E} [u(1, \theta) | r = 1, PIV_i] = -\mathbb{E} [u(1, \theta) | r = 0, PIV_i] \) and \( Pr(PIV_i | r = 1) = Pr(PIV_i | r = 0) \), so we get

\[
V_r(q_r, q_s) = \frac{1}{2} Pr(PIV_i | r = 1) \mathbb{E} [u(1, \theta) | r = 1, PIV_i] = \frac{1}{2} Pr(PIV_i | r = 1, \theta = 1) Pr(r = 1 | \theta = 1) - \frac{1}{2} Pr(PIV_i | r = 1, \theta = 0) Pr(r = 1 | \theta = 0)
\]

\[
= \frac{1}{2} \left( Pr(PIV_i | r = 1, \theta = 1) - \frac{1}{2} Pr(PIV_i | r = 1, \theta = 0) \right) \pi - \frac{1}{2} \pi.
\]

Note that \( Pr(PIV_i | r = 1, \theta = 1) = P(q_r + q_s p + \frac{1}{2} q_n, N - 1, \frac{N-1}{2}) \) and \( Pr(PIV_i | r = 1, \theta = 0) = P(q_r - q_s p + \frac{1}{2} q_n, N - 1, \frac{N-1}{2}) \). Hence, \( V_r(q_r, q_s) \) is given by (10).

### 3. Supplementary analysis for the proof of Proposition 2: Derivation of the condition under which equilibrium \( w_s(s_i) = s_i, w_r(r) = r, \) and \( w_0 = \frac{1}{2} \) exists.

According to Proposition 2, we can restrict attention to subgames that follow the information acquisition stage at which each shareholder \( i \) acquires \( r \) with probability \( q_r \), acquires \( s_i \) with probability \( q_s \), and stays uninformed with probability \( q_0 = 1 - q_r - q_s \). Such an equilibrium only exists if given \( q_r, q_s \), it is optimal for a shareholder who acquired a signal to follow it. It will be useful to
compute the probabilities that a random shareholder $j$ votes for the proposal, conditional on the advisor’s recommendation $r$ and the true state $\theta$:

$$\Pr [v_j = 1 | r = 1, \theta = 1] = q_r + q_s p + q_n \frac{1}{2},$$  \hspace{1cm} (37)

$$\Pr [v_j = 1 | r = 0, \theta = 1] = q_s p + q_n \frac{1}{2},$$  \hspace{1cm} (38)

$$\Pr [v_j = 1 | r = 1, \theta = 0] = q_r + q_s (1 - p) + q_n \frac{1}{2},$$  \hspace{1cm} (39)

$$\Pr [v_j = 1 | r = 0, \theta = 0] = q_s (1 - p) + q_n \frac{1}{2}.$$  \hspace{1cm} (40)

First, consider a shareholder with private signal $s_i$. Since his vote affects the decision only when he is pivotal, he compares $\mathbb{E} [u (1, \theta) | s_i, PIV_i]$ with zero or, equivalently, $\Pr (\theta = 1 | s_i, PIV_i)$ with $\frac{1}{2}$, and votes “for” if and only if the former is higher. By Bayes’ rule,

$$\Pr (\theta = s_i | s_i, PIV_i) = \frac{\Pr (PIV_i | \theta = s_i) p}{\Pr (PIV_i | \theta = s_i) p + \Pr (PIV_i | \theta \neq s_i) (1 - p)} = p > \frac{1}{2},$$

where we used the independence of $s_i$ and $r$ from $s_i$ conditional on $\theta$: because of independence, $v_j$ is independent from $\theta = s_i$ or $\theta \neq s_i$ (i.e., from whether shareholder $i$’s private signal is correct or not). Therefore, it is always optimal for a shareholder who acquired a private signal to follow it.

Second, consider a shareholder that acquired $r$. A shareholder compares $\mathbb{E} [u (1, \theta) | r, PIV_i]$ with zero and votes “for” if and only if the former is higher. Using Bayes’ rule and $\Pr (\theta = 1) = \frac{1}{2} = \Pr (r)$, we get

$$\mathbb{E} [u (1, \theta) | r, PIV_i] \Pr (PIV_i | r) = \Pr (\theta = 1 | r, PIV_i) \Pr (PIV_i | r) - \Pr (\theta = 0 | r, PIV_i) \Pr (PIV_i | r)$$

$$= \Pr (PIV_i | r, \theta = 1) \Pr (r | \theta = 1) - \Pr (PIV_i | r, \theta = 0) \Pr (r | \theta = 0).$$

It is sufficient to consider $r = 1$: since the model is symmetric, voting “against” is optimal for $r = 0$ whenever voting “for” is optimal for $r = 1$. When $r = 1$, the shareholder finds it optimal to vote “for” if and only if

$$\frac{\Pr (PIV_i | r = 1, \theta = 1)}{\Pr (PIV_i | r = 1, \theta = 0)} \frac{\pi}{1 - \pi} \geq 1.$$  \hspace{1cm} (42)

By independence of $s_i$, $s_j$, $j \neq i$, and $r$ conditional on $\theta$,

$$\Pr (PIV_i | r, \theta) = \Pr \left( \sum_{j \neq i} v_j = \frac{N - 1}{2} | r, \theta \right) = P \left( \Pr [v_j = 1 | r, \theta], N - 1, \frac{N - 1}{2} \right).$$

Plugging this into (42) gives

$$\frac{\pi}{1 - \pi} P \left( \frac{1}{2} + \frac{q_r}{2} + q_s (p - \frac{1}{2}), N - 1, \frac{N - 1}{2} \right) \geq 1.$$  \hspace{1cm} (43)

The intuition for (43) is as follows. Consider a shareholder with the advisor’s recommendation deciding whether to follow it. If $q_s > 0$, a split vote is a signal that the advisor’s recommendation is more likely to be incorrect ($r \neq \theta$), since a split vote is more likely when private signals of shareholders disagree with the advisor’s recommendation than when they agree with it. Therefore,
as long as \( q_r > 0 \) and \( q_s > 0 \), the information content from being pivotal lowers the shareholder’s assessment of the precision of the advisor’s recommendation. This logic is reflected in the left-hand side of (43), which gives the ratio of probabilities that the advisor is correct and incorrect: the first term \( \frac{\pi}{1-\pi} \) is the prior, while the second term reflects additional information from the fact that the vote is split.

Finally, consider an uninformed shareholder. Since the event of being pivotal is uninformative about state \( \theta \), such a shareholder is indifferent between voting “for” and “against” the proposal, so it is optimal for him to mix between the two options.

Therefore, if \( q_r \) and \( q_s \) satisfy (43), then voting in the direction of the signal that a shareholder has (private or advisor’s) is an equilibrium. If (43) is violated, there is no equilibrium with a positive value of the advisor’s recommendation. However, since all these sub-games imply zero value of recommendation of the advisor, they are not reached on equilibrium path if \( q_r > 0 \). In particular, whenever \( V_r(q_r, q_s) - f \geq 0 \), which is implied by any equilibrium with \( q_r > 0 \) (where \( V_r(q_r, q_s) \) is the value of the advisor’s recommendation to a shareholder), this condition is satisfied. Therefore, we do not verify (43) in subsequent derivations.

### 4. Supplementary analysis for the proof of Lemma 1.

The solutions to (24), if they exist, are given by

\[
q_r^1(\psi) = \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}} - \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}},
\]

\[
q_s^1(\psi) = \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}} + \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}},
\]

\[
q_r^2(\psi) = \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}} + \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}},
\]

\[
q_s^2(\psi) = \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}} - \frac{1}{2p-1}\left[ \frac{f+\frac{c}{2p-1}+\frac{2p-1}{2p-1}\psi}{\pi C_{N-1}} \right]^{\frac{2}{N-1}}.
\]

Note that \( q_r^j(0) = q_r^a \) and \( q_s^j(0) = q_s^a \) for \( j \in \{r, s\} \). Since \( p \in (\frac{1}{2}, 1) \), it is easy to see that \( q_r^2(\psi) + q_s^2(\psi) \leq q_r^1(\psi) + q_s^1(\psi) \) and that \( q_r^1(\psi) + q_s^1(\psi) \) is strictly decreasing in \( \psi \). Each solution satisfies \( (q_r, q_s) > 0 \) if and only if \( f + \psi < \frac{2\pi-1}{2p-1}(c + \psi) \Leftrightarrow f < \bar{f} + \frac{2(\pi-p)\psi}{2p-1} \).

**Proof of Claim 1:** If \( f \geq \bar{f} \), then there is no equilibrium \((q_r, q_s) > 0\).

First, since strictly positive solutions (22)-(23) do not exist for \( f \geq \bar{f} \), there is no equilibrium \((q_r, q_s) > 0 \) satisfying \( q_r + q_s < 1 \). Second, by contradiction, suppose there is an equilibrium \((q_r, q_s) > 0 \) with \( q_r + q_s = 1 \). Then, \( f + \frac{c}{2p-1} + \frac{2p-1}{2p-1} \psi \leq \frac{c}{2p-1} + \frac{2(\pi-p)\psi}{1-\pi} \) and \( q_r^i(\psi) + q_s^i(\psi) = 1 \) for some \( \psi \geq 0 \) and some \( i \in \{1, 2\} \). Since \( q_r^2(\psi) + q_s^2(\psi) \leq q_r^1(\psi) + q_s^1(\psi) \), we have \( q_r^1(\psi) + q_s^1(\psi) \geq 1 \).
This, together with the inequality above, implies

\[
1 \leq \frac{2p}{2p-1} \left[ \frac{1}{4} \left( f + \frac{C}{2p-1} \psi \right) \right]_{N-1}^2 + \frac{2(1-p)}{2p-1} \left[ \frac{1}{4} \left( \frac{C}{2p-1} \right)_{N-1}^2 \right] \leq \frac{2}{2p-1} \left[ \frac{1}{4} \left( f + c \psi \right)_{N-1}^2 \right] + \frac{2(1-p)}{2p-1} \left[ \frac{1}{4} \left( \frac{C}{2p-1} \right)_{N-1}^2 \right]
\]

\[
= \frac{2}{2p-1} \Lambda = q_0^*.
\]

which contradicts Assumption 1 that \(q_0^* \in (0,1)\).

**Proof of Claim 2:** If \(\frac{2p}{2p-1} \left[ \frac{1}{4} \left( f + \frac{c}{2p-1} \psi \right) \right]_{N-1}^2 \leq 1\), there is an equilibrium \((q_r, q_s) > 0\) if and only if \(f \in \left[ f_1, \hat{f} \right]\), where \(f_1\) is given by (21).

Note that

\[
q_r^b + q_s^b = \frac{2p}{2p-1} \left[ \frac{1}{4} \left( f + \frac{c}{2p-1} \psi \right) \right]_{N-1}^2 - \frac{2(1-p)}{2p-1} \left[ \frac{1}{4} \left( \frac{C}{2p-1} \right)_{N-1}^2 \right] (46)
\]

is strictly decreasing in \(f\). Also, when \(f = f_1\), the second term is zero and hence, given the inequality assumed by the claim, \(q_r^b + q_s^b \leq 1\) for \(f = f_1\). Hence, \(q_r^b + q_s^b \leq 1\) for any \(f \in \left[ f_1, \hat{f} \right]\) with strict inequality for \(f \neq f_1\). As shown above, \((q_r^b, q_s^b) > 0\) for \(f < \hat{f}\). Hence, there is an equilibrium \((q_r^b, q_s^b) > 0\) if \(f \in \left[ f_1, \hat{f} \right]\). By Claim 1, there is no equilibrium \((q_r, q_s) > 0\) if \(f \geq \hat{f}\). If \(f < f_1\), system (20) has no solution, so there is no equilibrium \((q_r, q_s) > 0\) with \(q_r + q_s < 1\). Finally, (20) with \(c + \psi\) and \(f + \psi\) instead of \(c\) and \(f\) does not have a solution if \(f < f_1\), while \(f + \psi\) instead of \(f\) does not have a solution if \(f < f_1\), so there is no equilibrium \((q_r, q_s) > 0\) with \(q_r + q_s = 1\) in this case either.

**Proof of Claim 3:** If \(\frac{2p}{2p-1} \left[ \frac{1}{4} \left( f + \frac{c}{2p-1} \psi \right) \right]_{N-1}^2 > 1\), there exists \(f_2 \geq f_1\) such that there is an equilibrium \((q_r, q_s) > 0\) if and only if \(f \in \left[ f_2, \hat{f} \right]\).

By Claim 1, there is no equilibrium \((q_r, q_s) > 0\) if \(f \geq \hat{f}\). Note that when \(f = \hat{f}\), the two roots in (46) are equal, and hence \(q_r^b + q_s^b\), given by (46), is below one. Also, when \(f = f_1\), the second term in (46) is zero and hence, given the inequality assumed by the claim, \(q_r^b + q_s^b > 1\) for \(f = f_1\). Since \(q_r^b + q_s^b\) is strictly decreasing in \(f\), there is a unique \(\hat{f}_1 \in \left( f_1, \hat{f} \right)\) at which (46) equals one (and since \(f < \hat{f}\), both \(q_r^b\) and \(q_s^b\) are strictly positive). Hence, if \(f \in \left( \hat{f}_1, \hat{f} \right)\), there is an equilibrium \((q_r, q_s) > 0\) if \(q_r + q_s < 1\). If \(f = \hat{f}_1\), there is an equilibrium \((q_r, q_s) > 0\) with \(q_r + q_s = 1\). Finally, if \(f < \hat{f}_1\), then \(q_r^b + q_s^b \geq q_r^b + q_s^b > 1\), so there is no equilibrium of type \((q_r^b, q_s^b)\) or \((q_r^b, q_s^b)\).
Next, consider $f \leq \hat{f}_1$. Consider equilibria with $q_r + q_s = 1$. Define

$$
\hat{f}_2 = c + C_{N-1}^{N-1} 2^{1-N} \left( \pi (1-p) \left( \frac{(1-p)(3p-1)}{p^2} \right) \right)^{N-1} - p(1-\pi). \tag{47}
$$

We next show that if $\hat{f}_2 \leq \hat{f}_1$, then the necessary and sufficient conditions for equilibrium of the type $(q^1_r(\psi), q^1_s(\psi)) > 0$ with $q^1_r(\psi) + q^1_s(\psi) = 1$ to exist (for some $\psi \geq 0$) is $f \in [\hat{f}_2, \hat{f}_1]$. To prove this, note that such an equilibrium exists if and only if $f$ is such that equation $q^1_r(\psi) + q^1_s(\psi) = 1$ has a solution $\psi \geq 0$ with $q^1_r(\psi) > 0$ (condition $q^1_s(\psi) > 0$ is implied by it from (44)). Hence, $\psi$ must satisfy

$$
\frac{2p}{2p-1} \left[ \frac{1}{4} - \left( \frac{f + \frac{c}{2p-1} + \frac{2p-1}{2p-1} \psi}{\pi C_{N-1}^{N-1}} \right)^{\frac{2}{N-1}} \right] \leq 1 \iff \psi \geq \psi_1, \tag{48}
$$

$$
\frac{1}{4} - \left( \frac{\frac{c}{2p-1} - f + \frac{2(1-p)}{2p-1} \psi}{(1-\pi) C_{N-1}^{N-1}} \right)^{\frac{2}{N-1}} \geq 0 \iff \psi \leq \psi_h, \tag{49}
$$

where the first inequality follows from $q^1_r(\psi) + q^1_s(\psi) = 1$ and

$$
\psi_1 = \frac{2p-1}{2p} \left( \left( \frac{(1-p)(3p-1)}{4p^2} \right)^{\frac{N-1}{2}} \pi C_{N-1}^{N-1} - f - \frac{c}{2p-1} \right),
$$

$$
\psi_h = \frac{2p-1}{2(1-p)} \left( 2^{1-N} (1-\pi) C_{N-1}^{N-1} + f - \frac{c}{2p-1} \right). \tag{50}
$$

Hence, this system is equivalent to

$$
\psi_1 \leq \psi \leq \psi_h. \tag{51}
$$

Note that $\psi_h \geq \psi_1 \iff f \geq \hat{f}_2$, given by (47). Therefore, if $f < \hat{f}_2$, there is no equilibrium $(q^1_r(\psi), q^1_s(\psi)) > 0$. We next show that if $f \in [\hat{f}_2, \hat{f}_1]$, so that (51) is non-empty, such an equilibrium exists. When $\psi = \psi_h$, (49) binds and since $\psi_1 \leq \psi_h$, then (48) is satisfied and hence $q^1_r(\psi_h) + q^1_s(\psi_h) \leq 1$ (since it equals the left-hand side of (48) when (49) binds). When $\psi = \psi_1$, (48) binds and hence $q^1_r(\psi_1) + q^1_s(\psi_1) \geq 1$ (since it equals the left-hand side of (48) plus a non-negative number). As shown above, $q^1_r(0) + q^1_s(0) = q^a_r + q^a_s \geq q^b_r + q^b_s \geq 1$ for $f \leq \hat{f}_1$. Hence, when $\psi = \max \{0, \psi_1\}$, we have $q^1_r(\psi) + q^1_s(\psi) \geq 1$ for $f \leq \hat{f}_1$. Since $q^1_r(\psi) + q^1_s(\psi)$ is strictly decreasing in $\psi$, it must be that $\psi_h \geq 0$ (otherwise, $q^1_r(\psi_h) + q^1_s(\psi_h) > q^1_r(0) + q^1_s(0) \geq 1$).

Thus, the interval $[\max \{0, \psi_1\}, \psi_h]$ is non-empty and by the intermediate value theorem there exists a unique $\psi^* \in [\max \{0, \psi_1\}, \psi_h]$ at which $q^1_r(\psi^*) + q^1_s(\psi^*) = 1$. Note also that for this $\psi^*$, $q^1_r(\psi^*) > 0$ (and $q^1_s(\psi^*) > 0$ follows from (44)). Indeed, suppose by contradiction that $q^1_s(\psi^*) \leq 0$. Since $q^1_s(0) = q^a_s > 0$ for $f < \hat{f}_1$, then by the intermediate value theorem, there exists $\psi^{**} \in (0, \psi^*)$ such that $q^1_s(\psi^{**}) = 0$. Since $\psi^{**} \leq \psi^*$ and $q^1_r(\psi) + q^1_s(\psi)$ is strictly decreasing in $\psi$, $q^1_r(\psi^{**}) + q^1_s(\psi^{**}) \geq q^1_r(\psi^*) + q^1_s(\psi^*) = 1$, and hence $q^1_s(\psi^{**}) \geq 1$. Since $q^1_r(\psi^{**})$ and $q^1_s(\psi^{**})$ satisfy $V_s(q_r, q_s) - c = V_r(q_r, q_s) - f = \psi^{**} + q^1_s(\psi^{**}) = 0$, we have $V_s(0, q^1_s(\psi^{**})) = c + \psi^{**}$, and hence $q^1_s(\psi^{**})$ is the equilibrium of the benchmark case with no advisor but with a higher cost, $\hat{c} = c + \psi^{**}$. Since the cost if higher, it must be that $q^1_s(\psi^{**}) \leq q^a_s(0) = q^a_s$, but then $q^a_s \geq 1$, which contradicts Assumption 1. Hence, indeed, $q^1_s(\psi^*) > 0$. Therefore, there exists an equilibrium
(q^1_r(\psi), q^1_s(\psi)) > 0 with q^1_r(\psi) + q^1_s(\psi) = 1 (for some \psi \geq 0) if and only if \( f \in [\tilde{f}_2, \tilde{f}_1] \).

Since \( \psi_t = \psi_h \) for \( f = \tilde{f}_2 \), then (48) and (49) bind for \( \psi = \psi_h \). Thus, \( q^2_r(\psi_h) + q^2_s(\psi_h) \). By (44), \( q^2_r(\psi_h) \in (0, 1) \), and hence \( q^2_r(\psi_h) = 1 - q^2_s(\psi_h) \in (0, 1) \). Hence, equilibrium of the type \((q^2_r(\psi), q^2_s(\psi)) > 0 with q^2_r(\psi) + q^2_s(\psi) = 1 (for some \psi \geq 0) exists for \( f = \tilde{f}_2 \). We next prove that there exists a cutoff level \( \bar{f}_3 \leq \tilde{f}_2 \) such that equilibrium of the type \((q^2_r(\psi), q^2_s(\psi)) > 0 with q^2_r(\psi) + q^2_s(\psi) = 1 (for some \psi \geq 0) exists for \( f \in [\bar{f}_3, \tilde{f}_2] \) and does not exist for \( f < \bar{f}_3 \). To see this, define

\[ V(f) = \min_{\psi \in [0, \psi_h(f)]} \left\{ q^2_r(\psi, f) + q^2_s(\psi, f) \right\} = - \max_{\psi \in [0, \psi_h(f)]} \left\{ -q^2_r(\psi, f) - q^2_s(\psi, f) \right\}, \]

where \( \psi_h(f) \) is given by (50). Define \( \Phi = \{ f \in [0, \tilde{f}_2] : V(f) \leq 1 \} \) and note that equilibrium of the type \((q^2_r(\psi), q^2_s(\psi)) > 0 with q^2_r(\psi) + q^2_s(\psi) = 1 \) exists if and only if \( f \in \Phi \). Indeed, if \( V(f) > 1 \), then \( q^2_r(\psi, f) + q^2_s(\psi, f) > 1 \) for any \( \psi \geq 0 \) (since for \( \psi > \psi_h(f) \), this function is not well defined) and hence no such equilibrium exists. On the other hand, suppose that \( V(f) \leq 1 \) and is achieved at \( \psi^*(f) \). Then \( q^2_r(\psi^*(f), f) + q^2_s(\psi^*(f), f) \leq 1 \). In addition, since \( \psi_h(f) < \psi_1(f) \) for \( f < \tilde{f}_2 \), (48) is violated and (49) binds for \( \psi = \psi_h(f) \), and hence \( q^2_r(\psi_h(f), f) + q^2_s(\psi_h(f), f) > 1 \) for \( f < \tilde{f}_2 \). By the intermediate value theorem, there then exists \( \psi \in [\psi^*(f), \psi_h(f)] \) such that \( q^2_r(\psi, f) + q^2_s(\psi, f) = 1 \). Since (45) implies that \( q^2_r(\psi) \in (0, 1) \), and hence \( q^2_s(\psi) = 1 - q^2_r(\psi) \in (0, 1) \) as well, this constitutes an equilibrium.

Next, note that \( V(f) \) is decreasing in \( f \). Indeed, define the Lagrangian \( L(f, \psi, \lambda, \mu) = -q^2_r(\psi) - q^2_s(\psi, f) + \lambda \psi + \mu (\psi_h(f) - \psi) \) and note that \( V(f) = -\max_{\psi, \lambda, \mu} L(f, \psi, \lambda, \mu) \). By Envelope theorem, \( V'(f) = -L'_f(f, \psi^*, \lambda^*, \mu^*) = - (q^2_r(\psi^*, f') + q^2_s(\psi^*, f') + \mu^* \psi'_h(f)) \). Note that \( \mu^* \geq 0 \) according to the Kuhn-Tucker conditions. Because \( q^2_r(\psi, f) + q^2_s(\psi, f) \) decreases in \( \psi \) and \( \psi'_h(f) \geq 0 \), it follows that \( V'(f) \leq 0 \). The fact that \( V'(f) \leq 0 \) implies that \( \Phi = [\tilde{f}_3, \tilde{f}_2] \) for some \( \tilde{f}_3 \leq \tilde{f}_2 \). Hence, equilibrium \((q^2_r(\psi), q^2_s(\psi)) > 0 with q^2_r(\psi) + q^2_s(\psi) = 1 \) exists for \( f \in [\tilde{f}_3, \tilde{f}_2] \) and does not exist for \( f < \tilde{f}_3 \), as required. Moreover, note that \( \tilde{f}_3 \geq \tilde{f}_1 \). This is because, (45) does not have a solution if \( \psi > \psi_h(f) \) if \( f < \tilde{f}_1 + \frac{2(1-\nu)}{4\varphi} \psi \) and hence does not have a solution if \( f < \tilde{f}_1 \).

Consider two cases. First, if \( \tilde{f}_2 \leq \tilde{f}_1 \), then combining the results above, equilibrium \((q_r, q_s) \neq 0 \) with \( q_r + q_s < 1 \). This implies that equilibrium \((q_r, q_s) > 0 \) if and only if \( f \neq \tilde{f}_1 \), which is a contradiction. Combined with Claim 1, this implies that equilibrium \((q_r, q_s) > 0 \) if and only if \( f \in [\tilde{f}_3, \tilde{f}_1] \), which is also a contradiction. Combined with Claim 1, this implies that equilibrium \((q_r, q_s) > 0 \) if and only if \( f \in [\tilde{f}_3, \tilde{f}_1] \), which is also a contradiction. Finally, we conclude that equilibrium \((q_r, q_s) > 0 \) if and only if \( f \in [\tilde{f}_3, \tilde{f}_1] \), where \( \bar{f}_2 = \min(\tilde{f}_3, \tilde{f}_1) \).

5. Supplementary analysis for the proof of Proposition 3: Properties of (32).
Let us fix fee $f$ and vary $\pi$. Recall from Lemma 1 that equilibrium with complete crowding out exists if and only if $f < \tilde{f} = \frac{2\pi - 1}{2p - 1}c$, i.e., $\pi > \frac{1}{2} + \frac{f}{c} (p - \frac{1}{2})$. The derivative of the left-hand side of (32) in $\pi$ is:

$$2 \sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k) - 1 + (2\pi - 1) \frac{dp_a}{d\pi} \sum_{k=\frac{N+1}{2}}^{N} P_q(p_a, N, k) > 0,$$

since $\sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k) > \frac{1}{2}$, $\sum_{k=\frac{N+1}{2}}^{N} P_q(p_a, N, k) > 0$, and $\frac{dp_a}{d\pi} > 0$. Indeed, $\sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k) > \frac{1}{2}$ because $\sum_{k=\frac{N+1}{2}}^{N} P(p_a, N, k) + \sum_{k=\frac{N+1}{2}}^{N} P(1 - p_a, N, k) = 1$ and $P(p_a, N, k) > P(1 - p_a, N, k)$ for $p_a > \frac{1}{2}$ and $k > N$. Second, $\sum_{k=\frac{N+1}{2}}^{N} P_q(p_a, N, k) > 0$ for $p_a > \frac{1}{2}$, as shown in the proof of Part 1. Finally, $\frac{dp_a}{d\pi} > 0$ follows directly from (31). Therefore, the left-hand side of (32) is strictly increasing in $\pi$.

Note also that the advisor’s presence strictly decreases firm value for $\pi \rightarrow \frac{1}{2} + \frac{f}{c} (p - \frac{1}{2})$. Indeed, in this case, $p_a \rightarrow p^*$, so we obtain

$$(2\pi - 1) \sum_{k=\frac{N+1}{2}}^{N} P(p^*, N, k) - \pi < \sum_{k=\frac{N+1}{2}}^{N} P(p^*, N, k) - 1 \iff 1 < 2 \sum_{k=\frac{N+1}{2}}^{N} P(p^*, N, k),$$

which is true, as just shown above, since $p^* > \frac{1}{2}$. Finally, when $\pi \rightarrow 1$, the advisor’s presence strictly increases firm value. Indeed,

$$\lim_{\pi \rightarrow 1} p_a = \frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{2f}{C_{N-1}^{N-1}}\right)^2} > \frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{C_{N-1}^{N-1}}{2p - 1} - \frac{1}{2}\right)^2} = p^*,$$

so the left-hand side of (32) converges to

$$\sum_{k=\frac{N+1}{2}}^{N} P\left(\lim_{\pi \rightarrow 1} p_a, N, k\right) - 1 > \sum_{k=\frac{N+1}{2}}^{N} P(p^*, N, k) - 1$$

because $\sum_{k=\frac{N+1}{2}}^{N} P_q(q, N, k) > 0$ for $q > \frac{1}{2}$, as shown in the proof of Part 1.

6. Supplementary analysis for the proof of Lemma 2.

6.1. Proof that $V(c)$ is strictly decreasing in $c$.

Note that

$$S\left(\frac{f_1(c)}{c}, c\right) = \frac{2p}{2p-1} \sqrt{\frac{1}{4} - \left(\frac{2c}{2p-1} - \frac{2-\frac{1}{N}(1-\pi)C_{N-1}^{N-1}}{\pi C_{N-1}^{N-1}}\right)^2},$$

$$S\left(\frac{f(c)}{c}, c\right) = \frac{2}{2p-1} \sqrt{\frac{1}{4} - \left(\frac{2c}{2p-1} - \frac{2-\frac{1}{N}(1-\pi)C_{N-1}^{N-1}}{2p-1}\right)^2},$$

Let us prove that $V(c)$ is strictly decreasing in $c$. For any $c$, one of three cases must hold: (1)
\( V(c) = S(f(c), c); \) (2) \( V(c) = S(f, c) \) for some \( f \in \left( \tilde{f}_1(c), \tilde{f}(c) \right) \); (3) \( V(c) = S(\tilde{f}(c), c) \).

As clear from (52), \( S(\tilde{f}(c), c) \) and \( S(\tilde{f}(c), c) \), corresponding to cases (1) and (3), are strictly decreasing in \( c \). In case (2), i.e., when (33) reaches the maximum at an interior point \( f^*(c) \), we can apply the envelope theorem: \( V'(c) = \frac{\partial S(f^*(c), c)}{\partial c} < 0 \). Together, this implies that \( V(c) \) is strictly decreasing in \( c \).

6.2. Ranking of the three equilibria in shareholder welfare.

First, we show that the equilibrium with incomplete crowding out of private information acquisition and \( q_r > (2p - 1)q_s \), denoted \( (q^b_s, q^b_r) \), has lower shareholder welfare than the equilibrium with incomplete crowding out of private information acquisition and \( q_r < (2p - 1)q_s \), denoted \( (q^o_s, q^o_r) \).

As shown above, \( q_r + q_s < 1 \). Using (26), we get \( q_r = p_a - p_d \) and \( q_s = \frac{p_a + p_d - 1}{2p - 1} \), and plugging these into (34), \( W(q_s, q_r) \) can be rewritten as

\[
\sum_{k=N+1}^{N} \left( \pi P(p_a, N, k) + (1 - \pi) P(p_d, N, k) \right) - \left( f + \frac{c}{2p - 1} \right) p_a - \left( \frac{c}{2p - 1} - f \right) p_d - \frac{1}{2} + \frac{c}{2p - 1}.
\]

Using (19),

\[
W(q_s, q_r) = \pi \left( \sum_{k=N+1}^{N} P(p_a, N, k) - \Omega_1 p_a \right) + (1 - \pi) \left( \sum_{k=N+1}^{N} P(p_d, N, k) - \Omega_2 p_d \right) - \frac{1}{2} + \frac{c}{2p - 1}.
\]

According to (19), (20), and (26), \( p_a, \Omega_1, \) and \( \Omega_2 \) are identical in both equilibria and \( p_d(q^b_s, q^b_r) = 1 - p_d(q^o_s, q^o_r) < \frac{1}{2} \). Therefore, to show that \( W(q^b_s, q^b_r) > W(q^o_s, q^o_r) \), it is necessary and sufficient to show that for \( p_d \geq \frac{1}{2} \)

\[
\sum_{k=N+1}^{N} P(p_d, N, k) - \Omega_2 p_d > \sum_{k=N+1}^{N} P(1 - p_d, N, k) - \Omega_2 (1 - p_d).
\]

Using \( \sum_{k=N+1}^{N} P(1 - q, N, k) = \sum_{k=N+1}^{N} P(q, N, N - k) = 1 - \sum_{k=N+1}^{N} P(q, N, k) \) and \( \Omega_2 = P(p_d, N - 1, \frac{N-1}{2}) \), this is equivalent to

\[
\sum_{k=N+1}^{N} P(p_d, N, k) - \frac{1}{2} > (p_d - \frac{1}{2})P(p_d, N - 1, \frac{N-1}{2}). \quad (53)
\]

Denote the left-hand side and the right-hand side by \( L(p_d) \) and \( R(p_d) \), respectively. Differentiating the left-hand side of (53),

\[
L'(x) = \sum_{k=N+1}^{N} P_x(x, N, k) = -\sum_{k=0}^{N-1} P_x(x, N, k) = -\frac{1}{x(1-x)} \left( \sum_{k=0}^{N-1} P(x, N, k)(k - Nx) \right)
\]

\[
= -\frac{1}{x(1-x)} \left( \sum_{k=0}^{N-1} kP(x, N, k) - Nx \sum_{k=0}^{N-1} P(x, N, k) \right)
\]

Note that \( \sum_{k=0}^{N-1} P(x, N, k) = I_{1-x} \left( \frac{N+1}{2}, \frac{N+1}{2} \right) \), where \( I_x(a, b) \) is the regularized incomplete beta
function. In addition, according to (64) and (65) and (66) in the proof of Auxiliary Lemma A1,
\[
\frac{N^{-1}}{2}\sum_{k=0}^{N^{-1}} kP (x, N, k) = NxI_{1-x} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) = Nx \left[ I_{1-x} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) - \frac{(1-x)^{N+1}}{2} B \left( \frac{N+1}{2}, \frac{N-1}{2} \right) \right],
\]
where \( B (a, b) \) is the beta function. Hence,
\[
x (1-x) L' (x) = Nx \left( \frac{1}{2} \right)^{N+1} x^{N-1} \frac{B (N+1/2, N-1/2)}{N^{-1}} = \left( (1-x) \right)^{N+1} N! \left( \frac{N-1}{2} \right)! = Nx (1-x) P \left( x, N-1, \frac{N-1}{2} \right).
\]
Differentiating the right-hand side of (53),
\[
R' (x) = P_x (x, N-1, \frac{N-1}{2}) (x-\frac{1}{2}) + P (x, N-1, \frac{N-1}{2}) < P (x, N-1, \frac{N-1}{2}) \frac{1}{2} = L' (x).
\]
Since \( L \left( \frac{1}{2} \right) = R \left( \frac{1}{2} \right) = 0 \), it follows that \( L (x) > R (x) \) for any \( x > \frac{1}{2} \). Hence, indeed, \( W(q^a_s, q^b_s) > W(q^b_s, q^c_s) \).

Second, we show that the equilibrium with incomplete crowding out of private information acquisition and \( q_r > (2p - 1) q_s \), denoted \((q^a_s, q^b_s)\), has higher shareholder welfare than the equilibrium with complete crowding out of private information acquisition, denoted \((0, q^c_s)\), whenever the two co-exist, i.e., \( f \in \left( f, \tilde{f} \right) \). Consider function \( \varphi (x) \in \left( \frac{1}{2}, 1 \right) \) defined as the higher root of \( x = P (\varphi (x), N-1, \frac{N-1}{2}) \) and given by (29). Since \( \Omega_1 = P (p_a, N-1, \frac{N-1}{2}) \), \( \Omega_2 = P (p_d, N-1, \frac{N-1}{2}) \), and since \( p_a > \frac{1}{2} \) and \( p_d < \frac{1}{2} \) in both of these equilibria, we have \( p_a = \varphi (\Omega_1) \) and \( p_d = 1 - \varphi (\Omega_2) \). Plugging these expressions for \( p_a \) and \( p_d \), we can re-write (34) as
\[
\sum_{k=0}^{N^{-1}} \frac{1}{2} P (x, N, k) = \frac{1}{2} - \sqrt{\Omega_1} < \frac{1}{2} - \sqrt{\Omega_2} < \frac{1}{2} - \sqrt{\Omega_1}
\]
where we used \( \sum_{k=0}^{N^{-1}} P (x, N, k) = \sum_{k=0}^{N^{-1}} P (x, N, k) = 1 - \sum_{k=0}^{N^{-1}} P (x, N, k) \) to get to the second line.

For equilibrium \((q^a_s, q^b_s)\), let us plug \( q_r = p_a - p_d \), \( q_s = \frac{p_a + p_d - 1}{2p - 1} \), \( p_a = \varphi (\Omega_1) \), and \( p_d = 1 - \varphi (\Omega_2) \) into (56). Then, using (19) and simplifying, we can write shareholder welfare \( W(q^a_s, q^b_s) \) as the following function of \( \Omega_1 \) and \( \Omega_2 \):
\[
\tilde{W} (\Omega_1, \Omega_2) = \pi \tilde{f} (\Omega_1) - (1-\pi) \tilde{f} (\Omega_2) + \frac{1}{2} - \pi,
\]
where
\[
\tilde{f} (x) = \sum_{k=0}^{N^{-1}} \frac{1}{2} P (x, N, k) = \frac{1}{2} \left( \varphi (x) - \frac{1}{2} \right)
\]
and \( \Omega_1 \) and \( \Omega_2 \) are given by (19).

Similarly, for equilibrium \((0, q^c_s)\), let us plug \( q_r = p_a - p_d \), \( q_s = 0 \), \( p_a = \varphi (\Omega_1) \), and \( p_d = 1 - \varphi (\Omega_2) \)
into (56). Using the fact that in this equilibrium, $\Omega_1 = \Omega_2 = \Omega_r = \frac{2f}{2\pi - 1}$ and simplifying, we can write shareholder welfare $W(0, q_r^e)$ as $\tilde{W}(\Omega_r, \Omega_r)$, where $\Omega_r = \frac{2f}{2\pi - 1}$.

Note next that $\Omega_1 = \Omega_r + \frac{1}{2\pi - 1} \varepsilon$ and $\Omega_2 = \Omega_r + \frac{\pi}{2\pi - 1} \varepsilon$, where $\varepsilon \equiv \frac{c}{2\pi - 1} - f \frac{1}{\pi(1 - \pi)}$ since $f < f$. Thus, in order to prove that $W(q_s^b, q_r^b) > W(0, q_r^e)$, it is necessary and sufficient to prove that $\tilde{W}(\Omega_r, \frac{1}{2\pi - 1} \varepsilon, \Omega_r + \frac{\pi}{2\pi - 1} \varepsilon) > \tilde{W}(\Omega_r, \Omega_r)$. Define function $\tilde{W}(x) \equiv \tilde{W}(\Omega_r + \frac{1}{2\pi - 1} x, \Omega_r + \frac{\pi}{2\pi - 1} x)$ for $x \geq 0$. Differentiating,

$$
\tilde{W}'(x) = \frac{\pi (1 - \pi)}{2\pi - 1} \left( f' \left( \Omega_r + \frac{1 - \pi}{2\pi - 1} x \right) - f' \left( \Omega_r + \frac{\pi}{2\pi - 1} x \right) \right) = -\pi \frac{(1 - \pi)}{2\pi - 1} \int_{\frac{1}{2\pi - 1} x}^{\frac{\pi}{2\pi - 1} x} f''(y) dy.
$$

Auxiliary Lemma A2 shows that function $f'(\cdot)$ is strictly concave, and hence $\tilde{W}'(x) > 0$ for any $x > 0$. Thus, $\tilde{W}(\Omega_r + \frac{1}{2\pi - 1} \varepsilon, \Omega_r + \frac{\pi}{2\pi - 1} \varepsilon) = \tilde{W}(\varepsilon) > \tilde{W}(0) = \tilde{W}(\Omega_r, \Omega_r)$, which proves the statement.

Combining the two results above, we can conclude that when $c \in (\hat{c}, \tilde{c})$ and when multiple equilibria exist, i.e., when $f \in [\hat{f}, \tilde{f}]$, they rank in shareholder welfare in the following way: The equilibrium with incomplete crowding out of private information acquisition and $q_r < (2p - 1) q_s$ has the highest shareholder welfare, followed by the equilibrium with incomplete crowding out of private information acquisition and $q_r > (2p - 1) q_s$, which is followed by the equilibrium with complete crowding out of private information acquisition.

7. Supplementary analysis for the proof of Proposition 5.

7.1. Proof that when $f = f_{-1}$, firm value is strictly higher in equilibrium with incomplete crowding out than in equilibrium with complete crowding out.

To see this, consider any equilibrium with $p_a > \frac{1}{2}$ and $p_d < \frac{1}{2}$. Since $\Omega_1 (q_r, q_s) = P(p_a, N - 1, \frac{N - 1}{2})$, $\Omega_2 (q_r, q_s) = P(p_d, N - 1, \frac{N - 1}{2})$ and since $p_a > \frac{1}{2}$ and $p_d < \frac{1}{2}$, we have $p_a = \varphi (\Omega_1)$ and $p_d = 1 - \varphi (\Omega_2)$, where $\varphi$ is given by (29). According to (27), firm value is

$$
\hat{U}(\Omega_1, \Omega_2) = \sum_{k=0}^{N} \frac{\pi P (\varphi (\Omega_1), N, k) + (1 - \pi) P (1 - \varphi (\Omega_2), N, k)}{2} = \sum_{k=0}^{N} \frac{\pi P (\varphi (\Omega_1), N, k) - (1 - \pi) P (\varphi (\Omega_2), N, k)}{2} + \frac{1}{2} - \pi = \pi f(\Omega_1) - (1 - \pi) f(\Omega_2) + \frac{1}{2} - \pi,
$$

where $f(x) \equiv \sum_{k=0}^{N} \frac{\pi P (\varphi (x), N, k)}{2}$. In equilibrium with complete crowding out and $f = f_{-1}$, we have $p_a = \frac{1}{2} + \frac{1}{2} q_r > \frac{1}{2}$, $p_d = \frac{1}{2} - \frac{1}{2} q_r < \frac{1}{2}$, and (according to (18)) $\Omega_1 (q_r, 0) = \Omega_2 (q_r, 0) = \frac{2f}{2\pi - 1} \equiv \Omega_r$. Consider $\Omega_1 \equiv \Omega_r + \frac{1 - \pi}{2\pi - 1} \varepsilon$ and $\Omega_2 \equiv \Omega_r + \frac{\pi}{2\pi - 1} \varepsilon$ with $\varepsilon \equiv \frac{c}{2\pi - 1} - f \frac{1}{\pi(1 - \pi)} > 0$. Note that $\pi \Omega_1 - (1 - \pi) \Omega_2 = 2 f_{-1}$ and $\pi \Omega_1 + (1 - \pi) \Omega_2 = \frac{c}{p - 0.5}$, i.e., $\Omega_1$ and $\Omega_2$ satisfy (19). Hence, for $f = f_{-1}$, equilibrium with incomplete crowding out is characterized by probabilities of being pivotal $\Omega_1$ and $\Omega_2$. We next prove that $\hat{U}(\Omega_r + \frac{1 - \pi}{2\pi - 1} \varepsilon, \Omega_r + \frac{\pi}{2\pi - 1} \varepsilon) > \hat{U}(\Omega_r, \Omega_r)$. Indeed, function $\hat{U}(x) \equiv \hat{U}(\Omega_r + \frac{1 - \pi}{2\pi - 1} x, \Omega_r + \frac{\pi}{2\pi - 1} x)$ for $x \geq 0$ is increasing because

$$
\hat{U}'(x) = \frac{\pi (1 - \pi)}{2\pi - 1} \left( f' \left( \Omega_r + \frac{1 - \pi}{2\pi - 1} x \right) - f' \left( \Omega_r + \frac{\pi}{2\pi - 1} x \right) \right) = -\pi \frac{(1 - \pi)}{2\pi - 1} \int_{\frac{1}{2\pi - 1} x}^{\frac{\pi}{2\pi - 1} x} f''(y) dy > 0.
$$
by Auxiliary Lemma A1. Hence, indeed, when \( f = \frac{x}{2^N} \), firm value is strictly higher in equilibrium with incomplete crowding out than in equilibrium with complete crowding out.

7.2. Comparison of \( \pi \) and \( \bar{\pi} \). Simplifying,

\[
P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}} N - 1, \frac{N-1}{2} \right) = C_{N-1}^{\frac{N-1}{2}} \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}} \right) \left( \frac{1}{2} - \frac{1}{2^{\sqrt{N}}} \right) \]

and hence

\[
\bar{\pi} \equiv \frac{1}{2} \left( 1 + \frac{C_{N-1}^{\frac{N-1}{2}} 2^{1-N} - \frac{2c}{2^{p-1}}}{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N - 1, \frac{N-1}{2} \right)} \right).
\]

Since \( \pi^* = \sum_{k=N}^{N} P(p_0, N, k) \), where \( p_0 = \left( p - \frac{1}{2} \right) q_0 + \frac{1}{2} = \Lambda + \frac{1}{2} \), we have

\[
\hat{\pi} \equiv \frac{1}{2} \left( 1 + \frac{\sum_{k=N}^{N} P \left( \frac{1}{2} + \Lambda, N, k \right) - \frac{1}{2}}{\sum_{k=N}^{N} P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N, k \right) - \frac{1}{2}} \right).
\]  

(59)

Hence, \( \hat{\pi} \leq \bar{\pi} \) if and only if

\[
\frac{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - \frac{2c}{2^{p-1}}}{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N - 1, \frac{N-1}{2} \right)} \leq \frac{\sum_{k=N}^{N} P \left( \frac{1}{2} + \Lambda, N, k \right) - \frac{1}{2}}{\sum_{k=N}^{N} P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N, k \right) - \frac{1}{2}}.
\]

Furthermore, from the indifference condition in the benchmark case, \( P \left( \frac{1}{2} + \Lambda, N - 1, \frac{N-1}{2} \right) = \frac{2c}{2^{p-1}} \), and hence, \( \hat{\pi} \leq \bar{\pi} \) if and only if

\[
\frac{\sum_{k=N}^{N} P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N, k \right) - \frac{1}{2}}{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}}, N - 1, \frac{N-1}{2} \right)} \leq \frac{\sum_{k=N}^{N} P \left( \frac{1}{2} + \Lambda, N, k \right) - \frac{1}{2}}{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( \frac{1}{2} + \Lambda, N - 1, \frac{N-1}{2} \right)}.
\]

Consider function

\[
g(x) = \frac{L(x)}{P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( x, N - 1, \frac{N-1}{2} \right)},
\]

where \( L(x) = \sum_{k=N}^{N} P(x, N, k) - \frac{1}{2} \) is the same as defined in the proof of Lemma 2. Then, the above inequality is equivalent to \( g \left( \frac{1}{2} + \frac{1}{2^{\sqrt{N}}} \right) \leq g \left( \frac{1}{2} + \Lambda \right) \). Differentiating,

\[
g'(x) = \frac{L'(x) \left( P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( x, N - 1, \frac{N-1}{2} \right) \right) + P_x (x, N - 1, \frac{N-1}{2}) L(x)}{\left( P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( x, N - 1, \frac{N-1}{2} \right) \right)^2}.
\]

Using the expressions for \( L'(x) \) and \( P_x (x, N - 1, \frac{N-1}{2}) \) in (54) and (55) in the proof of Lemma 2, it follows that the sign of \( g'(x) \) coincides with the sign of

\[
\tilde{g}(x) = N \left( P \left( \frac{1}{2}, N - 1, \frac{N-1}{2} \right) - P \left( x, N - 1, \frac{N-1}{2} \right) \right) - \frac{(N-1) \left( x - \frac{1}{2} \right) L(x)}{x \left( 1 - x \right)}.
\]
Note that
\[ \tilde{g}'(x) = -NP_x(x, N - 1, \frac{N-1}{2}) - (N - 1) \left[ \frac{x - \frac{1}{2}}{x(1-x)} \right]'L(x) - \frac{(N-1)(x-\frac{1}{2})}{x(1-x)}L'(x) \]
\[ = - (N - 1) \frac{x(1-x)+2(x-\frac{1}{2})^2}{x^2(1-x)^2}L(x) < 0 \]

Since \( \tilde{g}(\frac{1}{2}) = 0 \), \( \tilde{g}(x) < 0 \) for \( x \in (\frac{1}{2}, 1) \). Therefore, \( g(x) \) is strictly decreasing in \( x \in (\frac{1}{2}, 1) \). Hence, \( \tilde{\pi} \leq \bar{\pi} \iff g\left(\frac{1}{2} + \frac{1}{2\sqrt{N}}\right) \leq g\left(\frac{1}{2} + \Lambda\right) \) is satisfied if and only if \( \frac{1}{2} + \frac{1}{2\sqrt{N}} \geq \frac{1}{2} + \Lambda \iff \Lambda \leq \frac{1}{2\sqrt{N}} \). Note also that \( \Lambda \leq \frac{1}{2\sqrt{N}} \iff \tilde{\pi} \leq 1 \), as follows from (36). Hence, if \( \Lambda \leq \frac{1}{2\sqrt{N}} \), then \( \tilde{\pi} \leq \bar{\pi} \) and \( \bar{\pi} \leq 1 \), so the advisor improves the quality of decision-making compared to the benchmark case if and only if \( \pi > \bar{\pi} \). If \( \Lambda > \frac{1}{2\sqrt{N}} \), then \( \tilde{\pi} > \bar{\pi} \) and \( \bar{\pi} \geq 1 \), so the advisor never improves the quality of decision-making. Hence, in both cases, the advisor improves the quality of decision-making compared to the benchmark case if and only if \( \pi > \bar{\pi} \).

8. Auxiliary Lemma A1. Function \( f(x) \equiv \sum_{k=\frac{N+1}{2}}^N P(\varphi(x), N, k) \), where \( \varphi(x) \) is defined by (29), is strictly decreasing and strictly concave.

Proof of Auxiliary Lemma A1. It will be useful to compute the derivative:

\[ \varphi'(x) = -\frac{1}{C_{N-1}^N(N-1)\psi(x)}, \quad (60) \]

where

\[ \psi(x) \equiv \left( \frac{x}{C_{N-1}^N} \right)^{\frac{N}{4}-\frac{3}{2}} \sqrt{\frac{1}{4} - \left( \frac{x}{C_{N-1}^N} \right)^2} \sqrt{\pi^{-1}}. \quad (61) \]

The first derivative of \( f(x) \) is

\[ f'(x) = \sum_{k=\frac{N+1}{2}}^N P_q(\varphi(x), N, k) \varphi'(x) < 0, \]

since \( \varphi'(x) < 0 \) and \( \sum_{k=\frac{N+1}{2}}^N P_q(q, N, k) > 0 \) for any \( q > \frac{1}{2} \), including \( q = \varphi(x) \). The former follows from (60). The latter follows from \( \sum_{k=\frac{N+1}{2}}^N P_q(q, N, k) = -\sum_{k=0}^{\frac{N-1}{2}} P_q(q, N, k) \) and \( P_q(q, N, k) = P(q, N, k) \frac{k-Nq}{q(1-q)} < 0 \) for any \( k < \frac{N}{2} \) because \( q > \frac{1}{2} \). Therefore, \( f(x) \) is strictly decreasing. The second derivative of \( f(x) \) is

\[ f''(x) = \left( \frac{d\varphi}{dx} \right)^2 \sum_{k=\frac{N+1}{2}}^N P_{qq}(\varphi(x), N, k) + \frac{d^2\varphi}{dx^2} \sum_{k=\frac{N+1}{2}}^N P_q(\varphi(x), N, k) \]
Since $\sum_{k=0}^{N} P_q (q, N, k) = 0$ and $\sum_{k=0}^{N} P_{qq} (q, N, k) = 0$,

$$f'' (x) = - \left( \frac{d^2}{dx^2} \right) ^2 \left( \sum_{k=0}^{N-1} P_q (\varphi (x), N, k) \right) - \frac{d^2}{dx^2} \left( \sum_{k=0}^{N-1} P_q (\varphi (x), N, k) \right)$$

$$= - \frac{1}{C_{N-1}^{N-1} (N-1)^2} \sum_{k=0}^{N-1} P_q (\varphi (x), N, k) - \frac{\psi' (x)}{C_{N-1}^{N-1} (N-1) \psi (x)^2} \sum_{k=0}^{N-1} P_q (\varphi (x), N, k)$$

Plugging in $P_q, P_{qq}$ and simplifying,

$$= - \sum_{k=0}^{N-1} P (\varphi (x), N, k) \left[ \frac{N-3}{4} \left( x \frac{N-1}{C_{N-1}^{N-1}} - N + 2 \right) \left( \frac{N-1}{4} - \frac{N-1}{N-1} \frac{1}{\varphi (x)(1-\varphi (x))} \right) \right]$$

Next, we can calculate $\psi' (x)$:

$$C_{N-1}^{N-1} (N-1) \psi' (x) = \left( \frac{N-3}{4} \left( x \frac{N-1}{C_{N-1}^{N-1}} - N + 2 \right) \left( \frac{N-1}{4} - \frac{N-1}{N-1} \frac{1}{\varphi (x)(1-\varphi (x))} \right) \right)^{-\frac{1}{2}}$$

$$= \frac{1}{\varphi (x)-\frac{1}{2}} \left( \frac{N-3}{4} \frac{1}{\varphi (x)(1-\varphi (x))} - N + 2 \right) .$$

Thus,

$$- \sum_{k=0}^{N-1} P (\varphi (x), N, k) \left[ \frac{N-3}{4} \left( x \frac{N-1}{C_{N-1}^{N-1}} - N + 2 \right) \left( \frac{N-1}{4} - \frac{N-1}{N-1} \frac{1}{\varphi (x)(1-\varphi (x))} \right) \right]$$

Multiplying by $(\varphi (x) (1-\varphi (x)))^2$:

$$- \left( \frac{N-3}{4} \left( x \frac{N-1}{C_{N-1}^{N-1}} - N + 2 \right) \left( \frac{N-1}{4} - \frac{N-1}{N-1} \frac{1}{\varphi (x)(1-\varphi (x))} \right) \right)^2 f'' (x)$$

$$= \sum_{k=0}^{N-1} P (q, N, k) \left( (k-Nq)^2 - k (1-q)^2 - (N-k) q^2 + C (k-Nq) \right) \equiv L (q) ,$$

where we denote $\varphi (x)$ by $q \in \left( \frac{1}{2}, 1 \right)$. It follows that $f'' (x) < 0$ if $L (q) > 0$ for any $q \in \left( \frac{1}{2}, 1 \right)$. To prove it, denote

$$\zeta (q, k) \equiv (k-Nq)^2 - k (1-q)^2 - (N-k) q^2 + C (k-Nq)$$

$$= k (k-1) - (2 (N-1) q - C) k + N (N-1) q^2 - CNq ,$$

where $C \equiv \frac{2}{2q-1} \left( \frac{N-3}{4} - (N-2) q (1-q) \right)$. Then,

$$L (q) = \sum_{k=0}^{N-1} P (q, N, k) k (k-1) - (2 (N-1) q - C) \sum_{k=0}^{N-1} P (q, N, k) k + (N (N-1) q^2 - CNq) \sum_{k=0}^{N-1} P (q, N, k) .$$
Consider the first two terms:

1. Term 1:
\[
\sum_{k=0}^{N-1} k (k-1) C_N^k q^k (1-q)^{N-k} = \sum_{k=2}^{N-1} k (k-1) \frac{N!}{k! (N-k)!} q^k (1-q)^{N-k}
\]

\[
= N (N-1) q^2 \sum_{m=0}^{N-1} q^{N-1-m} P (q, N-2, m) = N (N-1) q^2 \Pr [k \leq \frac{N-1}{2} - 1 | k \sim B (N-2, q)].
\]

2. Term 2:
\[
\sum_{k=0}^{N-1} k C_N^k q^k (1-q)^{N-k} = \sum_{k=1}^{N-1} k \frac{N!}{k! (N-k)!} q^k (1-q)^{N-k}
\]

\[
= qN \left( \sum_{k=0}^{N-2} P (q, N-1, k) \right) = qN \Pr [k \leq \frac{N-1}{2} - 1 | k \sim B (N-1, q)].
\]

Hence,
\[
\frac{L(q)}{qN} = (N-1) q \Pr [k \leq \frac{N-1}{2} - 2 | k \sim B (N-2, q)] - 2 (N-1) q - C) \Pr [k \leq \frac{N-1}{2} - 1 | k \sim B (N-1, q)]
\]

\[
+ ((N-1) q - C) \Pr [k \leq \frac{N-1}{2} - 1 | k \sim B (N-1, q)] = I_{1-q} \left( \frac{N+1}{2}, \frac{N+1}{2} \right).
\]

Note that
\[
\Pr [k \leq \frac{N-1}{2} - 1 | k \sim B (N, q)] = I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right),
\]

\[
\Pr [k \leq \frac{N-1}{2} - 2 | k \sim B (N-2, q)] = I_{1-q} \left( \frac{N+1}{2}, \frac{N-3}{2} \right).
\]

where \( I_{1-q} (\cdot) \) is the regularized incomplete beta function. According to the property of the regularized incomplete beta function, \( I_x (a, b+1) = I_x (a, b) + \frac{x^a (1-x)^b}{bB(a,b)} \), where \( B(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!} \) is the beta function. Hence,

\[
I_{1-q} \left( \frac{N+1}{2}, \frac{N+1}{2} \right) = I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) + \frac{(1-q) \frac{N+1}{2} q^{N-1}}{N^2 B \left( \frac{N+1}{2}, \frac{N-1}{2} \right)}
\]

\[
I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) = I_{1-q} \left( \frac{N+1}{2}, \frac{N-3}{2} \right) + \frac{(1-q) \frac{N+1}{2} q^{N-3}}{N^2 B \left( \frac{N+1}{2}, \frac{N-3}{2} \right)}.
\]

Plugging into the expression for \( \frac{L(q)}{qN} \):

\[
\frac{L(q)}{qN} = (N-1) q \left( I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) - \frac{(1-q) \frac{N+1}{2} q^{N-3}}{N^2 B \left( \frac{N+1}{2}, \frac{N-3}{2} \right)} \right) - 2 (N-1) q - C) I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right)
\]

\[
+ ((N-1) q - C) \left( I_{1-q} \left( \frac{N+1}{2}, \frac{N-1}{2} \right) + \frac{(1-q) \frac{N+1}{2} q^{N-3}}{N^2 B \left( \frac{N+1}{2}, \frac{N-3}{2} \right)} \right)
\]

\[
= - (N-1) q \frac{(1-q) \frac{N+1}{2} q^{N-3}}{N^2 B \left( \frac{N+1}{2}, \frac{N-3}{2} \right)} + ((N-1) q - C) \frac{(1-q) \frac{N+1}{2} q^{N-3}}{N^2 B \left( \frac{N+1}{2}, \frac{N-3}{2} \right)}.
\]

Dividing by \( (1-q) \frac{N+1}{2} q^{N-3} \) and simplifying,

\[
\frac{L(q)}{(1-q) \frac{N+1}{2} q^{N-3} N} = q \frac{(N-1)!}{(N-1)! (\frac{N-3}{2})!} (2q - 1) - C q \frac{(N-1)!}{(N-1)! (\frac{N-3}{2})!}.
\]
Hence,
\[
\begin{align*}
L(q)(\frac{N-3}{2})! & (\frac{N-1}{2})!(2q-1) \\
& = (2q - 1)^2 - \frac{2N-1}{N} (N - 2) q (1 - q) \\
& = 4q^2 - 4N + 2N^2 q + 2N - 2q^2 - 2q + 1.
\end{align*}
\]

(67)

Since \(2q^2 - 2q + 1 > 0\), we conclude that \(L(q) > 0\) for any \(q \in (\frac{1}{2}, 1)\). Therefore, \(f''(x) < 0\), which completes the proof.

9. Auxiliary Lemma A2. Function \(\tilde{f}(x)\), defined by (58), is strictly concave.

Proof of Auxiliary Lemma A2. Differentiating \(\tilde{f}(x)\) and using the definition of \(f(x)\),
\[
\tilde{f}''(x) = f''(x) - 2\varphi'(x) - x\varphi''(x).
\]

Using \(f''(x)\) from the proof of Auxiliary Lemma A1 above, in particular, expressions (63), (67), (60), and the derivative of (60), we can write
\[
\tilde{f}''(x) = -x \frac{(2\varphi(x)^2 - 2\varphi(x) + 1)N}{(2\varphi(x) - 1) \varphi(x) (1 - \varphi(x)) (C_{N-1}^{N-1} (N - 1) \psi(x))^2} + \frac{2}{C_{N-1}^{N-1} (N - 1) \psi(x)} - \frac{x\psi'(x)}{C_{N-1}^{N-1} (N - 1) \psi(x)^2}.
\]

Multiplying both sides by \((C_{N-1}^{N-1} (N - 1) \psi(x))^2\), using (62), (61), and (29) and simplifying gives
\[
\left(C_{N-1}^{N-1} (N - 1) \psi(x)\right)^2 \tilde{f}''(x) = -\frac{N - 1}{2} \frac{x}{\varphi(x) (1 - \varphi(x)) (2\varphi(x) - 1)} < 0
\]
since \(\varphi(x) \in (\frac{1}{2}, 1)\). Therefore, \(\tilde{f}(x)\) is strictly concave.